Self-referential options

From toy examples to carbon markets

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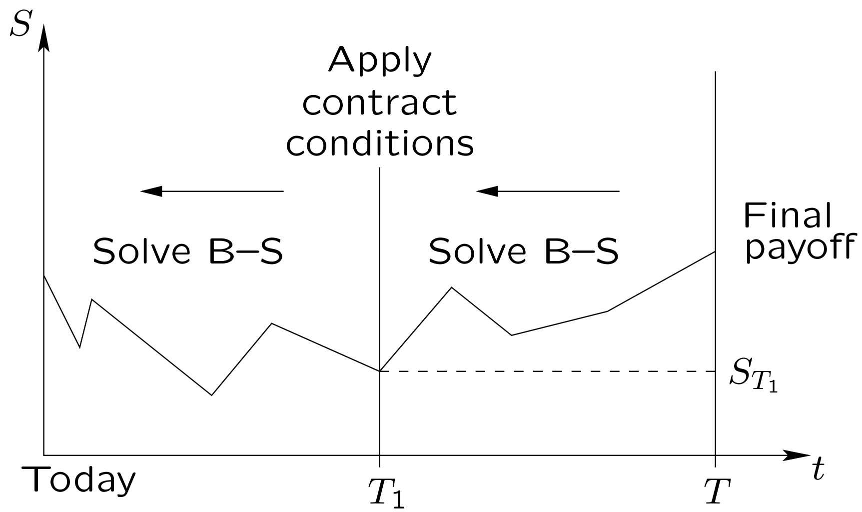
The serpent that eateth his tail

The worm Ouroboros.

Classical forward start option

This contract

- Is a call option expiring at time T
- With strike S_{T_1} where $T_1 < T$
- So the payoff is $\max(S_T - S_{T_1}, 0)$.


S
Apply
contract
conditions
Solve B–S
Solve B–S
Final
payoff
S_{T_1}
Today
T_1
T
t

Self-referential version

This contract

- Is a call option expiring at time T

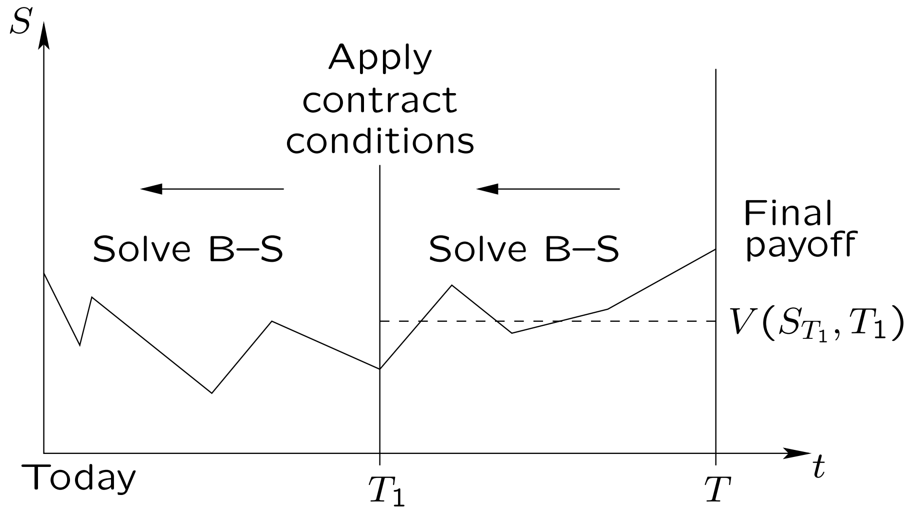
- Value function $V(t, S_t)$

- With strike $V(T_1, S_{T_1})$ where $T_1 < T$

- So the payoff is $\max(S_T - V(T_1, S_{T_1}), 0)$.

So the strike is the option's own value at time T_1.

Jeff Dewynne exam question . . .


S
Apply
contract
conditions
Solve B–S
Solve B–S
Final
payoff
$V(S_{T_1}, T_1)$
Today
T_1
T
t

Is this a sensible problem?

For $T_1 < t < T$, the strike K is known (once you have solved the problem!). So then the option is a vanilla call with value $C_{\mathsf{BS}}(S_t, t; K)$ where K is set at T_1 for each price path.

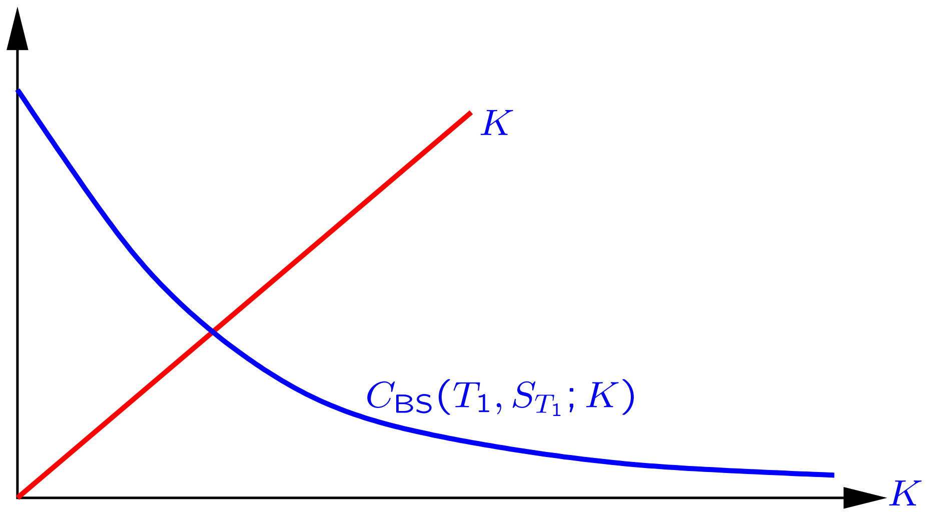
So at $t = T_1$, for each S_{T_1}, we have to solve

$$C_{\mathsf{BS}}(T_1 S_{T_1}; K, T) = K$$

for K (as a function of S_{T_1}). Then we carry that value forward.

For $0 < t < T_1$, solve BS PDE with terminal value (time T_1) given by

$$V(S_{T_1}, T_1) = C_{\mathsf{BS}}(T - T_1, S_{T_1}; K(S_{T_1}))$$



Unique solution for each S_{T_1}. Makes financial sense because the problem has negative feedback: increasing V at time T_1 increases the strike and makes the payoff less valuable.

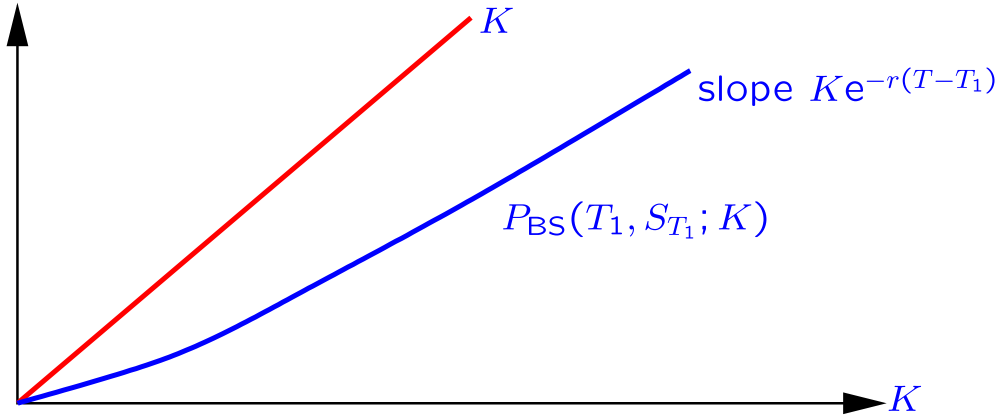
Self-referential forward start put

We could consider a self-referential forward-start put whose strike is the time-T_1 option value so the payoff is $\max(V(T_1, S_{T_1}) - S_T, 0)$.

Then we have to solve

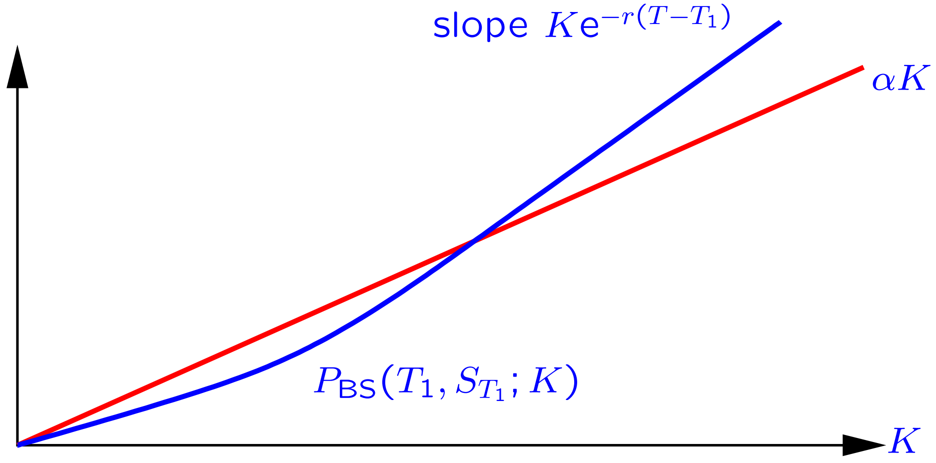
$$P_{\mathsf{BS}}(T_1, S_{T_1}; K, T) = K$$

and this has only the root $K = 0$:



$$P_{\mathsf{BS}}(S_{T_1}, T_1; K, T) = K$$

Aside: you can do this if the strike is $\alpha V(S_{T_1}, T_1)$ provided $\alpha > \mathrm{e}^{-r(T-T_1)}$ but this root is unstable to a feedback arbitrage:



Example: ordinary & self-referential Asians

Ordinary Asian option: payoff

- $\max(S_T - I_T/T, 0)$ (call)
- $\max(I_T/T - S_T, 0)$ (put)
- $S_T - I_T/T$ (forward)

where I_t is the moving average

$$I_t = \int_0^t S_u \, \mathrm{d}u.$$

So value function now $V(t, S_t, I_t)$.

The usual arguments:

$$\mathsf{d}I_t = S_t\,\mathsf{d}t$$

and then

$$\mathsf{d}V_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\right)\mathsf{d}t + \frac{\partial V}{\partial I}\mathsf{d}I_t + \frac{\partial V}{\partial S}\mathsf{d}S_t$$

has zero drift under $\mathbb{Q}$ and after discounting gives the PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + S\frac{\partial V}{\partial I} = 0$$

with the appropriate payoff.

Self-feeding (autophagous?) Asian option: payoff

- $\max(S_T - I_T/T, 0)$ (call)
- $\max(I_T/T - S_T, 0)$ (put)
- $S_T - I_T/T$ (forward)

where I_t is the average *of the option price*:

$$I_t = \int_0^t V(S_u, I_u, u)\,\mathrm{d}u.$$

By the same argument as for ordinary options they satisfy the PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + V\frac{\partial V}{\partial I} = 0.$$

Note that it is now *nonlinear*.

(If you don't like this you could consider an option whose strike is $v(t, S_t, I_t)$ where $v(\cdot)$ is a solution of the PDE.)

Forward contract has an explicit solution: if the payoff is

$$V(T, S_T, I_T) = \alpha_T S_T + \beta_T I_T / T$$

then

$$V(t, S_t, I_t) = \alpha(t) S_t + \beta(t) I_t$$

where the PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + V\frac{\partial V}{\partial I} = 0.$$

gives

$$\dot{\alpha} = -\alpha\beta + r\alpha, \qquad \dot{\beta} = -\beta^2.$$

Solution for $\beta(t)$:

$$\beta(t) = \frac{1}{t - T + \beta_T^{-1}} = \frac{1}{\beta_T^{-1} - (T-t)}, \qquad t \le T$$

- $\beta_T < 0$, solution exists for all $t < T$, negative feedback.

- $\beta_T > 0$, finite-time blow-up (that is, no solution if $T - t$ is too large) because of positive feedback. Blow-up due to interaction of nonlinear terms
$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial I} + \cdots$$
with 'wrong' gradient of final data.

Conjecture similar blow-up for puts but not calls.

Cap & trade markets

General principle is simple:

- A *cap* is set on emissions of a pollutant over a set period.

- *Allowance certificates* are created to equal the cap in quantity, and distributed (somehow: see later). Allowances may be freely traded.

- At the end of the period, polluters must

  - *Either* produce an allowance for each unit of emissions

  - *Or* pay a fixed penalty π per unit

Incentives

These rules create incentives:

- Invest in emission-reduction technology: trade cost of investment for lower allowance costs. Emission-free technologies are especially attractive if all producers receive the same price (eg electricity).

- Use a fuel with lower emissions (fuel-switching) eg coal/gas for CO2 or low sulphur for SOx..

- Consumers reduce consumption in response to higher prices (esp. CO2).

In the next period the cap may be set lower and/or the penalty increased to encourage permanent change.

Free trading increases flexibility and (in theory) allows the market to find the lowest-emission solution. The lowest cost reductions should be found first.

Cap and trade worked well for SOx and NOx (1990s, USA).

Scientific & political background

- Global warming *is** happening . . .
  - Temperature rises
  - Ice melts
  - Sea levels rise
  - Summers start earlier (on average!)
- . . . and is caused by greenhouse gases, esp. CO2.

*Scientific consensus not accepted by large parts of population (and some politicians). also timescales are long in political terms.

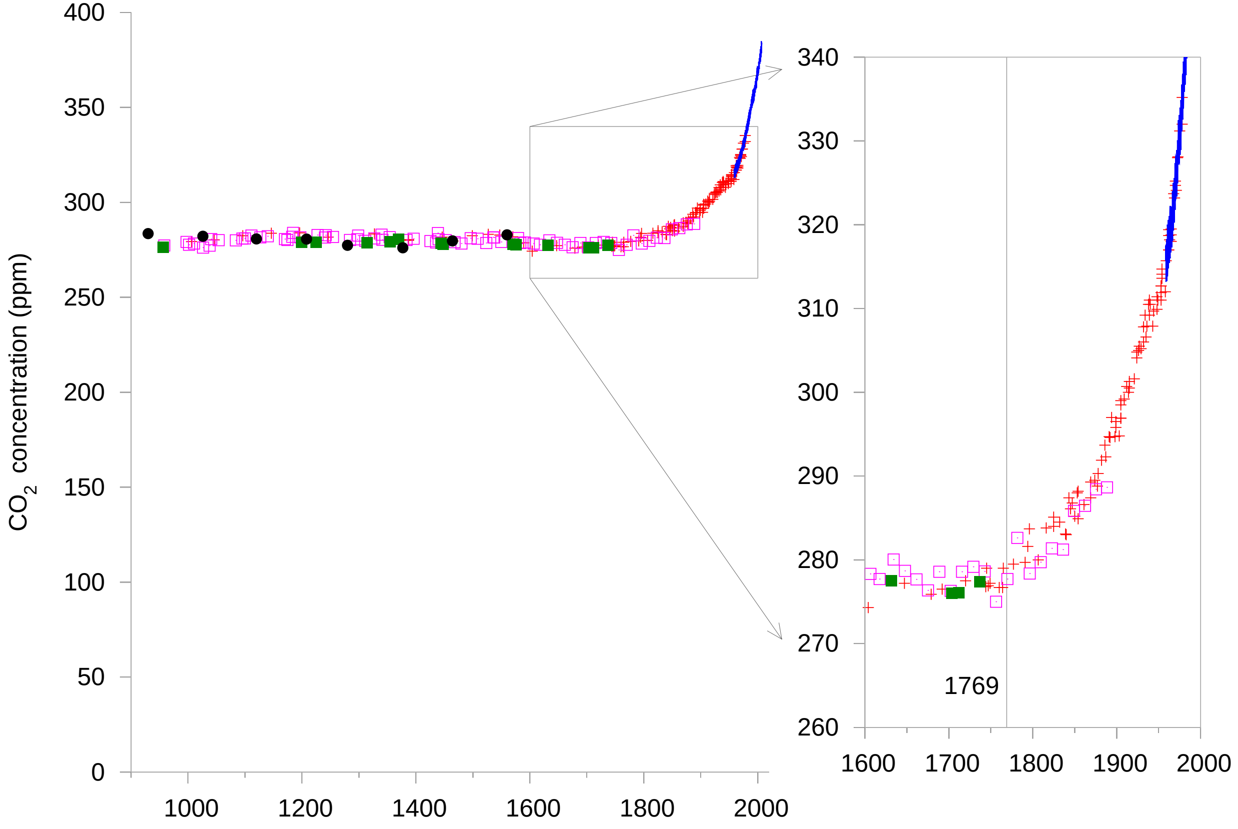


Figure by David Mackay.

Mechanisms

- Rio 1994 "Earth Summit"

- Kyoto 1997 Protocol: signatory countries committed to reduce CO_2 emissions. Cap and Trade markets set up to achieve this esp. in the EU.

- Big polluters did not all ratify (eg USA) and is self-policed but still important.

- Copenhagen 2009 —– inconclusive?

Kyoto's additional features

Notation: *Annexe 1 countries* are those committed by Kyoto to reduce emissions.

- The basic unit is 1 tonne of CO2 (tCO2).

- *Clean development Mechanism (CDM)*: companies in Annexe 1 countries can obtain Certified Emission Reduction (CER) credits, which can be used as allowances, by achieving emission reduction in non-Annexe 1 (developing) countries.

- *Joint Implementation (JI)*: as for CDM except that reductions can be in another Annexe 1 country (often eastern Europe or Russia).

The European Union

The EU has a single market, the EU Emission Trading Scheme (EU-ETS) which sets caps and oversees trading, CDM and JI.

- First period 2005–2007 (warm-up). At end of first period, completely new start. Penalty was 40 euros/tCO2.

- Second period 2008–2012 (end of Kyoto period).

- ?? Third period 2013 – . . . .

- EU allowances (EUA) and CER both created.

- Allocation devolved to national agencies. Free in first period 2005–2007, some can be auctioned in second period 2008–2012.

- Exchange trading of EUA and CER with standardised contracts; also futures and options.

- CER trades at a discount to EUA because of operational/political/regulat risk (projects may not be approved, may fail, . . .).

EUAs vs CERs

Figure: Prices of the December 2012 EUA futures contract (EU-ETS second phase), together with the price of the corresponding CER futures contract.

Figure from Carmona IPAM notes.

Other complications

- In second period:
  - Emissions tallied each year and allowances must be produced for that year
  - If you do not have enough allowances you must
    - Pay the penalty
    - AND produce the allowances next year.
  - Unused allowances can be carried forward (banked) to the next year. (Not clear what happens at end 2012.)

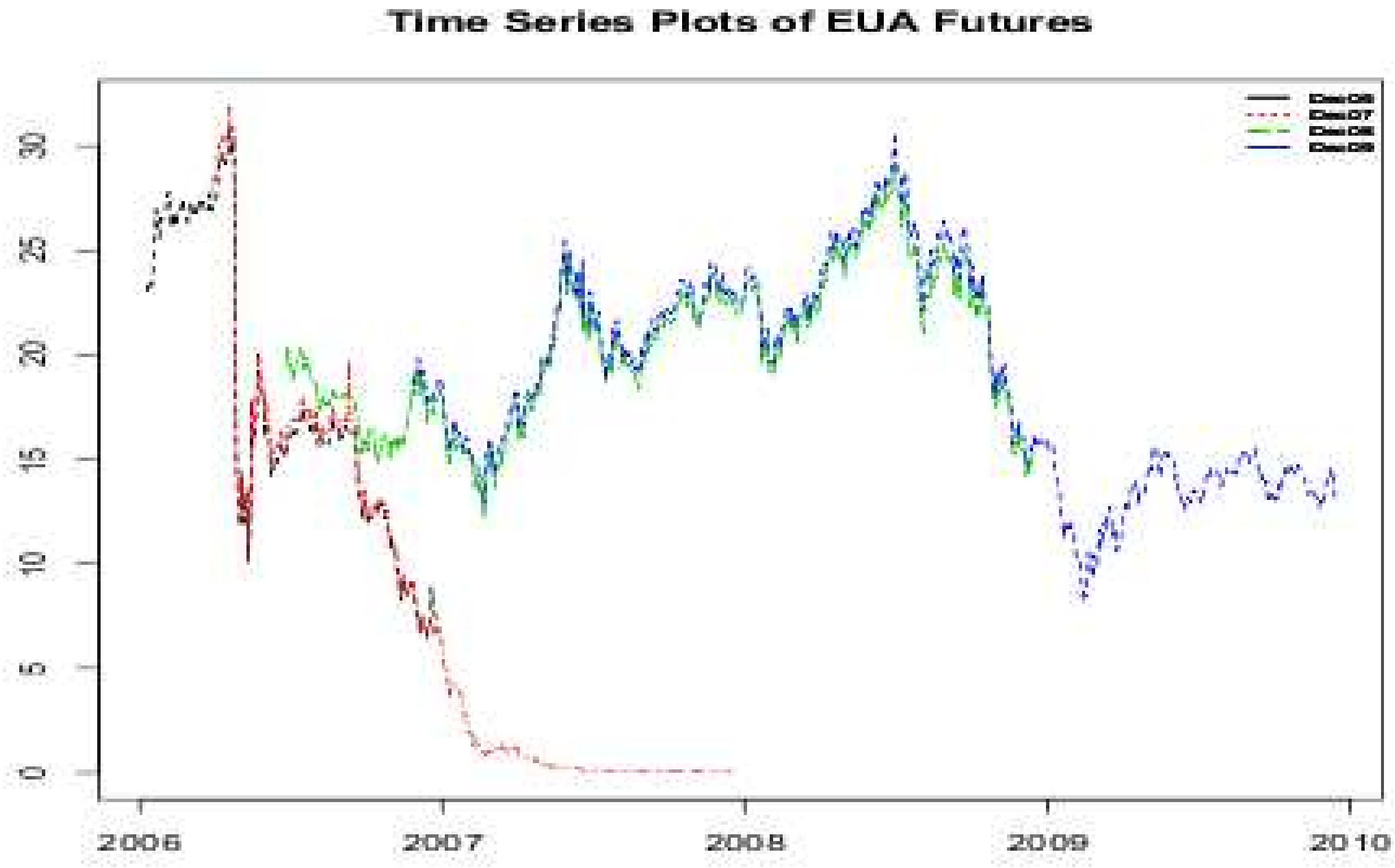
Stylised facts about EUA prices

- Without banking, EUA price should be between zero and π (penalty).*

- Without banking, price must converge to either 0 or π at period end.

- Price should reflect demand to date and forecast future demand. Note price crash in 1st series (next page) when it was realised that emissions would not reach the cap.

*If the penalty in a later period is larger, the requirement to produce allowances for over-emission might mean EUA $> \pi$.



EUA futures prices (first and second series).

Modelling frameworks

- ‘Equilibrium’ approach deriving price from fundamentals.

- ’Math finance’ models with exogenous specification of EUA price process

- Hybrid model based on merit orders.

- Derivatives.

Equilibrium approaches

See eg papers by R. Carmona.

- Formulate optimisation problem for a collection of agents (often risk neutral).

- Agents choose production schedules of energy taking allowance price into account and meeting the (inelastic) demand (exogenously given stochastic process).

- Large scale optimisation/dynamic programming problem.

- Can be cumbersome models.

Math finance style approach

- Simply prescribe a risk-neutral process for allowance price A_t
- Use vol specification to keep A_t between 0 and π (penalty).
- Various martingale tricks to get suitable candidate processes
- Calibrate to data.
- Use for derivative pricing.

- Rather arbitrary approach (no connection to fundamentals).
- Difficult to implement with banking and carryover.
- Calibration?
- Does not really model affect of abatement.

See Carmona IPAM notes for some ideas.

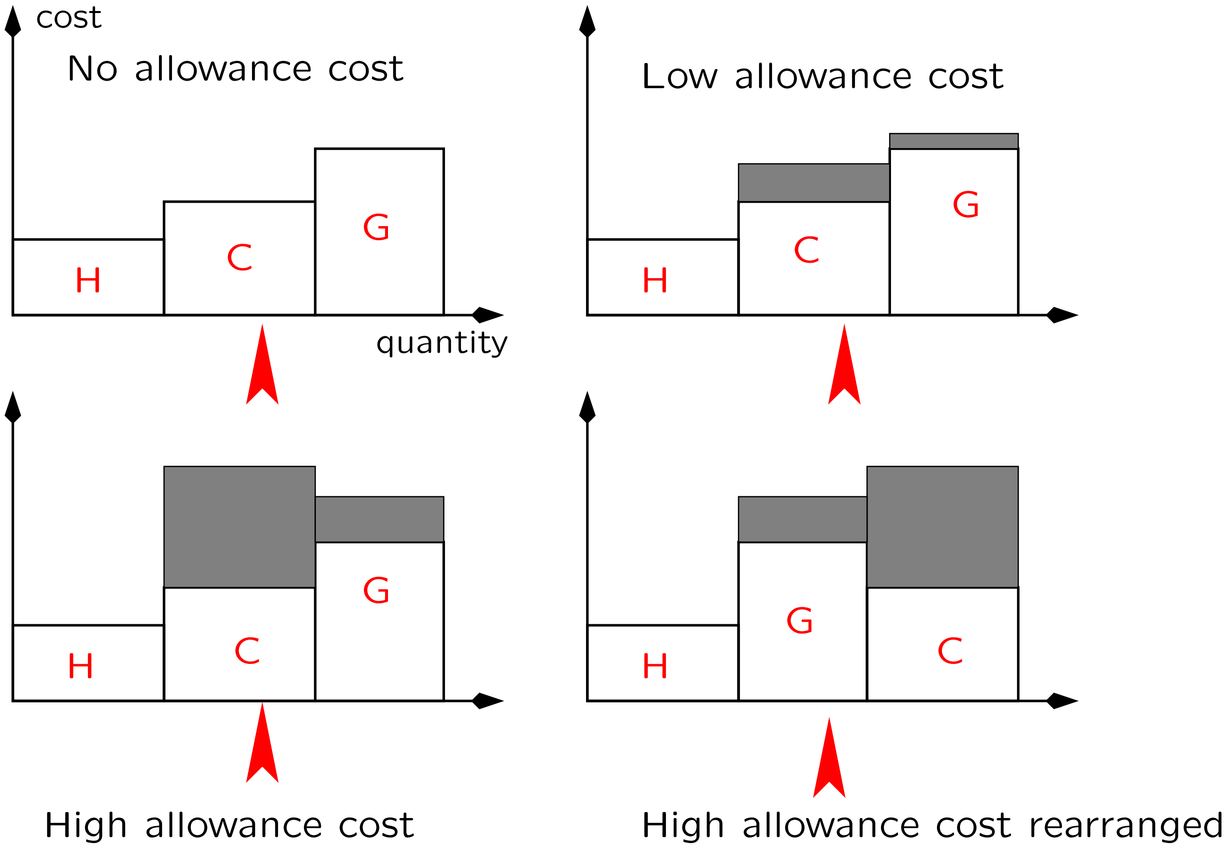
Hybrid approach

- Demand D_t is the only stochastic input and is exogenous.

- Market is inelastic so demand is met.

- An instantaneous merit order (taking into account the allowance price A_t) determines which producers produce. This sets the emission rate e_t as a function of D_t and A_t.

- That is, producers look at A_t and their own costs and submit a bid to the market, and all producers receive the same price. This gives a rearrangement of the merit order (for each value of A_t.


cost
No allowance cost
H
C
G
quantity
Low allowance cost
H
C
G
H
C
G
High allowance cost
H
G
C
High allowance cost rearranged

- Allowance price A_t is then a derivative on
  - Instantaneous demand D_t;
  - Total emissions to date,

$$E_t = \int_0^t e_s(A_s, D_s)\,\mathrm{d}s.$$

  (this is like an Asian option).

- Note that the determination of e_t is previsible (given D_t and A_t.

Assume that we have a process (under $\mathbb{Q}$) for D_t:

$$\mathsf{d}D_t = \mu_D \, \mathsf{d}t + \sigma_D \, \mathsf{d}W_t.$$

(For example, mean reverting, and incorporating the condition of positivity).

We also have

$$\mathsf{d}E_t = e(A_t, D_t) \, \mathsf{d}t$$

which is (as just noted) previsible.

The PDE for A

Now regard A_t as a function $A(t, D_t, E_t)$. Because A_t is the price of a traded security, its risk-neutral drift is r. So,

$$\begin{aligned}
\mathrm{d}A_t &= \frac{\partial A}{\partial t}\mathrm{d}t + \frac{\partial A}{\partial D}\mathrm{d}D_t + \frac{1}{2}\sigma_D^2\frac{\partial^2 A}{\partial D^2}\mathrm{d}t + \frac{\partial A}{\partial E}\mathrm{d}E_t \\
&= \left(\frac{\partial A}{\partial t} + \frac{1}{2}\sigma_D^2\frac{\partial^2 A}{\partial D^2} + \mu_D\frac{\partial A}{\partial D} + e(A, D)\frac{\partial A}{\partial E}\right)\mathrm{d}t + \sigma_D\frac{\partial A}{\partial D}\mathrm{d}W_t
\end{aligned}$$

and equating the drift to r means that $A(t, D, E)$ satisfies the *nonlinear* PDE

$$\frac{\partial A}{\partial t} + \tfrac{1}{2}\sigma_D^2\frac{\partial^2 A}{\partial D^2} + \mu_D\frac{\partial A}{\partial D} + e(A, D)\frac{\partial A}{\partial E} - rA = 0.$$

This PDE is *nonlinear* because the allowance price affects its own evolution through $e(A, D)$.

Terminal conditions

In the basic situation of a penalty π at the end of the period we have

$$A(T, D_T, E_T) = \begin{cases} \pi & \text{if } E_T > E_{\text{cap}} \\ 0 & \text{otherwise.} \end{cases}$$

So the allowance price is a digital option on total emissions! Other situations will be more complicated (eg with banking/carry forward we need to compute the price of allowances expiring each year simultaneously).

Illustrative numerics

These were produced (by DS) with a regularised model in which

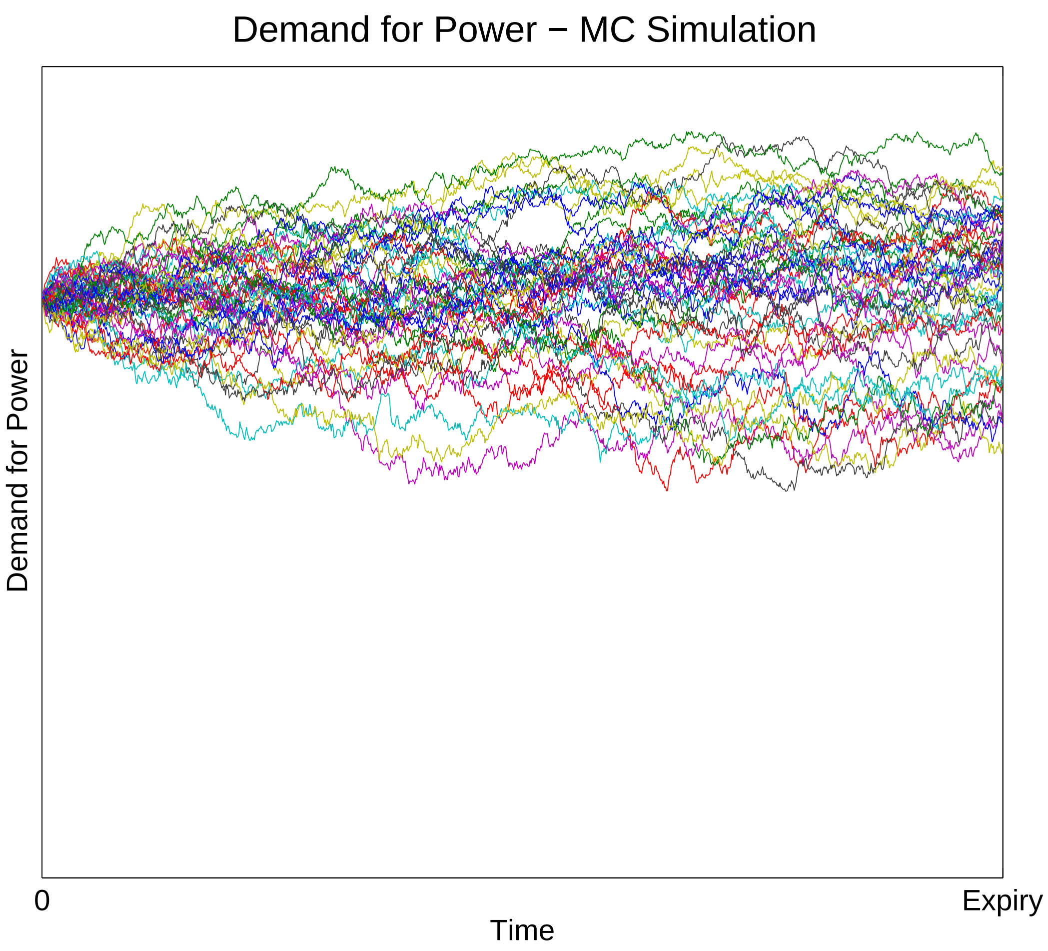
$$\mathrm{d}E_t = e(A_t, D_t)\,\mathrm{d}t + \sigma_E\,\mathrm{d}\widetilde{W}_t.$$

The BM $\widetilde{W}_t$, independent of W_t, represents measurement error/uncertainty in emissions. PDE now fully parabolic ($\longrightarrow$ conventional forward/bacward SDE problem).

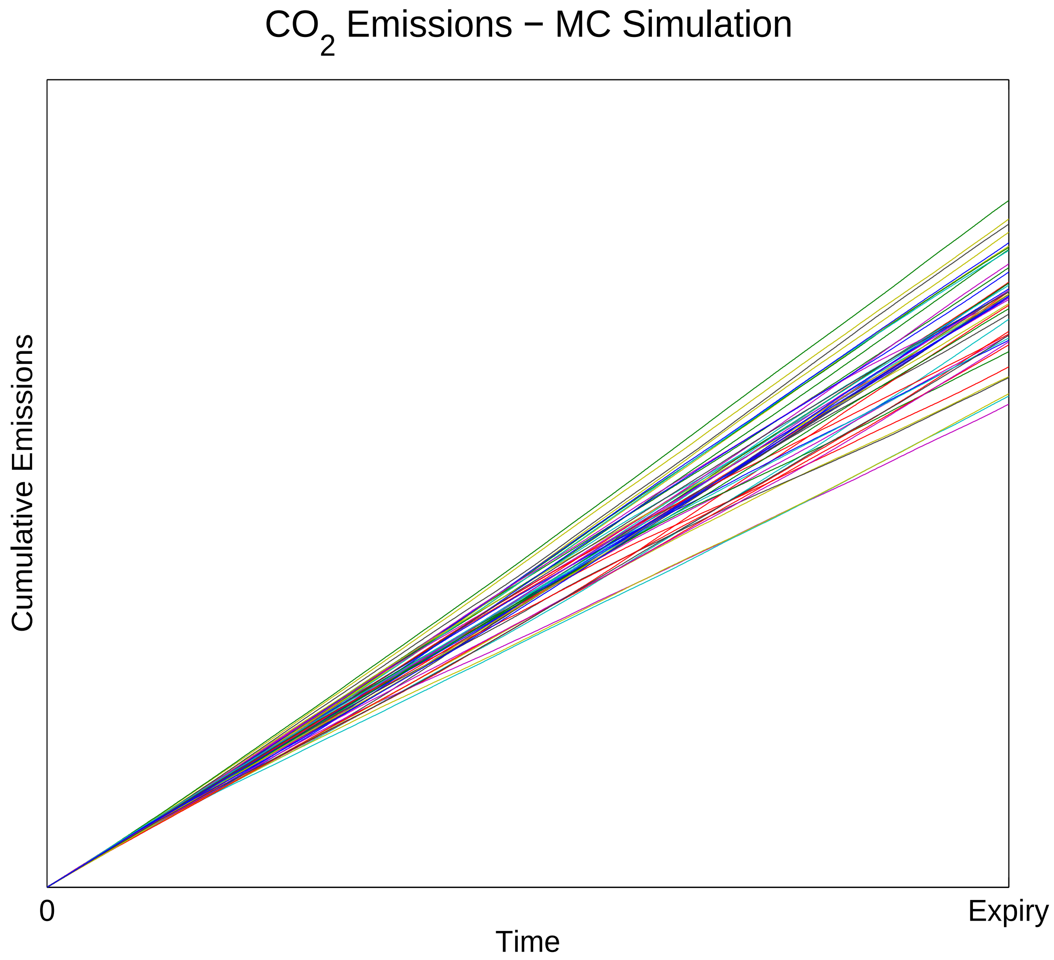
Note this may not prevent blow-up. In the Asian example if we say $\mathrm{d}I_t = V_t\,\mathrm{d}t + \sigma_I\,\mathrm{d}\widetilde{W}_t$ we get

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \sigma_I^2 \frac{\partial^2 V}{\partial I^2} + rS\frac{\partial V}{\partial S} - rV + V\frac{\partial V}{\partial I} = 0$$

but the forward contract $\alpha(t)S_t + \beta(t)I_t$ is unchanged!

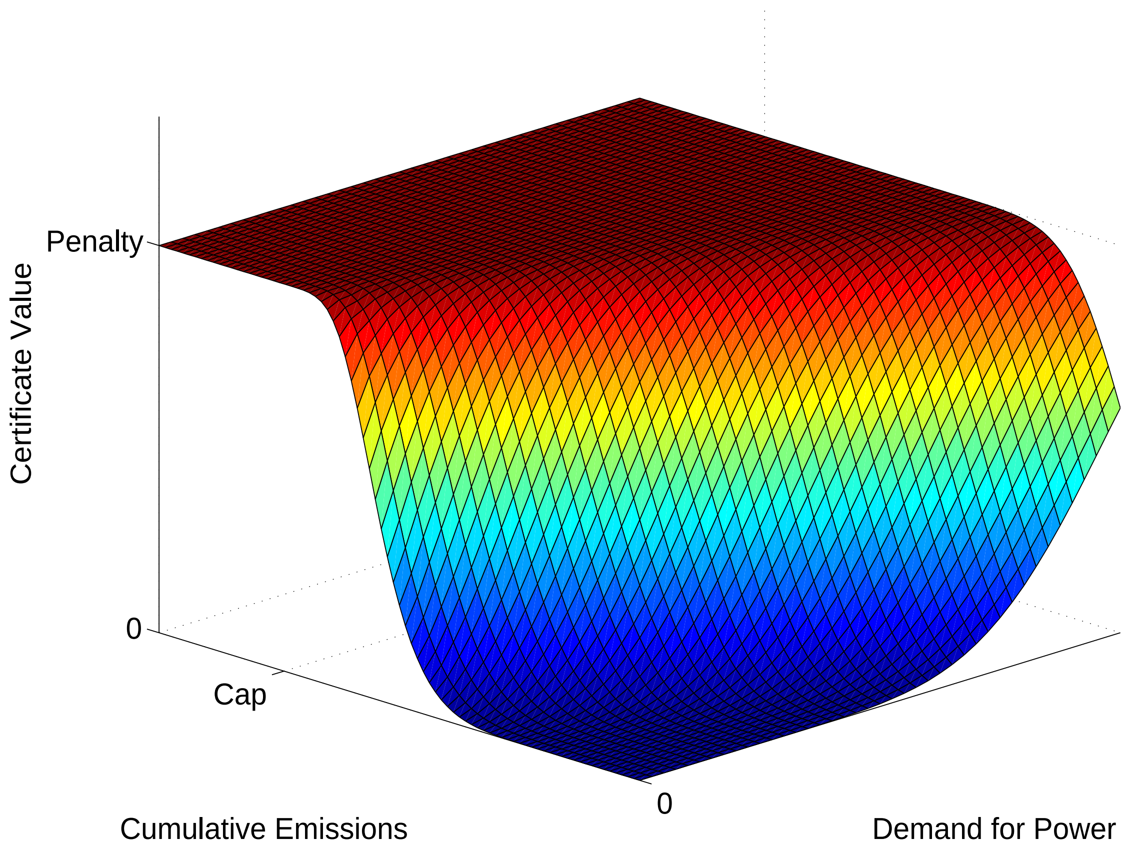


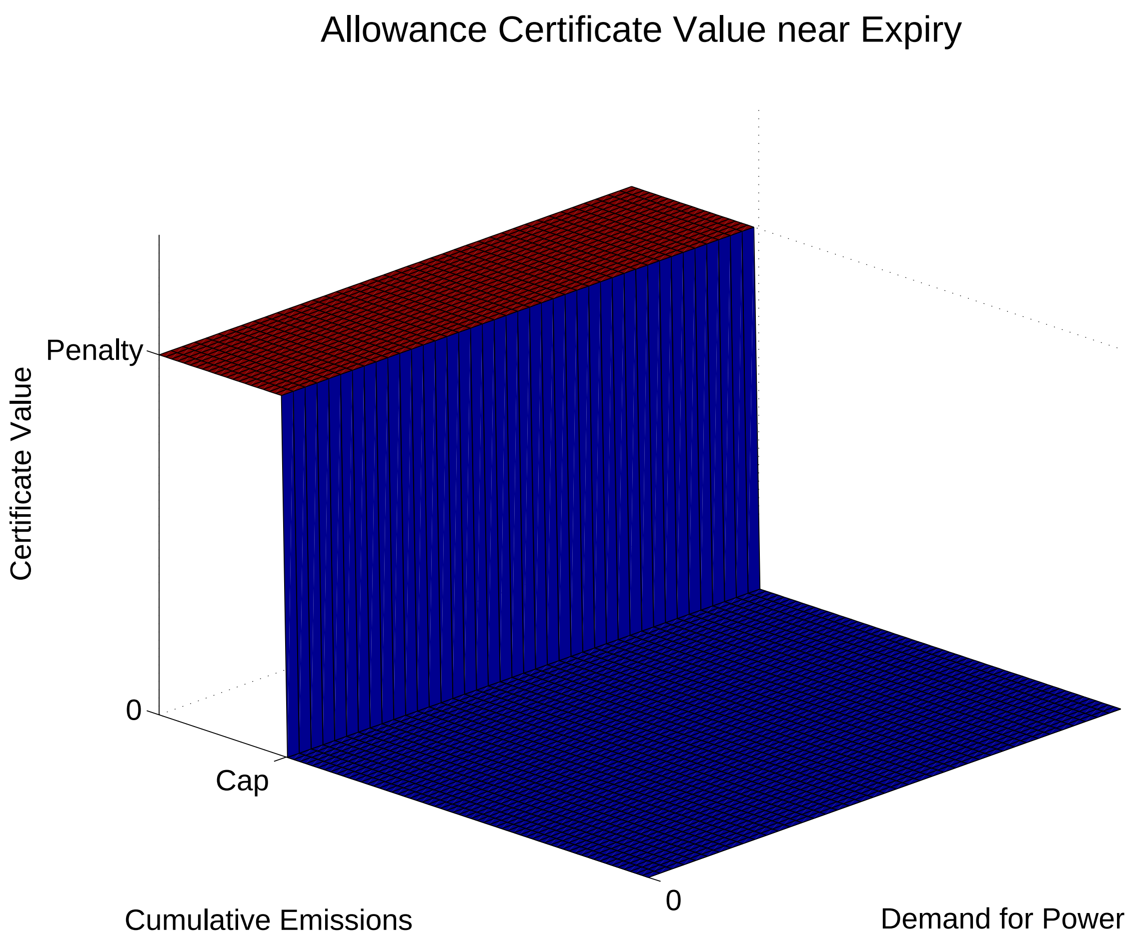
Demand for Power − MC Simulation
Demand for Power
0
Expiry
Time

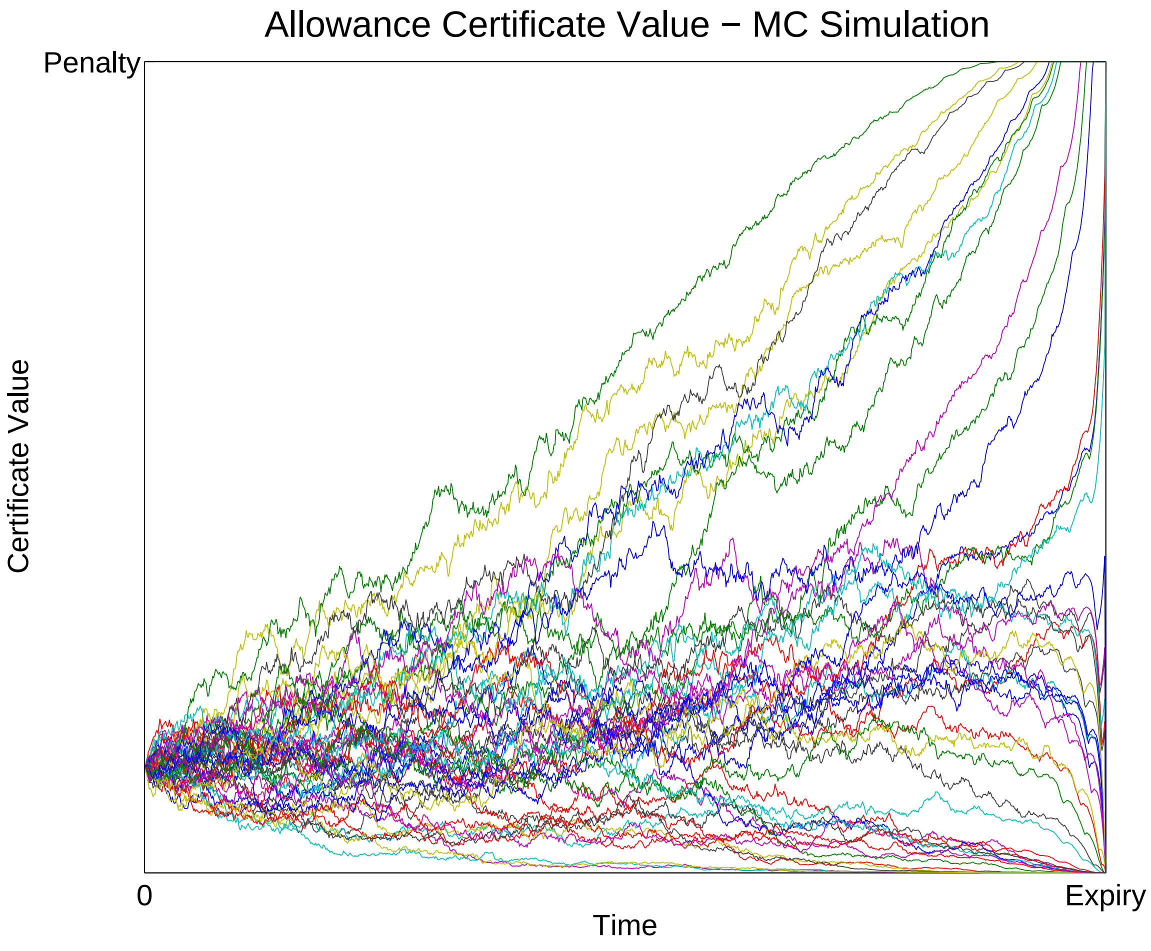


CO$_2$ Emissions − MC Simulation
Cumulative Emissions
0
Expiry
Time

Allowance Certificate Value at t=0




Allowance Certificate Value near Expiry
Penalty
Certificate Value
0
Cap
0
Cumulative Emissions
Demand for Power


Allowance Certificate Value − MC Simulation
Penalty
Certificate Value
0
Expiry
Time

Derivatives

The PDE above is nonlinear *so we cannot add solutions!* What it tells us is the price of exactly *one allowance certificate*, with a price scale corresponding to the penalty. The function $e(A, D)$ also has a price scale built into it which is consistent with a PDE for just one allowance; if we were to write down a PDE for two certificates we would have to rescale the variables appropriately.

Other derivatives have a price $V(t, D, E)$ and by the usual argument this satisfies the PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma_D^2 \frac{\partial^2 V}{\partial D^2} + \mu_D \frac{\partial V}{\partial D} + e(A, D)\frac{\partial V}{\partial E} - rV = 0,$$

where $A(t, D, E)$ is the single-allowance price already calculated as a functions of t, D, E. Note that $V = A$ recovers the original PDE for A.

Using this PDE we can set up

- Real options problems for valuation of power plant
- Real options problems for abatement costs
- Models for market design and operation
- Models for impact of consumer behaviour (this needs us to calculate processes for energy prices as well as emissions).

Sources: Look for papers by Carmona, Fehr, Hinz, Cetin. Part of these notes is based on Carmona's notes for the IPAM workshop on Math Finance at UCLA, January 2010.