Modal logic for metric and topology

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TANCL
August, 2007

Some Modal Logics for Distance/Metric Spaces

- Topology: modal operators as closure/interior operators, as derived set operator, etc.

$$\mathbb{I}P = \{w \mid \exists\epsilon > 0\ \forall v\ d(w, v) < \epsilon \Rightarrow v \in P\}.$$

  S4 is the logic of all metric spaces, the real line $\mathbb{R}$, and any Euclidean space.

- Conditional Logic/Nonmonotonic Logics/Belief Revision 'if it had been the case that φ, it would have been the case that ψ.'

$$w \models \varphi > \psi \;\;\Leftrightarrow\;\; \psi \text{ is true in all closest } \varphi\text{-worlds.}$$

  Mostly interpreted in distance spaces with limit assumption:

$$d(P, Q) = \inf\{d(v, w) \mid v \in P, w \in Q\} = \min\{d(v, w) \mid v \in P, w \in Q\}$$

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- Comparative Similarity Logic: 'more similar to a P-object than any Q-object.'

$$w \in P \leftleftarrows Q \;\Leftrightarrow\; d(w,P) < d(w,Q).$$

- Absolute Similarity Logic: 'similar to a P-object with degree at least $a \in \mathbb{R}^{\geq 0}$.'

$$w \in \exists^{\leq a} P \;\Leftrightarrow\; \exists v\; d(w,v) \leq a \wedge v \in P.$$

- Metric Temporal Logic over $\mathbb{R}$: 'within a time-units P.'

$$w \in \exists^{< a} P \;\Leftrightarrow \exists v\; v > w \wedge d(v,w) < a \wedge v \in P$$

Topology:

$$\mathbb{I}P = S(P,\top) \wedge P \wedge U(P,\top).$$

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Aim

A modal logic framework covering large parts of these lines of research, thus enabling a comparison of logics for distances and a systematic investigation of their semantics, expressive power and complexity.

Distance models

A distance space is a structure (Δ, d) with $d : \Delta \times \Delta \to \mathbb{R}^{\geq 0}$ such that

- $d(x, y) = 0$ iff $x = y$.

(Δ, d) is a metric space if we have, in addition,

- triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$;
- symmetry: $d(x, y) = d(y, x)$.

A distance model is a relational structure

$$M = (\Delta, d, p_1^M, \ldots),$$

in which (Δ, d) is a distance space and $p_i^M \subseteq \Delta$.

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Operators on metric/distance spaces

$a \in \mathbb{R}^{>0}$:

- $\exists^{<a} P = \{w \mid \exists v\ d(w,v) < a \wedge v \in P\}$
- $\forall^{<a} P = \{w \mid \forall v\ d(w,v) < a \rightarrow v \in P\}$
- $\forall^{<a}_{>0} P = \{w \mid \forall v\ 0 < d(w,v) < a \rightarrow v \in P\}$
- Interior of P: $\mathbb{I}P \ = \ \exists x \forall^{<x} P$
- Universal box: $\Box P \ = \ \forall x \forall^{<x} P$
- Derived set of P: $\partial P \ = \ \forall x \exists^{<x}_{>0} P$
- Closer operator $P \leftleftarrows Q \ = \ \exists x (\exists^{<x} P \sqcap \neg \exists^{<x} Q)$
- Conditional implication (with and without limit assumption):

$$P > Q \ = \ \neg \exists x \exists^{<x} P \sqcup \exists x (\exists^{<x} P \sqcap \neg \exists^{<x} (P \sqcap \neg Q))$$

$$P > Q \ = \ \exists x \exists^{<x} P \rightarrow \exists x \left(\exists^{\leq x} P \wedge \neg (\exists^{\leq x} (P \wedge \neg Q) \vee (P \wedge \neg Q))\right).$$

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General framework: qualitative metric system $\mathcal{QMS}$

Distance variables $x_1, x_2, \ldots$

Set variables $p_1, p_2, \ldots$

Constraints on relations between distance variables like, e.g.,

- the set Σ_0 of inequalities $x_i < x_j$,
- the set Σ_1 of linear rational equalities

  $a_1 x_1 + \cdots + a_n x_n = a_{n+1}$,

$\mathcal{QMS}[\Sigma]$-terms τ, for a set Σ of constraints $\varkappa$:

$$\tau \ ::= \ p_i \mid \varkappa \mid \neg\tau \mid \tau_1 \sqcap \tau_2 \mid \exists x_i \tau \mid \exists^{=x_i}\tau \mid \exists^{<x_i}\tau \mid \exists^{>x_i}\tau \mid \exists^{<x_i}_{>x_j}\tau$$

'Syntactic sugar:' $\tau_1 \sqsubseteq \tau_2 = \forall x \forall^{<x}(\neg\tau_1 \sqcup \tau_2)$.

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Expressive completeness

$\mathcal{FM}[\Sigma]$, the two-sorted first-order language $\mathcal{FM}[\Sigma]$ with

- individual variables $x_1, x_2, \dots$ of sort $\mathbb{R}^{\geq 0}$
- individual variables $w_1, w_2, \dots$ of sort object

$\mathcal{FM}[\Sigma]$-formulas φ:

$$\varphi \; ::= \; P_j(w_i) \mid \varkappa \mid d(w_i, w_j) < x_k \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \exists x_i \varphi \mid \exists w_i \varphi$$

$\mathcal{FM}_2[\Sigma]$ is the fragment of $\mathcal{FM}[\Sigma]$ with only two variables of sort object.

Expressive completeness

For each $\mathcal{QMS}[\Sigma]$-term τ, there is an $\mathcal{FM}_2[\Sigma]$-formula φ with one free variable of sort object such that, for all models M with assignments $\mathfrak{a}$ and all $o \in \Delta$,

$$o \in \tau^{M,\mathfrak{a}} \quad \text{iff} \quad (M, \mathfrak{a}) \models \varphi[o] \qquad (\bigstar)$$

Conversely, for each $\mathcal{FM}_2[\Sigma]$-formula φ with one free variable of sort object, there is a $\mathcal{QMS}[\Sigma]$-term τ such that ($\bigstar$) holds for all M with assignments $\mathfrak{a}$ and all $o \in \Delta$.

$\mathcal{FM}_2[\Sigma]$ is, however, exponentially more succinct than $\mathcal{QMS}[\Sigma]$.

Plan

- Logics without distance variables (constants for distances);
- The operators $\exists^{<x}$ and $\exists^{\leq x}$ (and their duals):
  - Logics of topology and absolute distance: operators

$$\exists^{<a},\ \exists^{\leq a},\ \exists x \forall^{<x}\tau,\ \exists x \forall^{\leq x}\tau,\ \forall x \forall^{<x}\tau,\ \forall x \forall^{\leq x}\tau$$

  - Logics of topology and comparative distance: operators

$$\exists x \mathrm{Bool}(\forall^{<x}\tau, \forall^{\leq x}\tau, \exists^{\leq x}\tau, \exists^{<x}\tau, p),$$

where τ is a set variable or again of the form $\exists x \mathrm{Bool}(\cdots)$.

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Fragments without distance variables

Terms τ are defined as ($a \in \mathbb{Q}^{\geq 0}$):

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Theorem Expressively complete for corresponding 2-variable fragment of FO-Logic. Satisfiability decidable for (symmetric) distance space. Undecidable for spaces with triangle inequality (three variables)!.

Operators	Space	Complexity
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$\exists^{=a}, \Box_F$	$\mathbb{R}$	undecidable

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$\exists^{<a}, \exists^{\leq a}$	Metric Spaces/$\mathbb{R}$/$\mathbb{R}^2$	ExpTime/PSpace/undecidable
$\exists^{>a}, \exists^{\leq a}$	Metric Spaces/$\mathbb{R}$/$\mathbb{R}^2$	in NExptime/PSpace/undecidable
$\exists^{=a}, \Box_F$	$\mathbb{R}$	undecidable

Topology and absolute distance

Terms τ are defined as ($a \in \mathbb{Q}^{\geq 0}$):

$$\tau = \; p_i \mid \neg\tau \mid \forall x \forall^{<x} \tau \mid \tau_1 \sqcap \tau_2 \mid \exists x \forall^{<x} \tau \mid \exists^{<a} \tau \mid \exists^{\leq a} \tau$$

equivalently:

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Interaction axioms:

$$\mathbb{C}p \to \exists^{<a} p, \;\; \exists^{<a} \mathbb{C}p \to \exists^{<a} p.$$

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Topology and absolute distance

The set of terms valid in metric spaces coincides with the set of terms valid in (finite) relational models of the form

$$\mathcal{F} = (\Delta, R, S_a^{<}, S_a^{\leq}),$$

where, for example,

$$uRv \Rightarrow uS_a^{<}v, \quad uS_a^{<}vRw \Rightarrow uS_a^{<}v$$

Representation Theorem: For every finite model $\mathcal{F}$ there exists a metric space $\mathcal{M}$ such that $\mathcal{F}$ is a 'p-morphic image' of $\mathcal{M}$.

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Topology and comparative distance

$\mathcal{QML}$-terms are constructed from set variables $p_1, p_2, \ldots$ using $\sqcap$, $\neg$, and the constructor

$$\exists x \text{Bool}(\forall^{<x}\tau, \forall^{\leq x}\tau, \exists^{\leq x}\tau, \exists^{<x}\tau, p),$$

where τ is a set variable or again of the form $\exists x\text{Bool}(\ldots)$.

Contains closure operator, universal modality and conditional implication (with and without limit assumption):

$$P > Q \ = \ \neg\exists x \exists^{<x} P \sqcup \exists x(\exists^{<x} P \sqcap \neg\exists^{<x}(P \sqcap \neg Q))$$

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An equivalent language

Set

- $\tau_1 \leftleftarrows \tau_2 = \exists x(\exists^{<x}\tau_1 \sqcap \neg\exists^{<x}\tau_2)$,
- $\tau_1 \overset{\leftarrow}{-} \tau_2 = \exists x(\exists^{\leq x}\tau_1 \sqcap \neg\exists^{\leq x}\tau_2)$,
- $\tau_1 \overset{\leftarrow}{=} \tau_2 = \exists x(\exists^{\leq x}\tau_1 \sqcap \neg\exists^{<x}\tau_2)$.

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$$\tau_1 \leftleftarrows \tau_2 \quad \Rightarrow \quad \tau_1 \overset{\leftarrow}{-} \tau_2 \quad \Rightarrow \quad \tau_1 \overset{\leftarrow}{=} \tau_2.$$

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Then

$$\tau_1 \leftleftarrows \tau_2 \quad \Rightarrow \quad \tau_1 \overset{\leftharpoonup}{-} \tau_2 \quad \Rightarrow \quad \tau_1 \overset{\leftharpoonup}{\equiv} \tau_2.$$

Comparison and ‘inf/min’

Set

$$\text{ⓡ}\tau = \tau \overset{\leftarrow}{=} \tau = \exists x(\exists^{\leq x}\tau \sqcap \neg\exists^{<x}\tau)$$

Then

$$(\text{ⓡ}\tau)^M \;=\; \big\{u \in \Delta \mid d(u,\tau^M) = \min\{d(u,v) \mid v \in \tau^M\}\big\}.$$

We obtain:

- $\tau_1 \overset{\leftarrow}{-} \tau_2 \;\equiv\; (\tau_1 \Leftarrow \tau_2) \sqcup (\neg(\tau_2 \Leftarrow \tau_1) \sqcap \text{ⓡ}\tau_1 \sqcap \neg\text{ⓡ}\tau_2)$;
- $\tau_1 \overset{\leftarrow}{=} \tau_2 \;\equiv\; (\tau_1 \Leftarrow \tau_2) \sqcup (\neg(\tau_2 \Leftarrow \tau_1) \sqcap \text{ⓡ}\tau_1)$.

Comparative distance logic

$\mathcal{CSL}$-terms τ are defined by

$$\tau \quad ::= \quad p_i \mid \neg\tau \mid \tau_1 \sqcap \tau_2 \mid ⓡ\tau \mid \tau_1 \leftleftarrows \tau_2,$$

or, equivalently,

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Theorem. For every $\mathcal{QML}$-term τ, there is a $\mathcal{CSL}$-term τ^* such that $\tau \equiv \tau^*$.

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Proof of $\mathcal{QML} \equiv \mathcal{CSL}$ (exp blowup)

Every $\mathcal{QML}$-term is equivalent to a term of the form

$$\tau \;\equiv\; \exists x \Big(\prod_{i \in I_0} \exists^{<x} \varphi_i \sqcap \prod_{i \in I_1} \exists^{\leq x} \varphi_i \sqcap \prod_{j \in J_0} \neg \exists^{\leq x} \psi_j \sqcap \prod_{j \in J_1} \neg \exists^{<x} \psi_j \Big) \sqcap \tau'.$$

Let $I = I_0 \cup I_1$, $J = J_0 \cup J_1$. Then τ is equivalent to the $\mathcal{CSL}$-term

$$\bar{\tau} \;=\; \prod_{i \in I_0, j \in J} (\bar{\varphi}_i \mathrel{\overset{\leftarrow}{-}} \bar{\psi}_j) \;\sqcap\; \prod_{i \in I_1, j \in J_0} (\bar{\varphi}_i \mathrel{\overset{\leftharpoonup}{-}} \bar{\psi}_j) \;\sqcap\; \prod_{i \in I_1, j \in J_1} (\bar{\varphi}_i \mathrel{\overset{\leftharpoonup}{=}} \bar{\psi}_j).$$

Observation:

$$\exists x (\prod_i \exists^{<x} \tau_i \sqcap \neg \exists^{<x} \rho) \equiv \prod_i \exists x \, (\exists^{<x} \tau_i \sqcap \neg \exists^{<x} \rho).$$

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$\mathcal{QML}$ and $\mathcal{CSL}$

The complexity of checking satisfiability of $\mathcal{QML}$/$\mathcal{CSL}$-terms:

Distance spaces	Complexity
All spaces/symmetric spaces	ExpTime
Triangle inequality	ExpTime
Metric spaces	ExpTime
$\mathbb{R}$	non r.e.
$\mathbb{Z}$	non r.e.
finite subspaces of $\mathbb{R}$	non r.e.

Proof: (i) Tree-like distance spaces are sufficient. (ii) Reduction of Diophantine equations.

Hilbert-style Axiomatization: (sym) distance spaces

$$((\varphi \leftleftarrows \psi) \sqcap (\psi \leftleftarrows \chi)) \;\to\; (\varphi \leftleftarrows \chi),$$
$$(\neg(\varphi \leftleftarrows \psi) \sqcap \neg(\psi \leftleftarrows \chi)) \;\to\; \neg(\varphi \leftleftarrows \chi), \tag{1}$$

$$\neg((\varphi \sqcup \psi) \leftleftarrows \varphi) \;\sqcup\; \neg((\varphi \sqcup \psi) \leftleftarrows \psi), \tag{2}$$

$$\forall(\varphi \to \psi) \;\to\; \neg(\varphi \leftleftarrows \psi), \tag{3}$$

$$ⓡ(\varphi \sqcup \psi) \;\to\; (ⓡ\varphi \sqcup ⓡ\psi), \tag{4}$$

$$(ⓡ(\varphi \sqcup \psi) \sqcap (\varphi \leftleftarrows \psi)) \;\to\; ⓡ\varphi \tag{5}$$

$$ⓡ\varphi \sqcap \neg(\psi \leftleftarrows \varphi) \;\to\; ⓡ(\varphi \sqcup \psi) \tag{6}$$

$$\forall(\varphi \leftrightarrow \psi) \;\to\; (ⓡ\varphi \leftrightarrow ⓡ\psi), \tag{7}$$

$$\varphi \;\leftrightarrow\; (ⓡ\varphi \sqcap \neg(\top \leftleftarrows \varphi)), \tag{8}$$

$$\top \leftleftarrows \bot, \tag{9}$$

$$\negⓡ\bot, \tag{10}$$

Axiomatization for spaces with triangle inequality

Add

$$\tau = \neg(\Diamond p \leftleftarrows p).$$

τ valid in distance spaces with the triangle inequality but not valid in symmetric distance spaces: $\tau^{\mathfrak{S}} \neq \emptyset$ in the symmetric model $\mathfrak{S}$, where

$$\Delta^{\mathfrak{S}} = \{a, b, c_i \mid i \in \mathbb{N}\},$$

$$p^{\mathfrak{S}} = \{c_i \mid i \in \mathbb{N}\},$$

$$d^{\mathfrak{S}}(a, c_i) = 2, \quad i \in \mathbb{N},$$

$$d^{\mathfrak{S}}(a, b) = 1, \quad d^{\mathfrak{S}}(b, c_i) = 1/2^i, \quad i \in \mathbb{N},$$

and all other distances are defined by symmetry.

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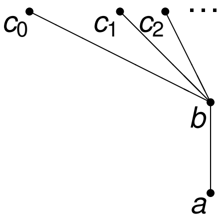

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Axiomatization for Metric Spaces

Add

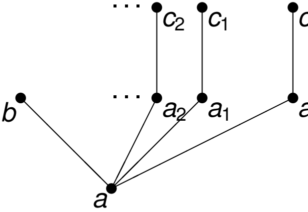
$$\tau = (p \leftleftarrows q) \to \Box(p \leftleftarrows q)$$

τ is valid in metric spaces but not in the following non-symmetric model $\mathfrak{T}$ satisfying the triangle inequality:

$\Delta^{\mathfrak{T}} = \{a, a_i, b, c_i \mid i \in \mathbb{N}\}$;

$p^{\mathfrak{T}} = \{b\}, \quad q^{\mathfrak{T}} = \{c_i \mid i \in \mathbb{N}\}$,

$d(a, b) = d(b, a) = 1, \quad d(a, a_i) = 1/2^i$,
$d(a_i, a) = 1, \quad d(a_i, c_i) = d(c_i, a_i) = 3/2, \quad i \in \mathbb{N}$;



and the other distances are computed as the lengths of the corresponding paths in graph above.

Open problems

- Is $\mathcal{QML}[\Sigma]$ 'the' bisimulation-invariant fragment of $\mathcal{QMS}[\Sigma]$ ($\mathcal{FM}[\Sigma]$)?
- Algebraic semantics for $\mathcal{QMS}[\Sigma]$? Does $\mathcal{QML}[\Sigma]$ have the finite model property?
- Relational semantics for $\mathcal{QMS}[\Sigma]$? Duality?
- Other intererresting classes of metric spaces: compact, connected?

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Articles

- Kutz, Sturm, Suzuki, Wolter, Zakharyaschev, TOCL, 2003 (absolute distances);
- Wolter and Zakharyaschev, JSL, 2005 (topology and absolute distance);
- Lutz, Walther, Wolter, Information and Computation, 2007 (metric temporal logic);
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