Free $\mu Ł\Pi$ algebras

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Topological and Algebraic Methods in Non Classical Logics III,
Oxford, 6-9 August 2007

Overview

1. Introduction
   - Preliminaries
   - $Ł\Pi$ algebras
   - $\mu Ł\Pi$ algebras
2. Free $\mu Ł\Pi$ algebras
   - Super-algebraic functions
   - Galois theory
   - $\mathcal{F}_\kappa(\mu Ł\Pi)$
3. Further topics and conclusion
   - Forgetful functor

T-norm based logics

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- associative: $x(y * z) = (x * y) * z$,
- commutative: $x * y = y * x$,
- non-decreasing: $x \leq y$ implies $x * z \leq y * z$,
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Definition

A residual $\rightarrow$, of a t-norm $*$, is a function from $[0,1]^2$ to $[0,1]$ such that

$$x*y \leq z \text{ if, and only if, } x \leq y \rightarrow z$$

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Gödel logic is the logic complete w.r.t the following connectives:

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Łukasiewicz logic is the logic complete w.r.t:

$$x \oplus y = \min\{x + y, 1\} \qquad \neg x = 1 - x$$

Mostert and Shields' Theorem

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Every continuous t-norm is locally isomorphic to either Łukasiewicz, product or Gödel t-norm.

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Such a decomposition applies also for the algebraic semantic of continuous t-norm based logic.

Algebraic semantics

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The algebraic semantic of Łukasiewicz logic is given by MV-algebras.

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Theorem (Hájek '98)

The algebraic semantic of product logic is given by Π-algebras.

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- $\varphi \cdot (\psi \ominus \xi) \leftrightarrow (\varphi \cdot \psi) \ominus (\varphi \cdot \xi)$,
- $\Delta(\varphi \rightarrow \psi) \rightarrow \varphi \rightarrow_\Pi \psi$.

Where $\Delta(\varphi)$ is defined as $(\neg\varphi) \rightarrow_\Pi 0$.

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- $\varphi \cdot (\psi \ominus \xi) \leftrightarrow (\varphi \cdot \psi) \ominus (\varphi \cdot \xi)$,
- $\Delta(\varphi \to \psi) \to \varphi \to_\Pi \psi$.

Where $\Delta(\varphi)$ is defined as $(\neg\varphi) \to_\Pi 0$.
The rules are modus ponens and necessitation:

- If φ and $\varphi \to \psi$ then ψ,
- if φ then $\Delta(\varphi)$.

Importance of ŁΠ logic

ŁΠ logic has a stronger expressive power than the above mentioned logics, indeed:

Theorem (Esteva, Godo, Montagna '01)

ŁΠ logic faithfully interprets Łukasiewicz, product and Gödel logic. Moreover, if limited to finite deduction, also Pavelka logic is interpretable in ŁΠ logic.

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More generally:

Theorem (Marchioni, Montagna '06)

Every logic based on a continuous t-norm with a finite number of idempotents is definable in ŁΠ logic.

ŁΠ algebras

Definition

ŁΠ algebras are the algebraic semantic of ŁΠ logic, so they are structures of type $\mathcal{A} = \langle A, \oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1 \rangle$.

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Example

The algebra $\langle [0,1], \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1\rangle$, where the operations are defined as follows:

- $x \oplus y = \min\{x+y, 1\}$ $\qquad \neg x = 1 - x$
- $x \cdot y = xy$ (ordinary product between reals)
- $x \rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$

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is a ŁΠ-algebra. Moreover it generates the variety of ŁΠ-algebras.

μŁΠ logic

We introduce now the main subject: namely the logic ŁΠ with fixed points.

Definition

The logic μŁΠ is obtained as an expansion of ŁΠ logic with a new (generalized) connective

$$\mu_{\varphi(p)}(\bar{\psi})$$

for any ŁΠ-formula $\varphi(p, \bar{\psi})$ in which the symbol $\to_{\Pi}$ does not appear.

Such connectives must satisfy a number of axioms which we give directly in their algebraic form.

$\mu Ł\Pi$ algebras

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which satisfy the following axioms:

$\langle A, \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1 \rangle$ is a $Ł\Pi$-algebra

$\mu x_{t(x)}(\bar{y}) = t(\mu x_{t(x)}(\bar{y}), (\bar{y}))$,

If $t(s(\bar{y}), \bar{y}) = s(\bar{y})$ then $\mu x_{t(x)}(\bar{y}) \leq s(\bar{y})$,

$$\bigwedge_{i \leq n} \Delta(p_i \leftrightarrow q_i) \leq (\mu x_{t(x,\bar{y})}(p_1, \ldots, p_n) \leftrightarrow \mu x_{t(x,\bar{y})}(q_1, \ldots, q_n))$$

The $\mu Ł\Pi$ algebra on [0,1]

Example

The algebra $\langle [0,1], \oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1, \{\mu x_{t(x)}(\bar{y})\}_{t(x,\bar{y}) \in CTerm} \rangle$ is a $\mu Ł\Pi$-algebra.

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Theorem

The $\mu Ł\Pi$-algebra on $[0,1]$ defined above generates the variety of $\mu Ł\Pi$-algebras.

Some notation

Notation

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^{alg}$ *are, respectively, the sets of integer, rational and real algebraic numbers.*
- $\mathbb{Z}[x_1, \ldots, x_n]$ *is the domain of polynomials in n variables and integer coefficients.*
- $\mathbb{Q}(x_1, \ldots, x_n)$ *is its fraction field.*
- *We use the same symbols to denote their members and the associated functions from* $\mathbb{R}^n$ *to* $\mathbb{R}$*.*
- *We will write* $\{P > 0\}$ *for* $\{v \in [0,1]^n \mid P(v) > 0\}$*.*

A remark

Remark

Notice that there is a tight link between fixed points and roots of polynomials. Indeed given a polynomial $P(x)$ its set of solutions is the set of fixed points of the polynomial $P(x) - x$. Viceversa, its set of fixed points can be seen as the set of solutions of the polynomial $P(x) + x$. This correspondence is preserved also when we restrict to the $[0, 1]$ interval.

Super-algebraic functions

Definition

Let us fix any $n \in \mathbb{N}$. We will call root function every function $f(y_1, \ldots, y_n)$ such that for any $P(x) = \sum_{i \leq n} a_i x^i \in \mathbb{R}^{alg}[x]$:

$$f(a_1, \ldots, a_n) = r \text{ iff } r \text{ is the minimum value such that } P(r) = 0.$$

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$$f(a_1, \ldots, a_n) = r \text{ iff } r \text{ is the minimum value such that } P(r) = 0.$$

We will call super-algebraic every function which is

- a rational polynomial function $P/Q \in \mathbb{Q}(x_1, \ldots, x_n)$ or,
- a root function, or
- a composition of the previous two kinds of function.

Super-algebraic functions

Note that root functions are not enough for our description since an element of $t \in \mathit{CTerm}$ can be represented as a member of $\mathbb{Z}[x_1, \ldots, x_n]$. Hence we need a different root function for every function represented by an element of $\mathbb{Z}[x_1, \ldots, x_n]$.

Definition

Let $\mathcal{R}$ be the set of all functions f_R such that given $f \in \mathbb{Z}[x, x_1, \ldots, x_n]$

$$f_R[a_1, \ldots, a_n] = r$$

iff

r is the minimum value for which $f(r, a_1, \ldots, a_n) = 0$.

Super-algebraic functions

Lemma

$\mathcal{R}$ is the set of super-algebraic functions.

Proof.

Notice that a rational polynomial function $P/Q \in \mathbb{Q}(x_1, \ldots, x_n)$ is the function in $\mathcal{R}$ associated to the polynomial $P(x_1, \ldots, x_n) - xQ(x_1, \ldots, x_n)$.
For the other direction notice that composing a root function with a suitable projection gives any desired function in $\mathcal{R}$. □

Galois' theorem

Theorem (Galois 1312)

A polynomial is solvable by radicals if, and only if, the group of automorphisms of the field of its solutions which fix the field of coefficients is solvable.

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In particular there exist polynomials which are not solvable by radicals. Hence super-algebraic functions are not only algebraic functions.
What we have to add to algebraic functions to get super-algebraic functions is unknown.

Free algebras

Notation

The free $\mu Ł\Pi$-algebra over κ generators will be denoted by $\mathcal{F}_\kappa(\mu Ł\Pi)$.

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$\mathcal{F}_\kappa(\mu Ł\Pi)$ is the subalgebra of the algebra of all functions from $[0,1]^\kappa$ to $[0,1]$ generated by the projections closed under $\mu Ł\Pi$ operations defined point-wise.

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In the case of MV-algebras, the characterization comes form a classical result:

Theorem (McNaughton '51)

The functions generated by the projections under MV-operations are exactly the continuous piecewise linear functions with integer coefficients.

$\mathcal{F}_0(\mu Ł\Pi)$

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Proposition

The $\mu Ł\Pi$-algebra on the $[0, 1]$ interval of $\mathbb{R}^{alg}$ can be embedded in every linearly ordered $\mu Ł\Pi$-algebra.

Semialgebraic sets

Definition

A subset S of $[0,1]^n$ is a $\mathbb{Q}$-semialgebraic if it is a boolean combination of sets of the form $\{P > 0\}$ for some $P \in \mathbb{Z}[x_1, \ldots, x_n]$.

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Hats

Definition

A *ŁΠ-hat* is a function $h : [0,1]^n \longrightarrow [0,1]$ such that there exists a $\mathbb{Q}$-semialgebraic set S and a function $f = P/Q \in \mathbb{Q}(x_1, \ldots, x_n)$ such that:

- Q has no zero in S,
- if $x \in S$ then $h(x) = f(x)$,
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- if $x \notin S$ then $h(x) = 0$.

A *μ-hat* is a function $h : [0,1]^n \longrightarrow [0,1]$ such that there exists a semialgebraic set S and a super-algebraic function f such that if $x \in S$ then $h(x) = f(x)$ and if $x \notin S$ then $h(x) = 0$.
If h is a function which satisfies either of these conditions we will indicate it by $\langle S, f \rangle$.

Basic functions

Definition

A basic ŁΠ-function and a basic μ-function over $[0,1]^n$ are, respectively, a finite sum of LΠ-hat and a finite sum of μ-hats

$$\langle S_2, f_1\rangle + \langle S_2, f_2\rangle + ... + \langle S_k, f_k\rangle$$

such that $S_i \cap S_j = \emptyset$ for any $i \neq j$.

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such that $S_i \cap S_j = \emptyset$ for any $i \neq j$.
We will denote by ŁΠB_n and B_n, respectively, the sets of ŁΠ-basic functions over $[0,1]^n$ and μ-basic functions over $[0,1]^n$

The algebra B_n

Theorem (Montagna, Panti '01)

$Ł\Pi B_n$ is the free $Ł\Pi$-algebra over n generators.

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Lemma

B_n contains the projection functions and is a $\mu Ł\Pi$-algebra under point-wise operations.

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Proof.

An easy adaptation of the proof of the previous theorem shows that B_n is closed under ŁΠ operations. □

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Lemma

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Proof.

An easy adaptation of the proof of the previous theorem shows that B_n is closed under $Ł\Pi$ operations. Given a term in $t \in Cterm$ let

$$g = \langle S_1, P_1 \rangle + \ldots + \langle S_r, P_r \rangle$$

be its associated function. □

The algebra B_n

Proof cont'd.

Then we claim

$$\mu x_{t(x,\bar{y})} = \langle T_1, Q_1 \rangle + \ldots + \langle T_r, Q_r \rangle$$

where T_i are semialgebraic sets and Q_i are μ-hat.

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Proof cont'd.

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where T_i are semialgebraic sets and Q_i are μ-hat. Indeed if we associate to any Q_i a new polynomial $Q_i' = Q_i - x$ and call $R_{Q_i'}$ the functions which give the minimum root of the polynomial Q_i', then it is easy to check that:

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where T_i are semialgebraic sets and Q_i are μ-hat. Indeed if we associate to any Q_i a new polynomial $Q_i' = Q_i - x$ and call $R_{Q_i'}$ the functions which give the minimum root of the polynomial Q_i', then it is easy to check that:

$$\begin{aligned} \mu x_{t(x,\bar{y})} = & \langle \{\bar{y} \mid \exists z\, (Q_1(z,\bar{y}) = z \wedge (z,\bar{y}) \in T_1)\}, R_{Q_1'} \rangle + \\ & \vdots \\ & + \langle \{\bar{y} \mid \exists z\, (Q_r(z,\bar{y}) = z \wedge (z,\bar{y}) \in T_r)\}, R_{Q_r'} \rangle \end{aligned}$$

□

The algebra B_n

Proof cont'd.

But for every $1 \leq i \leq r$ the set

$$\{\bar{y} \mid \exists z\,(Q_i(x,\bar{y}) = x \wedge (x,\bar{y}) \in T_i)\}$$

is a projection of a semialgebraic set.

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Proof cont'd.

But for every $1 \leq i \leq r$ the set

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is a projection of a semialgebraic set.
Hence by Tarski-Seidenberg theorem it is in turn a semialgebraic set. Moreover they are all disjoint, since the sets T_i are disjoint, and $R_{Q_i}(\bar{y}) \neq 0$ as $(x,\bar{y}) \in T_i$.

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Hence by Tarski-Seidenberg theorem it is in turn a semialgebraic set. Moreover they are all disjoint, since the sets T_i are disjoint, and $R_{Q_i}(\bar{y}) \neq 0$ as $(x, \bar{y}) \in T_i$.
From this follows that if $f_1, \dots, f_n$ are functions in B_n then $\mu x_{t(x)}(f_1, \dots, f_n)$ is also in B_n. □

Lemma

Let $P \in \mathbb{R}^{alg}[x_1, \ldots, x_n]$ *and let* $P^\sharp : [0,1]^n \longrightarrow [0,1]$ *be defined for any* $\bar{v} \in [0,1]^n$ *by* $P^\sharp(\bar{v}) = \min\{\max\{P(\bar{v}), 0\}, 1\}$. *The* $P^\sharp \in \mathcal{F}_\kappa(\mu Ł\Pi)$.

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Corollary

The characteristic function of every semialgebraic set is in $\mathcal{F}_\kappa(\mu Ł\Pi)$

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Corollary

The characteristic function of every semialgebraic set is in $\mathcal{F}_\kappa(\mu Ł\Pi)$

Proof.

Since $\mathcal{F}_\kappa(\mu Ł\Pi)$ is closed under the boolean operator it suffices to prove that the characteristic functions of the sets of the form $\{P > 0\}$ are in $\mathcal{F}_\kappa(\mu Ł\Pi)$. But such a function is just $\neg\Delta(P^\sharp)$ □

$\mathcal{F}_\kappa(\mu Ł\Pi)$

Theorem

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Proof.

We need to show that every basic function is in $\mathcal{F}_\kappa(\mu Ł\Pi)$.

□

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We need to show that every basic function is in $\mathcal{F}_\kappa(\mu Ł\Pi)$. Since the semialgebraic sets appearing in the definition of a basic function are pairwise disjoint we can substitute every $+$ with $\oplus$.

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Theorem

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Proof.

We need to show that every basic function is in $\mathcal{F}_\kappa(\mu Ł\Pi)$. Since the semialgebraic sets appearing in the definition of a basic function are pairwise disjoint we can substitute every $+$ with $\oplus$. Hence it is sufficient to show that every μ-hat is in $\mathcal{F}_\kappa(\mu Ł\Pi)$, which easily comes from the definition. □

The forgetful functor

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For general reasons this functor has an adjoint $\mathcal{G}$ which, given a ŁΠ-algebra $\mathcal{A}$, gives the free μŁΠ-algebra on $\mathcal{A}$. Since the new structure is based on old terms, this construction has a number nice porperties.

The forgetful functor

Proposition

The functor $\mathcal{G}$ creates isomorphisms.

The forgetful functor

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Proposition

Given two linearly ordered $\mu Ł\Pi$-algebras they are isomorphic if, and only if, their underlying ŁΠ-algebras are isomorphic.