Semisimplicity, EDPC and discriminator varieties of bounded commutative residuated lattices with S4-like modal operator

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Outline of my talk

• Substructural logics & Residuated lattices
  – Substructural logics
  – Residuated lattices
  – Extensions: +modality

• Main result:
  \( V \subseteq \Box_{BCRL} \), semisimple = discriminator
Substructural logics
&
Residuated lattices
Substructural logics

- Substructural logics: LJ (or LK) – structural rules,
  
  - rules

  - linear logic, relevant logic, fuzzy logic
Basic substructural logic : FL

No structural rules

FL = LJ – \{e, w, c\}

(CFL = LK – \{e, w, c\})

\[ \Gamma, A, B \vdash C \quad \Gamma \vdash C \quad \Gamma, A, A \vdash C \]

\[ \Gamma, B, A \vdash C \quad \Gamma, A \vdash C \quad \Gamma, A \vdash C \]
Sequent system: FL

\[
\begin{align*}
a & \vdash a, & 1, 0 & \vdash \\
\Gamma & \vdash A, & \Gamma & \vdash \\
\Delta, \Gamma, \Sigma & \vdash C & \Gamma, \Delta & \vdash C \\
\Delta, \Gamma, \Sigma & \vdash C & \Gamma, 1, \Delta & \vdash C \\
\Gamma & \vdash A, & \Delta, B, \Sigma & \vdash C & A, \Gamma & \vdash C \\
\Delta, \Gamma, A \rightarrow B, \Sigma & \vdash C & \Gamma & \vdash A \rightarrow C & \Gamma, A & \vdash C \\
\Gamma & \vdash A, & \Delta, B, \Sigma & \vdash C & \Delta, B & \vdash A, \Gamma, \Sigma & \vdash C \\
\Gamma, A & \vdash C & \Gamma & \vdash C & \Gamma & \vdash C \leftarrow A \\
\end{align*}
\]
Sequent system : FL

\[
\begin{align*}
\frac{\Gamma, A, B, \triangle \vdash C}{\Gamma, A \otimes B, \triangle \vdash C} & \quad \frac{\Gamma \vdash A \quad \triangle \vdash B}{\Gamma, \triangle \vdash A \otimes B} \\
\frac{\Gamma, A(B), \triangle \vdash C}{\Gamma, A \land B, \triangle \vdash C} & \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \\
\frac{\Gamma, A, \triangle \vdash C \quad \Gamma, B, \triangle \vdash C}{\Gamma, A \lor B, \triangle \vdash C} & \quad \frac{\Gamma \vdash A \quad B}{\Gamma \vdash A \lor B}
\end{align*}
\]
Basic substructural logics

- FL, FLe, FLw, FLew, ...

- FLew = FL + \{e, w\} = LJ – \{c\}
  
  Monoidal logic (Fuzzy logic)

- FLe = ILL – \{!, ?\}
Basic results

• Cut elimination theorem:
  FL, FLe, FLw, FLew, FLe, FLecw (= LJ)
  (CFLe, CFLew, CFLec, CFLecw (= LK))
Residuated lattices

- **Definition**: \( A = (A, \cdot, \rightarrow, \leftarrow, \land, \lor, 1) \)
  - \( (A, \cdot, 1) \): monoid
  - \( (A, \land, \lor) \): lattice
  - \( x \cdot y \leq z \iff x \leq z \leftarrow y \iff y \leq x \rightarrow z \)

- **Pointed residuated lattice = FL-algebra**
  - \( A = (A, \cdot, \rightarrow, \leftarrow, \land, \lor, 1, 0) \)
  - \( 0 \): arbitrary but fixed element of \( A \)
Basic facts

• The class of residuated lattices forms a variety: $\mathbf{RL}$

• Subvarieties:
  – $\mathbf{FL}$, $\mathbf{CRL}$, $\mathbf{IRL}$, …
  – Commutativity, integrality, increasing-idenpotency
Substructural logics & Residuated lattices

• Completeness theorem:
  Algebras for FLx is FLx-algebras
  \((x = e, w, ew, \ldots)\)

• Lindenbaum construction:
  \(\text{Frm} / \sim \quad A \sim B \equiv A \vdash B \text{ and } B \vdash A\)
Algebra - Logic

• commutativity ⇔ exchange
• integrality ⇔ weakening
• increasing-idempotency ⇔ contraction

• FLe, FLw-, FLew-algebra, …
• FLe, FLw, FLew, …
Book

- Residuated Lattices: an algebraic glimpse at substructural logics, P. Jipsen, T. Kowalski, N. Galatos and H. Ono
  - Residuated Lattices: an algebraic glimpse at logics without contraction, T. Kowalski and H. Ono (starting point for the book)
Extensions

- **Substructural logics + modalities**
  - What is natural modalities in substructural logics?
  - H. Ono, Modalities in substructural logics, a preliminary report
- Algebras for modal substructural logics = Residuated lattices + operators
  (cf. BAO’s)
\( \square \text{FLe} (\square \text{BCRL}) \)

- \( \square \text{FLe} = \text{FLe} + \text{S4-like modality} \)

\[
\begin{align*}
\square \Gamma & \vdash A \\
\square \Gamma & \vdash \square A \\
A, \Gamma & \vdash B \\
\square A, \Gamma & \vdash B
\end{align*}
\]

Cut elimination theorem holds for \( \square \text{FLe} \)
□FLe-algebras (□BCRL)

• $A = (A, \cdot, \rightarrow, \land, \lor, 1, 0, T, \perp, □)$
  – $(A, \cdot, \rightarrow, \land, \lor, 1, 0, T, \perp) : \text{FLe-algebra}$
  – S4-like modality
  – $1 \leq □1$,
  – $□x \cdot □y \leq □(x \cdot y)$
  – $□x \leq x$
  – $□x \leq □ □x$
  – $x \leq y \Rightarrow □x \leq □y$

• The class of □FLe-algebras forms a variety
☐ FLe & Modal FLe-algebras

• Completeness theorem:
  – Algebras for ☐ FLe is ☐ FLe-algebras
Congruence filter of □FLe-algebra

• F is a congruence filter:
  – 1 ∈ F
  – x, y ∈ F ⇒ x ∧ y ∈ F
  – x, x → y ∈ F ⇒ y ∈ F
  – x ∈ F ⇒ □x ∈ F
• <S> = \{x ∈ A: x ≥ □(s_1 ∧ 1) \ldots □(s_k ∧ 1), s_i ∈ S\}
Algebra basics

• $V$: variety is semisimple
  – All its algebras are semisimple

• $A$ in $\Box_{BCRL}$, $x \in \text{Rad}_A \iff \forall n \geq 1 \exists m$ s.t.,
  \[(\Box \neg (\Box (x \land 1))^n)^m = \bot, \neg x = x \rightarrow \bot\]

• $A$ is semisimple:
  \[\forall x \in A, \text{not greater than } 1, \exists n \geq 1, \text{s.t.},\]
  \[(\Box \neg (\Box x \land 1))^n)^m \neq \bot \text{ for any } m\]
Algebra basics

• V: variety is discriminator
  – The **ternary discriminator** is a term operation on every si algebra in V
    \[ t(x, y, z) = x \text{ if } x=y \]
    \[ z \text{ otherwise} \]
  – Algebra with discriminator term is simple
Algebra basics

• Discriminator variety $\Rightarrow$ semisimple variety
• Discriminator variety $V \Rightarrow V$ has the CEP
• **DPC** (definable principle congruence)
  – A first order formula $\Phi$, $a,b,c,d$ in $A$
  – $(c,d)$ in $\Theta(a,b) \iff A \models \Phi(a,b,c,d)$
• **EDPC** (equational definable principle congruence)
  – If $\Phi$ can be taken a finite set of equations
Facts

- $V$ is congruence-permutative $\Rightarrow$ discriminator = semisimple + EDPC

If semisimple $\Rightarrow$ EDPC
then discriminator = semisimple
Some historical remarks

• Every free classical FLew-algebras is semisimple (Grishin)
• The variety of FLew-algebras is generated by its finite simple members (Kowalski & Ono)
• Every free FLw-algebras is semisimple
• The variety of □FLew-algebras is generated by its finite simple members
Some historical remarks

• $V \subseteq FL_{ew}$, $V$ is discriminator
  
  = $V$ is semisimple

  = $V$ satisfies that $x \lor \neg(x^n) = 1$

  \[x^n = x \cdot \ldots \cdot x, \text{n-times}\]

  (Kowalski 2005)
Goal of my talk

- $V \subseteq \Box_{BCRL}$, $V$ is discriminator
  - $V$ is semisimple
  - $V \models \Box (x \land 1) \lor \neg (\Box (x \land 1))^n$
    for some natural number $n$
\[ \square E(1, n) \& \square EM(1, n) \]

- \( \square E(1, n) : \)
  \[ (\square (x \land 1))^n = \square (x \land 1))^{n+1} \]
  for any natural number \( n \)

- \( \square EM(1, n) : \)
  \[ \square (x \land 1) \lor \neg (\square (x \land 1))^n = 1 \]
  for any natural number \( n \)
Proposition

- $V \subseteq \Box^\text{BCRL}$, $V$ has EDPC
- $= V$ has DPC
- $= V \subseteq \Box^E(1,n)$
- for some natural number $n$
- $= V \models (\Box (x \land 1))^n = (\Box(x \land 1))^{n+1}$
- for some natural number $n$
Set up congruence

• $A$ in $V$ s.t. $(\Box(a \wedge 1))^n > \bot$, $a$ an element not greater than 1

• $\alpha = Cg(a, 1)$; nonzero, nonfull, principal

$\Rightarrow \exists \beta$ subcover

**Lemma** $\exists m$ s.t.,

$(\Box(a \wedge 1))^{m+1} \equiv \beta (\Box(a \wedge 1))^m$

$\neg(\Box(a \wedge 1))^m \equiv \beta (\neg(\Box(a \wedge 1))^m)^2$

$(\Box(a \wedge 1))^m \equiv \beta \neg\neg(\Box(a \wedge 1))^m$
A necessary condition for semisimplicity

• $V$ is semisimple subvariety of $\Box_{BCRL}$,
  $\forall \models \Box ?$

$\Box \equiv \Box (x \land 1) \geq (\neg (\neg \Box (x \land 1)_r)^k)_l$

Suppose $V$ falsifies $\Box$, Put $\Theta \equiv \forall V \theta_r$, $\theta_r = Cg (\neg (\neg \Box (x \land 1)_r)^K, 1)$

$K$ is the smallest number $V$ falsifies $\Box$
Some lemmas

• $0 < \Theta < \alpha$

• $V$ is semisimple subvariety of $\Box_{BCRL}$,
  $V \models \Box ? \quad YES!$

\[
V \models (\Box x \land 1) \geq (\neg(\neg\Box(x \land 1))^r)^k \mid_l
\]
for any $k$ there exist $r$ & $l$
Function $r$

- Suppose
  \[ V \models (\Box x \land 1) \geq (\neg (\neg \Box (x \land 1)^{r(i)})^{i})^{l} \]
- $r : \mathbb{N} \rightarrow \mathbb{N}$,
  
  $r(i)$ the smallest number s.t., $\exists l \in \mathbb{N}$ with
  \[ V \models (\Box (x \land 1)) \geq (\neg (\neg \Box (x \land 1)^{r(i)})^{i})^{l} \]
- Lemma: $r$ is non-decreasing function
Semisimple forces $\square^{EM}(1,n)$

- Lemma

$V \subseteq \square^{BCRL}$, semisimple,

$V \models (\square(x \land 1))^{n+1} = (\square(x \land 1))^n$

for some natural number $n$
Main theorem

• \( V \subseteq \square^{BCRL} \), \( V \) is discriminator

= \( V \) is semisimple

= \( V \models \square(x \land 1) \lor \neg \square(x \land 1)^n \)

for some natural number \( n \)
Corollary 1

- \( V \subseteq \square^{FL_e} \), \( V \) is discriminator
  = \( V \) is semisimple
  = \( V \models \square(x \land 1) \lor \lnot \square(x \land 1)^n \)
  for some natural number \( n \)
Corollary 2

- $V \subseteq \Box F_{Le}w$, $V$ is discriminator
  = $V$ is semisimple
  = $V \models \Box x \lor \neg (\Box x)^n$
    for some natural number $n$
Corollary 3

• $V \subseteq FLe_w$, $V$ is discriminator
  = $V$ is semisimple
  = $V \models x \lor \neg x^n$
    for some natural number $n$