Asymptotic approximations for Asian, European and American options with discrete averaging, or discrete dividend/coupon payments.

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Abstract

We develop approximations to the pricing of options on an asset which makes discrete dividend payments, focusing on the case of frequent payments. The principal mathematical tool is the method of multiple time scales, allied to matched asymptotic expansions. We first analyse European style options, deriving the continuously-paid dividend equation from the relevant discrete problem, and we analyse the range accrual note to compute the relevant 'continuity correction'. We also carry out the same analysis for Asian options with discrete averaging. We then give a detailed description of the intricate exercise policies that arise for American put (and, to a lesser extent, call) options when dividends are paid discretely, for the cases of proportionate and fixed-amount dividends.

1 Introduction

It is common in analysis of derivatives to assume that dividend or coupon payments are made continuously in time, or that averages for Asian options are computed continuously. In practice, however, these processes are discrete in time. This paper deals with asymptotic approximations, valid when the number of discrete events is large, for the difference between the Black–Scholes prices obtained with the two regimes. Using the method of multiple scales, we first consider two relatively straightforward cases: we analyse European contracts on an asset paying discrete dividends, showing how the continuous payment model is retrieved and characterising the difference between the two, and giving the same treatment to Asian options with continuous and discrete averaging. We extend the analysis to range accrual notes, calculating in addition the 'continuity correction' (an expression for the difference between the discrete and continuous option values) applicable to these contracts. We then turn to the more intricate case of American options.

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The Black–Scholes model for an American put option has been widely studied as a canonical early-exercise problem. The majority of these studies have assumed a continuously paid constant dividend yield (which is taken equal to zero in some cases). Apart from existence/uniqueness studies, attention has been paid to the properties of the early exercise boundary such as its convexity [4], and in particular its behaviour at times close to expiry, where approaches divide between those that use the Green's function to transform the problem into an integral equation (see, for example, [2, 9, 10]) and those that exploit matched asymptotic expansion methods directly on the underlying partial differential equation (for example, [8]).

Less attention has been paid to cases in which the contract has 'discrete' features. The relationship between a Bermudan option with frequent exercise opportunities and its continuously-exercisable equivalent is discussed in [6]; in this paper, we consider the behaviour of a put option on an asset which pays a large number of discrete dividends, which we refer to as the discrete version of the equivalent option with continuous dividend payments (we return to the precise definition of these terms below). Rather than deriving a continuity correction, as in [1, 5] for discretely and continuously activated barrier options and [6] for Bermudan and continuously-exercisable American ones, we focus on the structure of the exercise region between dividend dates, which is quite intricate. This has been recognised for some time: the current study was motivated by Figure 5-37 of the early text [3], and by the numerical calculations of [11] which are illustrated with similar figures. A sketch of the exercise boundary for a put option is given in Figure 1. The exercise region for the continuous option is to the left of the dashed line, which is the optimal exercise boundary for that contract. However, the exercise region for the discrete option on an asset paying proportionate dividends consists of many disjoint portions, one for each interval between dividend payments (plus final and initial ones), and its exercise boundary, shown solid, has correspondingly many components; each of these, save only the final one, extends to $S = 0$ as time approaches a dividend date from below, and then the next component starts from a finite value of S. Furthermore, the exercise boundary has very nearly constant speed for an appreciable fraction of the interval between payment dates. This is the structure that we seek to explain; and having done so, we repeat the analysis for call options, and for both puts and calls on an asset paying a fixed dividend.

2 General background

We work in a standard Black–Scholes model, considering American and European options with start date $t = 0$ and expiry $t = T$ on an asset that pays an annualised dividend yield q. We compare two forms of dividend payment. In the first, the dividend is paid continuously and the risk-neutral asset price satisfies

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

where the interest rate r and volatility σ are constant and W_t is a standard Brownian motion. For this process,

$$S_T|S_t = S_t \exp\left((r - q - \frac{1}{2}\sigma^2)(T - t) + \sigma W_{T-t}\right).$$

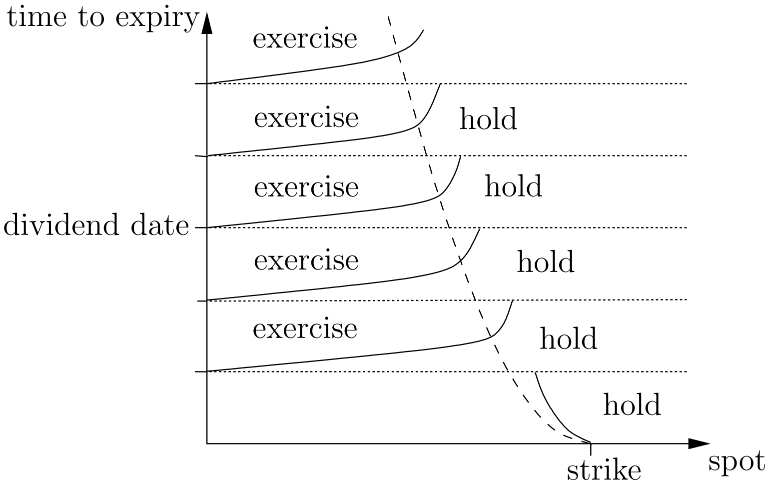

Figure 1: Sketch of the exercise boundaries (solid curves) for a put option with discrete dividend payments. The exercise regions are indicated, and the exercise boundary for the continuously paid case is the dashed curve.

We shall write $P^{\mathrm{e}}_{\mathrm{c}}(S,t)$ (resp., $P^{\mathrm{a}}_{\mathrm{c}}(S,t)$) for the value of a European (resp., American) put option in this model, the strike K being implicit. When making statements about a generic option whose only cashflow is a payoff at time T, we replace P with V. A European option satisfies the Black-Scholes partial differential equation (BSPDE)

$$\mathcal{L}^{q}_{\mathrm{BS}} V^{\mathrm{e}}_{\mathrm{c}} := \frac{\partial V^{\mathrm{e}}_{\mathrm{c}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^{\mathrm{e}}_{\mathrm{c}}}{\partial S^2} + (r-q)S\frac{\partial V^{\mathrm{e}}_{\mathrm{c}}}{\partial S} - rV^{\mathrm{e}}_{\mathrm{c}} = 0 \tag{1}$$

for $0 < t < T$, with the appropriate payoff condition.

In the second dividend model, dividends are paid at equal time intervals $\delta t = T/N$ with, for definiteness, the first payment immediately after the start, at $t = 0+$ and the last immediately after expiry, at $t = T+$. (Other configurations are clearly possible, but these details are not the main focus of this study. Our choice is consistent with Fig. 1.) For this model, between the dividend dates $t_i = i\,\delta t$, $i = 0, 1, \ldots, N-1$, the asset price evolves according to

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t,$$

while the usual no-arbitrage condition at $t = t_i$ means that

$$S_{t_i+} = S_{t_i-}\mathrm{e}^{-q\,\delta t}.$$

Hence

$$\begin{aligned} S_T &= S_0\left(\mathrm{e}^{-q\,\delta t}\right)^N \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W_T\right) \\ &= S_0 \exp\left((r - q - \frac{1}{2}\sigma^2)T + \sigma W_T\right) \end{aligned}$$

in agreement with the continuous case.

We shall write $P^{\mathrm{e,a}}_{\mathrm{d}}(S,t)$ for the value of European and American put options in this model, again replacing P with V for a generic contract. European contracts in this model satisfy the zero-dividend version of the BSPDE,

$$\mathcal{L}^0_{\mathrm{BS}} V^{\mathrm{e}}_{\mathrm{d}} = \frac{\partial V^{\mathrm{e}}_{\mathrm{d}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V^{\mathrm{e}}_{\mathrm{d}}}{\partial S^2} + rS\frac{\partial V^{\mathrm{e}}_{\mathrm{d}}}{\partial S} - rV^{\mathrm{e}}_{\mathrm{d}} = 0 \tag{2}$$

between dividend dates, at which times the value function is subject to jump/continuity conditions, as follows.

Although the asset value in the discrete model falls by the amount of the dividend at a dividend date, the option value is continuous in time at the dividend date because the option holder receives no cash flow. This transfer,

$$\begin{aligned}&\text{OptionValue(AssetBeforeDividend, TimeBeforeDividend)} = \\ &\qquad \text{OptionValue(AssetBeforeDividend} - \text{Dividend, TimeAfterDividend)},\end{aligned}$$

holds for all possible asset values and hence states that

$$V_{\mathrm{d}}(S, t_i-) = V_{\mathrm{d}}(S\mathrm{e}^{-q\,\delta t}, t_i+) \tag{3}$$

for all S. Note immediately that, *at fixed* S, a decreasing value function, for example of a contract such as a put option, has a jump *decrease* across the dividend time.

2.1 European contracts

We first establish some elementary properties of European contracts.

Proposition 1. *With our assumption on the timing of dividend payments, the values of generic European continuous and discrete options agree immediately before any dividend date:*

$$V^{\mathrm{e}}_{\mathrm{d}}(S, t_i-) = V^{\mathrm{e}}_{\mathrm{c}}(S, t_i-).$$

Proof. This follows directly from the representation of the option value as the discounted risk-neutral expectation of the payoff, because for each asset value S_{t_i-}, the conditional distribution of $S_T|S_{t_i-}$ is the same for both models. □

The equality does not hold in between dividend dates (because of the different drifts in the two models), nor does it hold for American option (because of path-dependency). If the final discrete dividend is paid immediately *before* the expiry time rather than after it, equality holds at t_i+ rather than at t_i-, while if the expiry time is strictly between two dividend payment dates, no simple general statement can be made, because of the final partial interval $t_N < t < T$.

Proposition 2. *With our convention on the timing of dividends, if the payoff is decreasing (resp. increasing) in S then the discrete option value is less (resp. greater) than the continuous value between dividend dates: if $q > 0$ and $V(S,T)$ is decreasing then, for $0 < S < \infty$, and for $t_{i-1} < t < t_i$, $i = 1, \ldots, N$,*

$$V_{\mathrm{d}}^{\mathrm{e}}(S,t) < V_{\mathrm{c}}^{\mathrm{e}}(S,t),$$

with the inequality reversed when $V(S,T)$ is increasing.

Proof. Take any i, noting that $V_{\mathrm{d}}^{\mathrm{e}}(S,t_i-) = V_{\mathrm{c}}^{\mathrm{e}}(S,t_i-)$. Set $\widetilde{V}(S,t) = V_{\mathrm{d}}^{\mathrm{e}}(S,t) - V_{\mathrm{c}}^{\mathrm{e}}(S,t)$. By subtracting the BSPDE (1) from (2), we have

$$\mathcal{L}_{\mathrm{BS}}^{0}\widetilde{V} = -qS\frac{\partial V_{\mathrm{c}}^{\mathrm{e}}}{\partial S},$$

with $\widetilde{V}(S,t_i-) = 0$. Because $\partial V_{\mathrm{c}}^{\mathrm{e}}/\partial S < 0$ (resp. > 0), solving the BSPDE backwards from t_i we have $\widetilde{V}(S,t) < 0$ (resp. > 0) for $S > 0$, $t_{i-1} < t < t_i$, as required. □

Note that other timings of the dividend payments lead to different orderings of continuous and discrete value functions. As we shall see below, the difference $\widetilde{V}(S,t)$ is approximately a sawtooth function of time for each S, with upward-sloping portions separated by jumps down. For the payment schedule of Proposition 2, this function is negative except at $t = t_i-$, where it vanishes. If the final dividend is paid immediately before the exercise time. the ordering is reversed and the sawtooth is shifted up to be positive everywhere except $t = t_i+$. For intermediate schedules, it straddles the value zero.

Further insight into these results follows from consideration of the dividend-sensitivity of the continuous option, a 'Greek' which surely should be termed Diva. Setting $\varpi = \partial V_{\mathrm{c}}^{\mathrm{e}}/\partial q$, differentiation of the BSPDE (1) with respect to q shows that

$$\mathcal{L}_{\mathrm{BS}}^{q}\varpi = S\frac{\partial V_{\mathrm{c}}^{\mathrm{e}}}{\partial S},$$

with $\varpi(S,T) = 0$. It follows that $\varpi > 0$, and in fact $\varpi = -(T-t)S\,\partial V_{\mathrm{c}}^{\mathrm{e}}/\partial S$. The effect of dividends is to increase the value of an option with decreasing payoff compared with the same option on a dividend-free asset (because the drift of the dividend-paying asset is smaller). This gives rise to the upward-sloping parts of the sawtooth function above, while the downward jumps stem from the drop in value of an option (for fixed S) at a dividend date.

3 Asymptotic approximation to discrete options

We now return to a general contract, either European or American, and consider the difference function $V_{\mathrm{d}}(S,t) - V_{\mathrm{c}}(S,t)$ between the continuously and discretely sampled versions. For frequent dividend payments, we give a detailed description of its structure (in the American case, this description holds for asset values in the hold region and far from any exercise boundary) and we obtain the continuous BSPDE (1) by a systematic asymptotic procedure. Unlike the analysis of the exercise region for the American problem to follow, this is not complicated; although it can presumably be established using standard homogenisation techniques, our interest is in the structure of the difference between the value functions in the continuous and discrete problems, and we therefore use the method of multiple scales.

The difference consists of two parts. One is a one-off 'offset' determined by the precise timing of the final dividend payment, as discussed briefly above; we do not consider this in detail. The other is the rapidly varying 'local' difference, which is more interesting, and we look at its structure below. We also show that the term

$$-qS\frac{\partial V_{\mathrm{c}}}{\partial S}$$

in the continuous BSPDE (1) has its origin in an expansion of the jump condition (3):

$$\begin{aligned}V_{\mathrm{d}}(S,t_i-) &= V_{\mathrm{d}}(S\mathrm{e}^{-q\,\delta t},t_i+)\\ &\sim V_{\mathrm{d}}(S,t_i+)+\left(\mathrm{e}^{-q\,\delta t}-1\right)S\frac{\partial V_{\mathrm{d}}}{\partial S}+\cdots\\ &\sim V_{\mathrm{d}}(S,t_i+)-q\,\delta t S\frac{\partial V_{\mathrm{d}}}{\partial S}+\cdots.\end{aligned}$$

We first carry out the preliminary scaling

$$t'=\sigma^2(T-t),$$

which we use throughout the paper (and we abuse notation by using $V(S,t)$ and $V(S,t')$ for the option value). It makes time nondimensional[1] on the volatility timescale σ^{-2}, and converts the backward BSPDEs into forward ones. The zero-dividend BSPDE (which holds between dividend dates for a generic function $V_{\mathrm{d}}(S,t')$) becomes

$$\frac{\partial V_{\mathrm{d}}}{\partial t'}=\tfrac{1}{2}S^2\frac{\partial^2 V_{\mathrm{d}}}{\partial S^2}+\rho S\frac{\partial V_{\mathrm{d}}}{\partial S}-\rho V_{\mathrm{d}}, \tag{4}$$

where

$$\rho=\frac{r}{\sigma^2}$$

is the dimensionless interest rate (percent per inverse-squared-volatility-time). Likewise, define the dimensionless dividend rate as

$$\gamma=\frac{q}{\sigma^2}.$$

Also define

$$\epsilon^2=\sigma^2\,\delta t,$$

the dimensionless dividend interval which, as is common for realistic parameter values, we assume to be small: $\epsilon^2\ll 1$. Lastly, label the dividend dates as t'_n.

[1]There is little to be gained in this problem from scaling either S or V, as the BSPDE is invariant under scaling these quantities with constants, so we are content with a semi-nondimensionalisation.

Now choose a specific dividend date t'_n and write

$$t' = t'_n + \epsilon^2\tau,$$

so that the interval to the 'next' dividend payment t'_{n+1} (counting backwards in calendar time of course) is $0 < \tau < 1$. We use the method of multiple scales, writing the discrete option value as a function of *three* variables, namely $V_{\mathrm{d}}(S, t', \tau)$, and thinking of τ as a 'fast' time.) Using the standard chain rule

$$\frac{\partial}{\partial t'} \mapsto \frac{\partial}{\partial t'} + \frac{1}{\epsilon^2}\frac{\partial}{\partial \tau},$$

the option value satisfies

$$\frac{1}{\epsilon^2}\frac{\partial V_{\mathrm{d}}}{\partial \tau} + \frac{\partial V_{\mathrm{d}}}{\partial t'} = \tfrac{1}{2}S^2\frac{\partial^2 V_{\mathrm{d}}}{\partial S^2} + \rho S\frac{\partial V_{\mathrm{d}}}{\partial S} - \rho V_{\mathrm{d}} \tag{5}$$

in the typical interval $0 < \tau < 1$, with the jump condition

$$V_{\mathrm{d}}(S\mathrm{e}^{-\epsilon^2\gamma}, t', 1-) = V_{\mathrm{d}}(S, t', 1+).$$

We now expand

$$V_{\mathrm{d}}(S, t', \tau) \sim V_0(S, t', \tau) + \epsilon^2 V_2(S, t', \tau) + \epsilon^4 V_4(S, t', \tau) + \cdots$$

and substitute in (5), equating coefficients of powers of ϵ to find that

$$\frac{\partial V_0}{\partial \tau} = 0, \qquad \frac{\partial V_2}{\partial \tau} = \mathcal{L}'V_0, \qquad \frac{\partial V_4}{\partial \tau} = \mathcal{L}'V_2,$$

where

$$\mathcal{L}' := -\frac{\partial}{\partial t'} + \tfrac{1}{2}S^2\frac{\partial^2}{\partial S^2} + \rho S\frac{\partial}{\partial S} - \rho$$

is the scaled zero-dividend Black-Scholes operator. We also expand the jump condition to $O(\epsilon^4)$ (including $\mathrm{e}^{-\epsilon^2\gamma}$) , giving

$$V_0(S, t', 1+) = V_0(S, t', 1-), \tag{6}$$

$$V_2(S, t', 1+) = V_2(S, t', 1-) - \gamma S\frac{\partial V_0}{\partial S}(S, t', 1-), \tag{7}$$

$$\begin{aligned} V_4(S, t', 1+) = {} & V_4(S, t', 1-) - \gamma S\frac{\partial V_2}{\partial S}(S, t', 1-) \\ & + \frac{1}{2}\gamma^2 S\frac{\partial V_0}{\partial S}(S, t', 1-) + \frac{1}{2}\gamma^2 S^2\frac{\partial^2 V_0}{\partial S^2}(S, t', 1-) \end{aligned} \tag{8}$$

In solving the problems for V_0 etc., we must avoid *secular terms*: functions of τ that grow unboundedly and invalidate the ordering of the expansion. We therefore impose the condition that

$$V_0(S, t', \tau),\ V_2(S, t', \tau) \text{ and } V_4(S, t', \tau) \text{ are } \textit{periodic in } \tau \text{ with period 1.} \tag{9}$$

This does not mean that $V_{\rm d}(S,t')$ is periodic, but rather that all the departure from strict periodicity is via a modulation on the slow time scale t' of the fast-timescale periodic behaviour. We can therefore replace $V_{0,2,4}(S,t',1+)$ with $V_{0,2,4}(S,t',0+)$ in the jump conditions (8). This allows us to close the system of equations, which was made indeterminate by the introduction of the new variable τ.

We start with $V_0(S,t',\tau)$ which, being independent of τ is an as-yet unknown function $V_{0u}(S,t')$ of the spot and slow time variables only; it automatically satisfies the first of the jump conditions (8). Then, we find that

$$V_2(S,t',\tau)=\tau\mathcal{L}'V_{0u}(S,t')+V_{2u}(S,t'),$$

where $V_{2u}(S,t')$ is also unknown at this stage. We now apply the second of the jump conditions (8) (replacing $1+$ with $0+$) to find that

$$\mathcal{L}'V_{0u}=\gamma S\frac{\partial V_{0u}}{\partial S},$$

which is the scaled *continuous* BSPDE for V_{0u}. Applying the payoff condition,

$$V_{0u}(S,t')=V_{\rm c}(S,t')$$

and we have thus established leading-order convergence of the discrete problem to the continuous one.

We are left with the function $V_{2u}(S,t')$. Repeating the procedure above for V_4, we find

$$V_4(S,t',\tau)=\tfrac{1}{2}\tau^2\mathcal{L}'^2V_{0u}(S,t')+\tau\mathcal{L}'V_{2u}(S,t')+V_{4u}(S,t'),$$

where $V_{4u}(S,t')$ is again unknown. The third jump condition above, together with the fact that if $V(S,t')$ is a solution of the BSPDE, so is $S\partial V/\partial S$ (that is, $\mathcal{L}'$ commutes with $S\partial/\partial S$), eventually shows that the solvability condition is

$$\mathcal{L}'V_{2u}=\gamma S\frac{\partial V_{2u}}{\partial S}.$$

That is, the $O(\epsilon^2)$ correction also satisfies the continuous BSPDE. We can identify V_{2u} with the correction attributable to the interaction of the dividend payment schedule with the expiry date, as discussed above.

With the dividend payment schedule of Proposition 1, both $V_2(S,t')$ and $V_4(S,t')$ vanish at $\tau=0$, and so the unknown functions V_{2u} and V_{4u} are absent. We have thus shown that the difference between the discrete and continuous prices is

$$V_{\rm d}(S,t',\tau)-V_{\rm c}(S,t')=\epsilon^2\tau\gamma S\frac{\partial V_{\rm c}}{\partial S}(S,t')+\tfrac{1}{2}\epsilon^4\tau^2\gamma^2S\frac{\partial}{\partial S}\left(S\frac{\partial V_{\rm c}}{\partial S}(S,t')\right)+O(\epsilon^6)$$

and is therefore $O(1/N)$. Reverting to original variables,

$$V_{\rm d}(S,t)-V_{\rm c}(S,t)=(t_i-t)qS\frac{\partial V_{\rm c}}{\partial S}+\tfrac{1}{2}(t_i-t)^2q^2S\frac{\partial}{\partial S}\left(S\frac{\partial V_{\rm c}}{\partial S}\right)+\cdots,$$

for $t_{i-1}<t<t_i$, with periodic repetition as above; so the first term has the form 'Diva $\times$ time to dividend' as one might expect.

3.1 Discrete and continuous Asian options

Let us now consider arithmetic (for simplicity only) Asian options. The continuous version of this option has price $V_{\mathrm{c}}^{\mathrm{Asian}}(S, A, t)$ where the variable A is the running average whose time-t value is

$$A_{\mathrm{c},t} = \frac{1}{T}\int_0^t S_u \,\mathrm{d}u.$$

(We have divided by T for simplicity in what follows; most texts use $I_t = \int_0^t S_u \,\mathrm{d}u$ and divide by T in the payoff.) Then $\mathrm{d}A_t = (S_t/T)\,\mathrm{d}t$ and the continuous Asian BSPDE is

$$\frac{\partial V_{\mathrm{c}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{\mathrm{c}}}{\partial S^2} + rS\frac{\partial V_{\mathrm{c}}}{\partial S} + \frac{S}{T}\frac{\partial V_{\mathrm{c}}}{\partial A} - rV_{\mathrm{c}} = 0.$$

The corresponding discretely averaged option uses the average

$$A_{\mathrm{d},t} = \frac{1}{N}\sum_{j=0}^{t_i} S_{t_j},$$

where the averaging dates are separated by a time interval δt and $i = \lfloor t/\delta t \rfloor$ labels the last averaging date before time t. In between the averaging dates, the option value satisfies the ordinary BSPDE (no averaging term). The updating rule for the average is

$$A_{\mathrm{d},t_i+} = A_{\mathrm{d},t_i-} + \frac{S_{t_i}}{N}.$$

Continuity at averaging dates of the option price for a price trajectory gives the updating rule

$$V_{\mathrm{d}}^{\mathrm{Asian}}(S, t_i-, A) = V_{\mathrm{d}}^{\mathrm{Asian}}(S, t_i+, A + S/N)$$

for all S and A.

In this problem, rather than scaling time with σ^{-2}, it is more transparent to scale it with T and to use

$$\varepsilon^2 = \frac{\delta t}{T} = \frac{1}{N}$$

as the small parameter. Then the updating rule is approximated by

$$V_{\mathrm{d}}^{\mathrm{Asian}}(S, t_i-, A) = V_{\mathrm{d}}^{\mathrm{Asian}}(S, t_i+, A) + \varepsilon^2 S\frac{\partial V_{\mathrm{d}}^{\mathrm{Asian}}}{\partial A}(S, t_i+, A),$$

and exactly the same machinery as was used for the case of dividends leads directly to the continuous version of the BSPDE and to a correction term which is, at leading order, equal to $\tau S \partial V_{\mathrm{c}}^{\mathrm{Asian}}/\partial A$. Although this analysis works for any averaging, it is particularly clear when the average is arithmetic and the payoff is affine in S and A, say $\max(S - A - K, 0)$, as following the similarity reduction $V(S, A, t) = S\mathcal{V}((A-K)/S, t)$ the effect of averaging is the same as that of a (non-proportionate) dividend payment in the equation for $\mathcal{V}$.

3.2 Continuity corrections for range accrual notes

We next consider discrete and continuous payment in a range accrual note. This is a contract which has no terminal payoff but pays a coupon whenever the underlying asset price S_t lies within a specified range, $S_L < S_t < S_U$. The coupon may be modelled as being continuously paid, or as a series of discrete payments at times t_i, separated by δt. We assume for simplicity that the coupon rate is a constant c in annualised units. A contract with a coupon paid in $S_L < S_t < S_U$ is the same as a long position in a contract paying coupons in $0 < S_t < S_U$ and a short position in one paying in $0 < S_t < S_L$, so henceforth we only consider the former.

It is apparent that the discrete contract is simply a strip of binary (digital) calls and has the model-independent representation

$$V_{\mathrm{d}}^{\mathrm{ran}}(S,t) = c\,\delta t \sum_i P^{\mathrm{b}}(S,t;S_U,t_i),$$

where $P^{\mathrm{b}}(S,t;S_{L,U},t_i)$ is the value of a binary put, the last two arguments indicating the strike and expiry, and the sum is over all remaining dividend times. Letting $\delta t \to 0$, we have

$$V_{\mathrm{c}}^{\mathrm{ran}}(S,t) = c \int_t^T P^{\mathrm{b}}(S,t;S_U,s)\,\mathrm{d}s,$$

a formula which can also be written using a Green's function if one is available, as in (say) the Black–Scholes model.

Now consider this contract in the Black–Scholes model (for simplicity, without dividends). The continuous contract satisfies

$$\mathcal{L}_{\mathrm{BS}}^0 V_{\mathrm{c}}^{\mathrm{ran}}(S,t) + c\mathbb{I}_{(0,S_U)} = 0 \tag{10}$$

with $V_{\mathrm{c}}^{\mathrm{ran}}(S,T) = 0$, and with continuity of $V_{\mathrm{c}}^{\mathrm{ran}}$ and $\partial V_{\mathrm{c}}^{\mathrm{ran}}/\partial S$ at $S = S_U$. The discrete version satisfies

$$\mathcal{L}_{\mathrm{BS}}^0 V_{\mathrm{d}}^{\mathrm{ran}}(S,t) + c\,\delta t \sum_i \delta(t-t_i)\mathbb{I}_{(0,S_U)} = 0,$$

where $\delta(\cdot)$ is the delta distribution; this implies the jump condition

$$V_{\mathrm{d}}^{\mathrm{ran}}(S,t_i+) - V_{\mathrm{d}}^{\mathrm{ran}}(S,t_i-) + c\,\delta t\mathbb{I}_{(0,S_U)} = 0,$$

so that, across coupon dates, the option value is continuous in time outside the coupon-paying interval, and has a jump equal to the cash flow, $-c\,\delta t$, within it.

The method of multiple scales can be used, just as above, to derive the continuous BSPDE with a coupon, (10), in the coupon range and the normal coupon-less BSPDE outside it. The correction to the continuously paid contract, $V_{\mathrm{d}}^{\mathrm{ran}}(S,t) - V_{\mathrm{c}}^{\mathrm{ran}}(S,t)$, consists of a sawtooth function of time within the coupon region (with boundary-layer smoothings at the ends of this region), plus a solution of the BSPDE which vanishes at expiry and is continuous, but has a derivative jump of $-\frac{1}{2}c\,\delta t$ at $S = S_U$. This constitutes the 'continuity correction' for the range accrual note. In the Appendix, we give its derivation and an explicit formula for it. Note that the correction is negative outside the coupon region and positive inside it. This reflects the fact that in the discrete case, asset paths can dip into the coupon region between coupon dates without triggering any coupon payment and, conversely, they can rise above S_U between coupon dates without forfeiting any payment. In the continuous case, the former paths would receive coupons while below S_U, and the latter would lose them above S_U.

4 American put options

We now return to the put option, this time in its American form. The early-exercise possibility forces the value function of both continuous and discrete options to lie on or above the payoff. For the continuous option, this leads to the usual early-exercise boundary as sketched in Fig. 1. For the discrete option, as noted above, at fixed S, the value function (which is decreasing) falls at a dividend date. The corollary to this is rather subtle: because the option value immediately after the dividend can be no less than the payoff, it follows that, immediately before the dividend time, the option value is strictly above the payoff (except, perhaps, at zero asset value) and at that instant, early exercise is not optimal for any asset value. In other words, one would never exercise before the dividend date when it is known that the asset is about to fall by the dividend, with a consequent increase in the payoff of exercise. In fact, for a put option, we shall see that the option value immediately after a dividend date has a conventional exercise region whose boundary tends to $S = 0$ as the succeeding dividend date is approached, and, as has been noted by several authors, the approximately straight part of the exercise boundary can be explained as the asset value at which the gain from the uplift to the payoff due to the dividend balances the time value of money on the strike.

Continuous options: basic properties. The continuous put option price $P_{\mathrm{c}}^{\mathrm{a}}(S,t)$ satisfies

$$\mathcal{L}_{\mathrm{BS}}^{q} P_{\mathrm{c}}^{\mathrm{a}} = \frac{\partial P_{\mathrm{c}}^{\mathrm{a}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_{\mathrm{c}}^{\mathrm{a}}}{\partial S^2} + (r-q)S\frac{\partial P_{\mathrm{c}}^{\mathrm{a}}}{\partial S} - rP_{\mathrm{c}}^{\mathrm{a}} = 0 \tag{11}$$

for $0 < t < T$, with the payoff constraint $P_{\mathrm{c}}^{\mathrm{a}}(S,t) \geq \max(K-S,0)$. The exercise region is $0 < S \leq S_{\mathrm{c}}^*(t)$, which is an increasing convex function of t; as $t \uparrow T$, it tends either to K if $0 \leq q \leq r$ (the usual case in practice), or to rK/q if $q > r$. At $S = S_{\mathrm{c}}^*(t)$, the smooth pasting conditions

$$P_{\mathrm{c}}^{\mathrm{a}}(S_{\mathrm{c}}^*(t),t) = K - S_{\mathrm{c}}^*(t), \qquad \frac{\partial P_{\mathrm{c}}^{\mathrm{a}}}{\partial S}(S_{\mathrm{c}}^*,t) = -1$$

apply; from them, it follows by differentiation and use of the BSPDE that

$$\frac{\partial P_{\mathrm{c}}^{\mathrm{a}}}{\partial t}(S_{\mathrm{c}}^*,t) = 0, \qquad \lim_{S\downarrow S_{\mathrm{c}}^*(t)} \frac{\partial^2 P_{\mathrm{c}}^{\mathrm{a}}}{\partial S^2}(S,t) = \frac{2(rK - qS_{\mathrm{c}}^*)}{\sigma^2 {S_{\mathrm{c}}^*}^2} = \frac{2(\rho K - \gamma S_{\mathrm{c}}^*(t))}{{S_{\mathrm{c}}^*}^2},$$

where $\rho = r/\sigma^2$ and $\gamma = q/\sigma^2$ are as before.

Discrete option: problem statement. In between dividend dates, the discrete option price satisfies

$$\mathcal{L}_{\mathrm{BS}}^{0} P_{\mathrm{d}}^{\mathrm{a}} = 0$$

for $S^*_{\rm d}(t) < S < \infty$, where $S^*_{\rm d}(t)$ is its exercise boundary, at which the smooth pasting conditions, and the corollary that $\partial P^{\rm a}_{\rm d}/\partial t = 0$ there, are the same as for the continuous case. However, the Gamma at the exercise boundary is now

$$\lim_{S\downarrow S^*_{\rm d}(t)} \frac{\partial^2 P^{\rm a}_{\rm d}}{\partial S^2}(S,t) = \frac{2rK}{\sigma^2 {S^*_{\rm c}}^2} = \frac{2\rho K}{{S^*_{\rm c}}^2}. \qquad (12)$$

The jump condition at a dividend date t_i is

$$P^{\rm a}_{\rm d}(S, t_i-) = P^{\rm a}_{\rm d}(S{\rm e}^{-q\,\delta t}, t_i+).$$

Supposing for the moment that there is indeed an exercise region at t_i+, we then have $P_{\rm d}(S, t_i+) = K - S$ for $0 \le S \le S^*_{\rm d}(t_i+)$. Thus,

$$P^{\rm a}_{\rm d}(S, t_i-) = \begin{cases} K - S{\rm e}^{-q\,\delta t} & 0 \le S \le S^*_{\rm d}(t_i+){\rm e}^{q\,\delta t}, \\ P^{\rm a}_{\rm d}(S{\rm e}^{-q\,\delta t}, t_i+) & S > S^*_{\rm d}(t_i+){\rm e}^{q\,\delta t}; \end{cases}$$

the latter of these update conditions, in the continuation region (in which the value function is above the payoff), is the same as for the European case. The former, however, shows (as noted earlier) that the value function lies above the payoff for *all* $S > 0$ at time t_i-. The value function at this instant, which forms the initial value for the solution of the BSPDE backwards to time t_{i-1}, is thus a C^1 function with a known jump, given by (12), in its second derivative at $S = S^*_{\rm d}(t_i+){\rm e}^{q\,\delta t}$.

4.1 Asymptotic analysis

We proceed as before to analyse this problem for small $\epsilon^2 = \sigma^2\,\delta t$, using the backwards time variables t' and τ in a multiple scales expansion of the discrete option value. In a typical time interval $0 < \tau < 1$ (with periodicity in τ), we distinguish two principal regimes, separated by a transition. As illustrated in Figure 2, the asset price range is also divided into regions. In the first time region, which starts at $\tau = 0$ and matches into the transition around $\tau = qS^*_{\rm c}(t')/rK$, the difference between the value function and the payoff has:

- A small 'travelling-wave' region I_S, of size $O(\epsilon^2 K)$, around the discrete exercise boundary; this moves from $S = 0$ at $\tau = 0$ and approaches $S^*_{\rm c}$ as τ approaches $qS^*_{\rm c}(t')$. In this region, the dominant balance in (13) below is between the first term on the left and the first and last on the right, the others being smaller.

- A region II_S, overlapping with region I_S and ending close to $S^*_{\rm c}$, in which the difference in the value function is approximately linear in S. Here the dominant balance is between the first term on the left of (13) and the last on the right, representing diffusive smoothing of the curvature discontinuity.

- A region III_S around $S = S^*_{\rm c}(t')$, in which we write

  $$S = S^*_{\rm c}(t')(1 + \epsilon x);$$

  in this region the value function makes the transition from linear to quadratic, and the dominant balance involves the first term on each side of (13) and the last on the right.

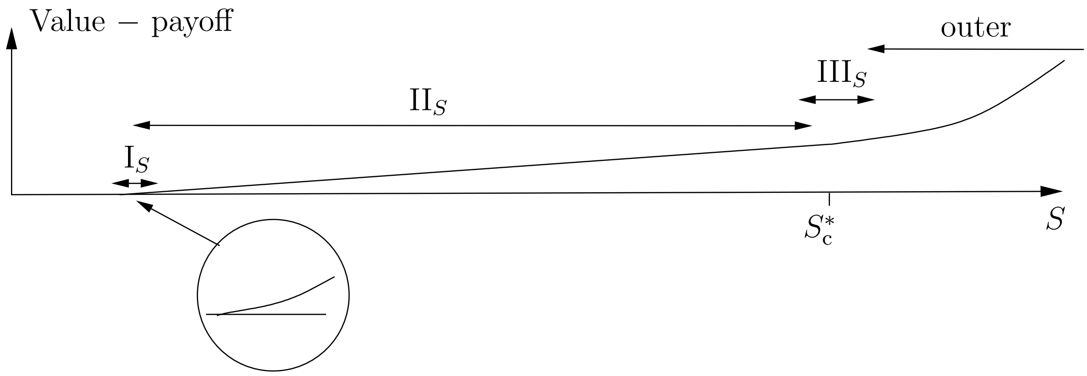

Figure 2: Schematic of the regions in an interval between two dividend payments.

- An outer region IV_S, in which we retrieve the continuous option value, matching with the value function in region III_S. Here there is no immediate dominant balance in (13); instead, the solvability condition for the first higher-order term in a regular expansion determines the solution.

In the second time interval, from τ near $qS_\mathrm{c}^*(t')/rK$ to $\tau = 1$, the discrete exercise boundary has reached region III_S and remains there; the value function has just two regions, regions I_S and II_S being subsumed into region III_S. The transition from the first regime to the second is more complicated, and is described below.

It is convenient to subtract the payoff from the option value, writing the value function and exercise boundary in the multiple-scales form

$$P_\mathrm{d}^\mathrm{a}(S,t') = K - S + W(S,t',\tau), \qquad S_\mathrm{d}^*(t') = s^*(t',\tau),$$

and we shall suppress the dependence on t' wherever it is not misleading to do so, in particular writing $S_\mathrm{c}^*(t') = S_\mathrm{c}^*$. Then,

$$\frac{1}{\epsilon^2}\frac{\partial W}{\partial \tau} + \frac{\partial W}{\partial t'} = \tfrac{1}{2}S^2\frac{\partial^2 W}{\partial S^2} + \rho S\frac{\partial W}{\partial S} - \rho W - \rho K, \qquad s^*(t',\tau) < S < \infty, \tag{13}$$

with the constraint $W \geq 0$ implying the smooth-pasting conditions

$$W = 0, \quad \frac{\partial W}{\partial S} = 0 \qquad \text{at} \quad S = s^*(t',\tau),$$

and the initial condition (approximating $\mathrm{e}^{-\epsilon^2\gamma}$ by $1-\epsilon^2\gamma$)

$$W(S,t',0) \sim \begin{cases} \epsilon^2\gamma S, & 0 \leq S \leq (1+\epsilon^2\gamma)s^*(t',1) \\ W(S(1-\epsilon^2\gamma),t',1), & (1+\epsilon^2\gamma)s^*(t',1) \leq S < \infty. \end{cases}$$

Note, again, that $W(S,t',0)$ is C^1 and its second derivative has a jump at the switch point from linear to nonlinear. It is also very important in the analysis to follow that the value surface is exactly linear at $\tau = 0+$, having been 'wiped clean' by exercise of the option. Lastly we have periodicity in τ (as already implicit in the second part of the initial condition above).

We observe immediately that $W(S,t',0)$ is of $O(\epsilon^2)$ relative to the payoff for $S < S_{\mathrm{c}}^*$. We thus expect $W(S,t',\tau)$ also to be of $O(\epsilon^2)$ in this range for S. This is why we subtracted the payoff earlier.

We begin with the two regions in which S is unscaled.

The outer region IV_S**.** In this region, the expansion procedure is the same as for a European contract, giving

$$W(S,t',\tau) \sim W_0(S,t') + \epsilon W_1(S,t') + \epsilon^2 W_2(S,t',\tau) + \cdots .$$

The analysis of §3 can be repeated to show that $W_0(S,t')$ satisfies the continuous BSPDE with a continuous dividend yield q. This solution is to be matched with the solution in region III_S. However, as argued above (and justified in more detail below), the solution in region III_S is of $O(\epsilon^2)$. Just as in the analysis of §3.2, we conclude from matching at $O(1)$ and at $O(\epsilon)$ that, as $S \downarrow S_{\mathrm{c}}^*$, $W_0(S,t')$, $\partial W_0/\partial S$ and $W_1(S,t')$ all tend to zero. This is enough to show that $W_0(S,t')$ satisfies the same smooth pasting conditions as the continuous option price, and is therefore equal to it (and so the choice of S_{c}^* as the leading-order location of the transition is justified *a posteriori*). It then follows that $W_1(S,t') \equiv 0$, and the 'continuity correction' for the discrete option is at $O(\epsilon^2)$ (that is, $O(1/N)$). This is to be expected, given that the value function and its Delta are continuous for both the continuous and discrete problems; for essentially the same reason, the correction is equally small for the range accrual note considered above and for Bermudan options [6], while for barrier options, with a discontinuous Delta at the barrier, the correction is of $O(\epsilon)$ (that is, $O(1/\sqrt{N})$).

It only remains to state the matching behaviour of the outer solution: as $S \downarrow S_{\mathrm{c}}^*$, and written in inner variables (that is, for large x),

$$W(S,t',\tau) \sim \epsilon^2 \left(2(\rho K - \gamma S_{\mathrm{c}}^*)x^2 + O(1) \right),$$

where '$O(1)$' is in fact equal to $W_2(S_{\mathrm{c}}^*,t',\tau)$; as (for reasons given below) we do not compute a continuity correction for this contract, we do not need this term.

Region II_S**.** The main driver of the difference between continuous and discrete option prices is region II_S, and the main driving term is $-\rho K$ in (13) (this discounting term is, indeed, responsible for the existence of an exercise region in the continuous problem).

The initial data suggests that solution takes the form

$$W(S,t',\tau) \sim \epsilon^2 W_2(S,t',\tau) + O(\epsilon^3),$$

where

$$\frac{\partial W_2}{\partial \tau} = -\rho K, \qquad W_2(S,t',0) = \gamma S, \quad 0 < S < S_{\mathrm{c}}^*,$$

so that

$$W_2(S,t',\tau) = \gamma S - \rho\tau K.$$

Because $W \geq 0$, this solution is only valid for $\rho\tau K/\gamma < S < S_{\mathrm{c}}^*$, and thus only for $0 < \tau < \gamma S_{\mathrm{c}}^*/\rho K = q S_{\mathrm{c}}^*/rK = \tau^*$, say. It is easy to show that $\tau^* < 1$: if $q < r$, this holds because $S_{\mathrm{c}}^* < K$, while if $q > r$, we use the fact that $S_{\mathrm{c}}^*(t')$ has initial (in t'; terminal in t) value rK/q and then decreases.

We therefore see the division of the interval $0 < \tau < 1$ described above. The exercise boundary is close (in a way to be made more precise below) to $S = \rho\tau K/\gamma$, which is where W_2 vanishes. It moves rapidly, taking the fraction τ^* of the dividend interval to move from $S = 0$ to a neighbourhood of S_{c}^*, and its speed is determined only by the competing effects of dividend payments (via the initial condition) and the interest rate (via the BSPDE); diffusion plays no role in this region at leading order. Indeed, iteration of the expansion (which essentially completes rt to $\mathrm{e}^{rt} - 1$) shows that the solution is affine in S at all orders in ϵ. The only way that curvature of the value surface can enter this region is by diffusion from its ends, via exponentially small terms, an important fact in what follows.

The exercise boundary for $0 < \tau < \tau^*$: Region I_S. While $0 < \tau < \tau^*$, the exercise boundary $S = s^*(\tau)$ is near $S = \rho K\tau/\gamma$, where $W_2 = 0$. It is necessary to restore diffusion in order to allow the two smooth pasting conditions to hold, and to rescale W_2 (because it is small). Specifically, set

$$S = \frac{\rho K}{\gamma}\tau + \epsilon^2 K\xi, \qquad W_2(S, t', \tau) = \epsilon^2 W_2^*(\xi, \tau),$$

note that the inner limit of the outer solution $W_2(S, t', \tau)$ in these variables is then $\gamma S - \rho K\tau \sim \epsilon^2\gamma\xi + O(1)$. The term $O(1)$ here simply determines any offset of the exercise boundary on the inner scale (the inner problem is autonomous in ξ); this is determined by higher-order terms (which we do not compute) in the outer solution, and we may assume that the exercise boundary is at $\xi = 0$.

The inner problem is then

$$-\frac{\rho}{\gamma}\frac{\partial W_2^*}{\partial \xi} = \frac{1}{2}\frac{\partial^2 W_2^*}{\partial \xi^2} - \rho K + o(\epsilon)$$

(the $o(\epsilon)$ terms include the τ-derivative of W_2^*), with $W_2^*(0, \tau) = 0$, $\partial W_2^*/\partial\xi(0, \tau) = 0$ and $W_2^*(\xi, \tau) \sim \gamma K\xi + O(1)$ as $\xi \to \infty$. The solution is a function only of the travelling-wave variable ξ, namely

$$W_2^*(\xi, \tau) = \gamma K \left(\xi - \frac{\gamma}{2\rho}\left(1 - \mathrm{e}^{-2\rho\xi/\gamma}\right) \right).$$

Region III_S: $0 < \tau < \tau^*$. Near $S = S_{\mathrm{c}}^*$, we again seek a balance between the first term on each side of (13), but as the initial data is prescribed, we cannot rescale W_2. We write

$$S = S_{\mathrm{c}}^*(1 + \epsilon x^* + \epsilon x), \qquad W_2(S, t', \tau) = w_2(x, \tau)$$

where x^* is defined below. The leading order problem for w_2 is then

$$\frac{\partial w_2}{\partial \tau} = \frac{1}{2}\frac{\partial^2 w_2}{\partial x^2} - \rho K, \tag{14}$$

with

$$w_2(x,0) = \gamma S_c^* + F(x)$$

where $F(x)$ is generated by the periodicity condition. Because the shift induced by the dividend payment is of $O(\epsilon^2)$, it can be ignored at this order of accuracy. However, the location of the exercise boundary at $\tau = 1$, namely $S = s^*(t',1) = S_c^*(1+\epsilon x^*(t'))$, say, corresponding to $x = x^*(t')$, say, is unknown, albeit constant on the fast timescale. Thus, $F(x)$ is smooth (analytic) except at $x = 0$, where its second derivative has a jump. Specifically,

- $F(x)$ vanishes for $x < 0$ (the exercise region at $\tau = 1-$);
- $F(0) = 0$, $F'(0) = 0$, and $F''(0+) = 2\rho K$ (the smooth pasting conditions and their corollary);
- $F(x) \sim (\rho S_c^* - \gamma K)(x + x^*)^2(1 + o(x))$ as $x \to \infty$ (the matching condition with the outer solution).

Determination of x^* should lead to a continuity correction for this contract, as in [6], probably via the vanishing of the linear component in the asymptotic behaviour of $F(x)$ as $x \to \infty$; unfortunately the added complexities due to the exercise boundary have precluded completing his part of the solution.

Because $F(x)$ is unknown, we cannot compute $w_2(x,\tau)$, but we can make progress by using its asymptotic behaviour as $x \to -\infty$. After subtracting the particular solution $\gamma S_c^* - \rho K\tau$, we are left with a solution of the heat equation with initial data that vanishes for $x < 0$. The far-field behaviour ($x \to -\infty$, $\tau = O(1)$ is determined entirely by the behaviour of the initial data near $x = 0$, and in this case takes the form $\rho K\tau f(x/\sqrt{\tau})$, where $f(-\infty) = 0$ and $f(\eta) \sim \eta^2$ as $\eta \to \infty$. We have

$$f(\eta) = \eta^2 + 1 + \frac{1}{\sqrt{2\pi}}\left(\eta \mathrm{e}^{-\eta^2/2} - (\eta^2+1)\int_\eta^\infty \mathrm{e}^{-s^2/2}\,\mathrm{d}s\right),$$

and the asymptotic behaviour of $f(\eta)$ as $\eta \to -\infty$ is

$$f(\eta) \sim -\sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-\eta^2/2}}{\eta^3},$$

so that the far-field behaviour of $w_2(x,\tau)$ as $x \to -\infty$ is

$$w_2(x,\tau) \sim \gamma S_c^* - \rho K\tau - \rho K\sqrt{\frac{2}{\pi}}\frac{\tau^{\frac{5}{2}}}{x^3}\mathrm{e}^{-x^2/2\tau}. \tag{15}$$

In what follows we shall also need some information about the next term in the inner expansion, namely $w_3(x,\tau)$. Its initial data $w_3(x,0)$ contains a term $\gamma S_c^* x$, $-\infty < x < \infty$, which arises from the dividend-shift of the payoff; the rest of its initial data is only non-zero for $x > 0$ (which matches with the outer solution as $x \to \infty$, growing cubically there). The function $w_3(x,\tau)$ satisfies (14) with a different inhomogeneous term having the important feature that it decays rapidly as $x \to -\infty$. It follows that $w_3(x,\tau)$ contains a term $\gamma S_c^* x$ for $0 < \tau < \tau^*$, and the far-field behaviour of the inner solution is

$$\gamma S_c^* - \rho K \left(\tau + \sqrt{\frac{2}{\pi}} \frac{\tau^{\frac{5}{2}}}{x^3} e^{-x^2/2\tau} \right) + \epsilon \gamma S_c^* x + \cdots .$$

It is worth noting that the term $\gamma S_c^* x$ matches directly with the corresponding term from the solution $W_2(S,\tau)$ in region II_S, when the latter is written in inner variables.

Region III_S: $\tau^* < \tau < 1$. In this time interval, the exercise boundary reaches region III_S, because at this time $w_2(x,\tau)$ first falls to zero, at $x = -\infty$. Its behaviour for τ shortly after τ^* is thus determined by the far-field behaviour of $w_2(x,\tau)$, specifically by where this function vanishes. setting $x = s^*(t',\tau)$ in (15) and solving approximately, we have

$$s^*(t',\tau) \sim -\sqrt{-2\tau^* \log(\tau - \tau^*)}$$

as τ increases away from τ^*. Thereafter, $s^*(t',\tau)$ increases, finally tending to $s^*(t',1) = 0$ as $\tau \to 1$, and the cycle repeats.

The transition though $\tau = \tau^*$: intermediate region $\mathrm{II}_S/\mathrm{III}_S$. The solution $w_2(x,\tau) + \epsilon w_3(x,\tau)$ with no exercise boundary in region III_S described above is valid from $\tau = 0$ until shortly before $\tau = \tau^*$, at which time it first approaches zero, for large negative values of x. The fall to zero of $w_2 + \epsilon w_3$ triggers the initiation, at $x = -\infty$, of the exercise boundary in region III_S, moving in from region II_S. In so doing, the exercise boundary must slow down greatly; having moved from $S = 0$ to near $S = S_c^*$ in the interval $0 < \tau < \tau^*$, it remains near $S = S_c^*$ for the remaining time interval $\tau^* < \tau < 1$.

Writing $\tau' = \tau - \tau^*$, which we expect to be small, we make the ansatz that the exercise boundary $x = s^*(t',\tau')$ satisfies the equation

$$-\rho K \sqrt{\frac{2}{\pi}} \frac{\tau^{*\frac{5}{2}}}{s^{*3}} e^{-s^{*2}/2\tau^*} + \epsilon \gamma S_c^* s^* = \rho K \tau'; \tag{16}$$

we have kept $\tau = \tau^*$ except in the discounting term because τ' is small. As in region II_S, this is to assume that the second x–derivative of w can be neglected on the appropriate length scales, and *a posteriori* verification shows this to be the case, so that smooth pasting is achieved via a small travelling wave region (which is, indeed, the continuation in τ of region I_S). The left-hand side of (16) is an increasing function of s^* which it is most convenient to examine working backwards in τ'. When τ' is small and positive, s^* is large and negative. Provided that τ' is not too small, the term $\epsilon \gamma S_c^* s^*$ can be neglected (because $\epsilon \ll 1$), and iteration shows that

$$s^*(t',\tau') \sim -\sqrt{-2\tau^* \log \tau'} + \frac{3}{2} \frac{\sqrt{2\tau^*} \log \sqrt{-2\tau^*}}{\sqrt{-\log \tau'}}$$

This putative balance holds until s^* is large enough that the neglected linear term comes into play. All three terms in (16) balance over a short interval $\tau' = \epsilon \sqrt{|\log \epsilon|}\, \widetilde{\tau}'$, with the appropriate scale for s^* being

$$s^* \sim -\sqrt{2\tau^*}\left(-\sqrt{|\log\epsilon|} - \frac{2\log\sqrt{|\log\epsilon|}}{\sqrt{|\log\epsilon|}} - \frac{\widetilde{s}^*}{2\sqrt{|\log\epsilon|}}\right),$$

and then we have, approximately,

$$\frac{\rho K\tau^*}{2\sqrt{\pi}}\mathrm{e}^{\widetilde{s}^*} + \gamma\sqrt{2\tau^*}S_\mathrm{c}^*\left(-1 - \frac{2\epsilon}{|\log\epsilon|} + \frac{\widetilde{s}^*(t',\tau')}{2|\log\epsilon|}\right) = \rho K\tau'.$$

It follows that in this regime,

$$\widetilde{s}^*(t',\widetilde{\tau}') \sim \log\left(\frac{2\sqrt{\pi}\tau'}{\tau^*}\right).$$

The final regime is found as $\widetilde{\tau}' \to 0$, being a balance between the last two terms in (16), and it gives

$$s^*(t',\tau) \sim \frac{\rho K}{\epsilon\gamma S_\mathrm{c}^*}\tau'$$

as $\tau' \to -\infty$, matching with the solution in region II_S.

4.2 Non-proportional dividend payments

We now consider the payment of a constant, rather than proportional, dividend at each dividend date. This case was also considered by [11], who reported an exercise boundary which, like the proportionate case, restarts (working in backward time) from $S = 0$ at each dividend date; however, instead of increasing linearly for part of the inter-dividend period as in the proportionate case, he reported an apparent jump in the exercise boundary from zero to a finite (order-one) value at a time which agreed well with the result of a simple ost-of-money calculation. We now investigate this structure in more detail; it is illustrated in Figure 3.

Suppose that the payment is $D\,\delta t$ (independently of S). Then we have

$$S_{t_i+} = S_{t_i-} - D\,\delta t$$

with the zero-dividend geometric Brownian motion between dividend dates. We see immediately that this model is problematic because S_{t_i+} is negative if $S_{t_i-} < D\,\delta t$. In [11], this issue was dealt with by applying a boundary condition that the put value is equal to the discounted strike at $S_D\mathrm{e}^{-r(t_i-t)}$, where $S_D = D\,\delta t$), for $t_{i-1} < t < t_i$ (an improved version of this condition is given in [12]). Here we use a slightly different condition which allows us to solve for all $0 < S < \infty$, arguing below that the overall impact of the specific choice is small. We assume that the dividend paid is $\max(D\,\delta t, S_{t_i-})$: that is, if the asset value just before a dividend date is less than the dividend, the asset pays out all its remaining value. Thereafter, the asset value remains at zero. Hence, just before the dividend date the value of a put option on the asset is equal to K if $0 \le S_{t_i-} \le D\,\delta t$. The jump condition (3) for a put option becomes

$$P_\mathrm{d}(S,t_i-) = \begin{cases} P_\mathrm{d}(S - D\,\delta t, t_i+), & D\,\delta t < S < \infty \\ K^{\mathrm{a,e}} & 0 \le S \le D\,\delta t, \end{cases}$$

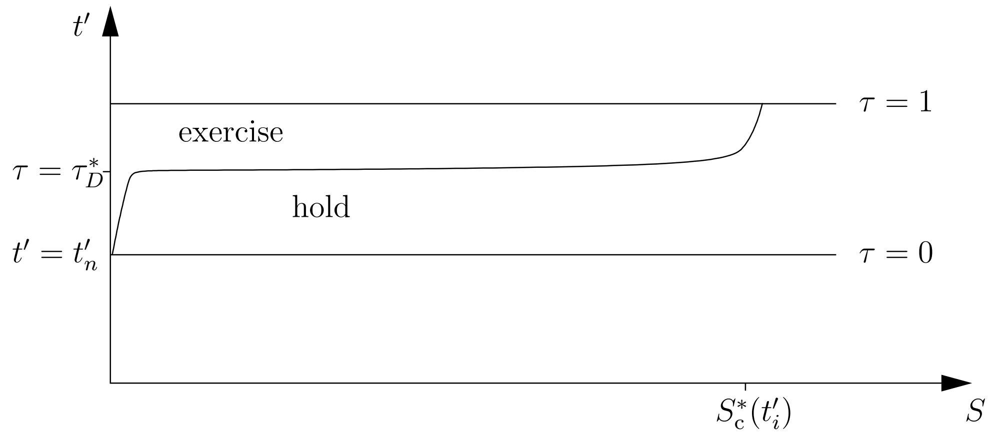

Figure 3: Schematic of the structure (in backwards time) of the exercise boundary for a fixed dividend payment.

where

$$K^{a,e} = \begin{cases} K & \text{for an American option,} \\ Ke^{-r(T-t_i)} & \text{for a European option,} \end{cases}$$

reflecting the time at which the payoff occurs. That is, as we work backwards from expiry, at each dividend date the value function is shifted to the right by D and the gap left for $0 \le S < D\,\delta t$ is filled by setting the value equal to K, as sketched in Figure 4.

The zero-dividend Black–Scholes equation is solved between dividend dates, but only in the hold region $S_d^*(t) < S < \infty$. Far away from $S = 0$ and, for American options, away from exercise boundaries, the multiple scales ideas used before show that, as $\delta t \to 0$, the appropriate limiting PDE is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial t^2} + (rS - D)\frac{\partial V}{\partial S} - rV = 0$$

(this equation has no simple explicit solutions except for cash and the discounted asset-dividend combination; a version of it follows from the similarity reduction of arithmetic Asian options with affine payoffs). We expect a corresponding continuously-sampled exercise boundary $S = S_c^*(t)$.

Turning to the regions near and below $S = S_c^*(t)$, matters are simplified by the way in which the American option wipes the slate clean at each dividend date via its exercise region.[2] We make the additional, and realistic, assumption that $D < rK$ (roughly equivalent to $q < r$ for a proportionate dividend yield: see below for why). We recall the scaled BSPDE

$$\frac{\partial P_d^a}{\partial t'} = \tfrac{1}{2}S^2 \frac{\partial^2 P_d^a}{\partial S^2} + \rho S \frac{\partial P_d^a}{\partial S} - \rho P_d^a,$$

to be solved between dividend dates. As before, it is convenient to set $P_d^a(S, t') = K - S + W(S, t', \tau)$, $\quad S_d^*(t') = s^*(t', \tau)$, where $W(S, t', \tau)$ satisfies (13). and then we expand

[2] For a European put, we have to impose a boundary condition on $S = 0$ and then the repeated shifting of the payoff leads to a complicated region near $S = 0$ which we do not analyse here.

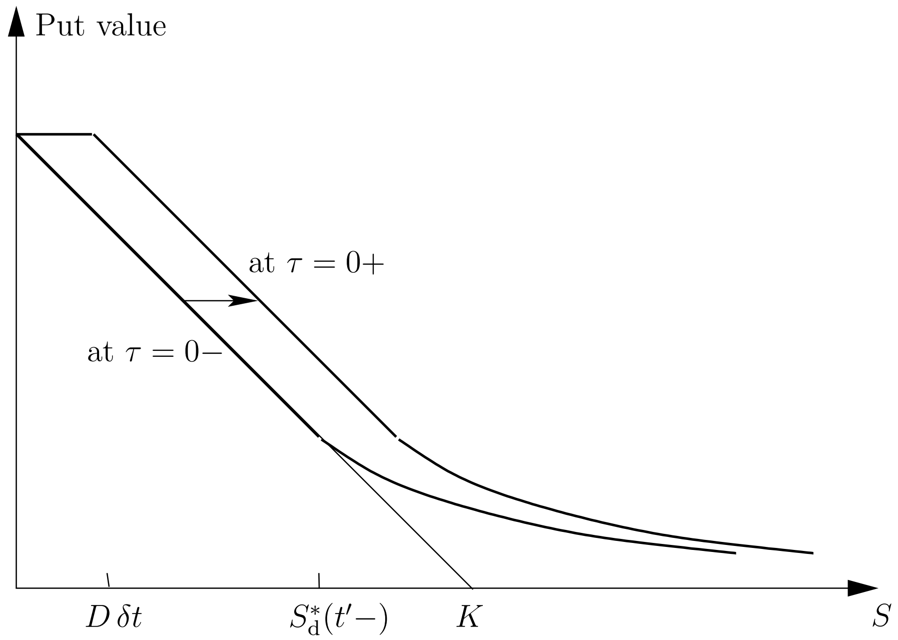

Figure 4: Shifting of the value function at a dividend date: constant dividend payment.

$W(S,t',\tau)$ as earlier. Now, however, the initial value of the excess over the payoff is

$$W(S,t',0) \sim \begin{cases} \epsilon^2 S, & 0 \le S \le S_D = \epsilon^2 s_D := \sigma^2 \delta t D, \\ \epsilon^2 s_D, & \epsilon^2 s_D < S < s^*(t',1) + \epsilon^2 s_D, \\ W(S - \epsilon^2 s_D, t', 1), & s^*(t',1) + \epsilon^2 s_D \le S < \infty \end{cases}$$

(see Figure 4): this function is the difference between the two bold curves). It is helpful to decompose this into the portfolio of one long asset, one short call with strike $\epsilon^2 s_D$, and one long power-type call with strike $K^* = s^*(t',1) + \epsilon^2 s_D$; the important feature of the payoff of the last of these is that, near K^*, it has the local behaviour $\rho K\left(\max(S-K^*)\right)^2/S^{*2}_{\mathrm{d}}$. The value of an option with this payoff can readily be calculated in terms of vanilla call prices. Also as before, it is crucial that the initial value is exactly constant in S over the majority of the interval $(0, s^*(t',0))$, as once again the behaviour of the exercise boundary will be determined by diffusion in from the ends of this interval.

The dominant feature of the behaviour is again that the option value decreases linearly in τ; the reader should envisage the upper bold curve in Figure 4 descending at a constant rate. The optimal exercise boundary then accommodates the payoff constraint. Specifically, the structure is as follows. The interval $0 < \tau < 1$ is divided into $0 < \tau < \tau^*_D = D/rK$ and $\tau^*_D < \tau < 1$. From $\tau = 0$ until shortly before $\tau = \tau^*_D$, the principal (leading order) features are:

- The 'horizontal part' of the initial value descends at the rate $-\epsilon^2 \rho K$, so that in, but away from the ends of, the interval $\epsilon^2 \rho K \tau < S < \epsilon^2 s_D$, $W(S,\tau) \sim S - \epsilon^2 \rho K \tau$.

- At the left-hand end of this interval there is a small travelling wave region $S = \epsilon^2 \rho K \tau\,(1 + O(\epsilon^2))$ in which the smooth-pasting conditions are accommodated.

- Meanwhile, the ‘corner’ of the initial value at $S = \epsilon^2 s_D$ smooths by diffusion (the short-time asymptotics of a call option, see [7]) in a region $S = \epsilon^2 s_D (1 + O(\epsilon)))$.
- The constant section of the initial value function decreases without changing shape (apart from exponentially small incursions from its ends) until it touches down at $\tau = \tau_D^*$ (to leading order).
- The curvature discontinuity at $S = K^*$ diffuses exactly as before. Now, however, we shall need its asymptotics for the whole range of S below the discontinuity, rather than simply in a small neighbourhood of it.

Now if there were no diffusion at all, at time[3] $\tau = \tau_D^* = D/rK$ the translated initial value of Figure 4 would sit exactly on its untranslated version for $\epsilon^2 s_D < S < s^*(t', 0)$ and the exercise boundary would jump instantaneously from $\epsilon^2 s_D$ to $s^*(t', 0)$, near K^*. However, both the discontinuities in the initial value contribute small tails which smooth off the jump over a very short timescale. The short call part of the initial value gives a negative contribution which has two effects. First, and before $\tau = \tau_D^*$, the exercise boundary rapidly (taking a (τ) time of $O(\epsilon)$) traverses the region around its ‘strike’ $S = \epsilon^2 s_D$ as the discounting term pulls the value function down to the payoff (except for a minute region very close to the corner, the location of the exercise boundary is still just determined by the intersection of the no-exercise solution with the payoff, smoothed off by a travelling wave near this intersection point). Then, the intersection point (with associated travelling wave region) moves much more rapidly to the right and its location is determined by the intersection of the right-hand tail of the call option with the payoff, as the discounting term pulls it down, much as for the proportionate case above. This stage, which is still before $\tau = \tau_D^*$ because the contribution from the call is negative, lasts until the influence of the left tail of the power call with strike at K^*, whose contribution is positive; there is a momentary transition when the two tails balance, very close to $\tau = \tau_D^*$, and then the exercise boundary is determined by the left tail of the power call (the tail of the other call then being exponentially negligible). It moves, still very rapidly, until it comes into the region $S = K^*(1 + O(\epsilon))$ where, just as for the proportionate yield, it slows down rapidly and remains close to $S = K^*$ until the next dividend date.

It would of course be possible to give more detail of all these stages in the evolution along the lines of the proportionate case. In view of the modelling uncertainties near $S = 0$, we simply calculate where the influence of this region becomes subordinate to that of the curvature discontinuity at $S = K^*$. This entails knowledge of the tail behaviour of the difference between a call option price and its asymptotes. A long (because terms cancel) calculation based on the explicit formulae, or a rather shorter one using the short-time asymptotics of the heat equation, shows that the value of a call option with expiry T and strike $\mathcal{K}$ on an asset with volatility σ differs from its asymptotes by

$$\frac{\mathcal{K}(\sigma^2 T)^{\frac{3}{2}}}{(\log(S/\mathcal{K}))^2} \left(\frac{S}{\mathcal{K}}\right)^{-r/\sigma^2 + \frac{1}{2}} \exp\left(-\left(\log(S/\mathcal{K})\right)^2 / 2\sigma^2 T\right),$$

[3] With exact discounting, this is $(\sigma^2/\epsilon^2 r)\log(1 + \epsilon^2 D/\sigma^2 K)$, which, after translation of notation, is the time at which the apparent jump in [11] occurred. With his boundary condition, the put value is specified at $S = \epsilon^2 s_D$ to be a decresaing function which first reaches the payoff at $\tau = \tau_D^*$ and there is no exercise boundary until this time.

as $T \to 0$, for both $S > \mathcal{K}$ and $S < \mathcal{K}$. The dominant feature of this expression is the exponential. For our problem, we wish to find the value of S at which the small-time values of an option with strike $\epsilon^2 s_D$ and one with strike K^* are comparable, and at leading order this means that we need $\left(\log(S/\epsilon^2 s_D)\right)^2 = \left(\log(S/K^*)\right)^2$, so that the switchover we seek is at $S \sim \epsilon\sqrt{s_D K^*}$. For larger values of S than this, the behaviour is likely to be independent of the particular modelling assumption made near $S = 0$, as any suitable modelling assumption leads to the same exponential behaviour, with only the prefactor differing according to the details near the origin.

5 American call options

5.1 Proportional dividend payments

American call options with frequent dividend payments are very different from put options. The continuously sampled option only has an exercise region when $q > 0$, and in this case its value $C_{\mathrm{c}}^{\mathrm{a}}(S,t)$ satisfies the BSPDE $\mathcal{L}_{\mathrm{BS}}^{q} C_{\mathrm{c}}^{\mathrm{a}} = 0$ for $0 < S < S_{\mathrm{c}}^{*}(t)$, with the value-matching and smooth pasting conditions $C_{\mathrm{c}}^{\mathrm{a}} = S - K$, $\partial C_{\mathrm{c}}^{\mathrm{a}}/\partial S = 1$ at $S = S_{\mathrm{c}}^{*}(t)$. For larger S, early exercise is optimal to capture the dividends on the asset, and so $C_{\mathrm{c}}^{\mathrm{a}} = S - K$ for $S > S_{\mathrm{c}}^{*}(t)$. When $q = 0$, there is no early exercise, and the American and European calls have the same value function; we recall that, as $S \to \infty$, it is asymptotic to $S - K\mathrm{e}^{-r(T-t)}$.

Now consider what happens to a discretely sampled contract, with value $C_{\mathrm{d}}^{\mathrm{a}}(S,t)$, as we move backwards in time from expiry to the final dividend date t_N. For $t_N < t < T$, with no dividends remaining, the option is a vanilla call. The value function satisfies the zero-dividend BSPDE and lifts off the payoff $\max(S-K,0)$ until at $t = t_N+$, just after the dividend date, it lies above the payoff everywhere and is asymptotic to $S - K\mathrm{e}^{-r\,\delta t}$. Now the jump condition (3) is applied, shifting the value function to the right. For large S, the shifted asymptotic behaviour lies below the payoff because S is replaced with $S\mathrm{e}^{-q\,\delta t}$, while for small S the shifted value function still lies above the payoff. Hence there is a value of S, say $S = S_N^*$ (which can easily be shown to be unique), where the payoff and the shifted value function are equal; for $S > S_N^*$, the option should be exercised at $t = t_N-$ (because holding would lead to an option value below the payoff at $t = t_N+$), while for $S < S_N^*$ the option should be held.

What now happens, for $t_{N-1} < t < t_N$: is there an exercise boundary (as for the put option) or not? We show that the latter is correct. Write

$$
\begin{aligned}
V_N(S) &= \max\left(\max(S-K,0), V_{\mathrm{d}}^{a}(S,t_N+)\right) \\
&= \begin{cases} S-K, & S \geq S_N^*, \\ V_{\mathrm{d}}^{a}(S,t_N+), & S < S_N^*; \end{cases}
\end{aligned}
$$

this is the value of the discretely sampled option at $t = t_N-$. Then clearly $V_N(S)$ is no less than $\max(S-K,0)$, and hence the value of an option with time-t_N payoff $V_N(S)$ is no less than the value of a call with the same expiry; and in turn, recalling that no continuous dividends are paid, this call option value lies (strictly) above $\max(S-K,0)$

for times $t_{N-1} < t < t_N$. Hence there is no early exercise here either: the only time that early exercise is optimal is at the dividend date.

This pattern is clearly repeated for all the other dividend dates, and hence the scenario that emerges is that the option is essentially a Bermudan one — not because the exercise dates are pre-specified, but because the optimal exercise policy dictates that exercise should only take place immediately before dividend dates t_i, and then only for $S > S_i^*$ calculated recursively as above. In the limit $\delta t \to 0$, we expect the discretely sampled values S_i^* to lie close to the continuously sampled exercise boundary $S = S_{\mathrm{c}}^*(t)$, while for $S < S_{\mathrm{c}}^*(t)$ we recover via multiple scales the non-zero dividend BSPDE $\mathcal{L}_{\mathrm{BS}}^q C_{\mathrm{c}}^a = 0$. The difference between the continuous and discrete values is of $O(\epsilon^2)$ and can be characterised as a combination of a 'sawtooth' correction as described in Section 3 and a Bermudan-style continuity correction almost identical to those analysed in [6], and we do not give details here.

5.2 Non-proportional dividend payments

The case of a constant dividend payment $D\,\delta t$ at each dividend date is very similar to that of proportional payments just discussed. One point to note is that the shift of the value function across a dividend date takes its argument from S to $S-D$, a constant shift rather than a proportional one. When this is applied at large S, it takes the asymptotic value $S - K\mathrm{e}^{-r\,\delta t}$ to $S - D\,\delta t - K\mathrm{e}^{-r\delta t}$, and this only lies below the payoff $S-K$ if

$$D\,\delta t > K(1 - \mathrm{e}^{-r\,\delta t}).$$

Hence, expanding for small δt, if $D < rK\,(1 + O(r\,\delta t))$, there is no early exercise at all. This simply states that the constant dividends are not large enough to trigger early exercise for dividend capture. With this caveat, the derivation of the appropriate version of the BSPDE, and the form of the correction terms, are sufficiently close to those of the proportionate case that no more details need be given.

6 Discussion

We have given a systematic presentation of the asymptotics for the difference between continuously and discretely sampled options in a Black–Scholes framework. The method of multiple scales reveals the local-in-time structure as being periodic, with slower modulations. When the options concerned are American, the exercise region has multiple components, and it is interesting to note that, although the continuous exercise boundary emerges naturally as $\delta t \to 0$, within the continuous exercise region the ratio of hold to exercise in a diagram such as Fig. 1 or Fig. 3 tends to a fixed value strictly between 0 and 1. This is reminiscent of the mushy regions found in Stefan problems with volumetric heating, but it should be noted that the probability that the asset ever enters the 'long thin' excursions of the hold region into the continuous exercise region is vanishingly small in the limit.

Our discussion has been confined to the simple Black–Scholes model, but in principle the method can be applied to other models such as local volatility, CEV etc., with the proviso that the analytic treatment is less clean; substantial parts of it will, however, carry over because of the relatively small role played by the volatility. in principle, jump diffusion models can also be at least partly treated because the contribution of the jump terms is tractable, and at least the general structure will carry over to multidimensional models (several assets, or stochastic volatility). Our final comment is that, even though we included a precautionary $O(\epsilon)$ (or $O(1/\sqrt{N})$) term at places in the analysis, all these terms vanished and the correction terms we have calculated are an order of magnitude smaller, namely $O(1/N)$. This is attributable to the extra degree of smoothness possessed by our contracts, in contrast with barrier options with their nonzero boundary Delta.

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Appendix: the range accrual note

Write the difference between the discretely and continuously paid range accrual notes as

$$V_{\mathrm{d}}^{\mathrm{ran}}(S,t') - V_{\mathrm{c}}^{\mathrm{ran}}(S,t') = \begin{cases} \overline{W}(S,t') & 0 < S < S_U \qquad \text{(inside the coupon region)}, \\ W(S,t') & S_U < S < \infty \qquad \text{(outside it)}. \end{cases}$$

Then using the two time scales t' and τ we have

$$\frac{1}{\epsilon^2}\frac{\partial W}{\partial \tau} = \mathcal{L}'W, \qquad \frac{1}{\epsilon^2}\frac{\partial \overline{W}}{\partial \tau} = \mathcal{L}'\overline{W} - \kappa$$

in their respective domains of definition; here, $\kappa = c/\sigma^2$ is the rescaled coupon payment. The cash-flow jump conditions are that that

$$\left[\overline{W}(S,t',\tau)\right]_{\tau=1-}^{\tau=1+} = \epsilon^2\kappa, \qquad \left[W(S,t',\tau)\right]_{\tau=1-}^{\tau=1+} = 0,$$

and periodicity in τ states that $W(S,t',0+) = W(S,t',1-)$, with the same condition for $\overline{W}$.

Much as in Section 3, we find that the first three terms in an expansion in powers of ϵ (including $O(\epsilon)$ in view of the boundary layer analysis to follow) have the form

$$W(S,t',\tau) \sim W_{0u}(S,t') + \epsilon W_{1u}(S,t') + \epsilon^2 W_{2u}(S,t'),$$

where the subscript u indicates that the function it decorates is, at this stage, unknown except that it satisfies the scaled BSPDE $\mathcal{L}'(\cdot) = 0$; and similarly

$$\overline{W}(S,t',\tau) \sim \overline{W}_{0u}(S,t') + \epsilon \overline{W}_{1u}(S,t') + \epsilon^2\left(\overline{W}_{2u}(S,t') - \kappa\tau\right), \tag{17}$$

in which the last term on the right is interpreted as being periodic in τ. We naturally expect, and show below, that $W_{0u}, \overline{W}_{0u} \equiv 0$; slightly less obviously, we have $W_{1u}, \overline{W}_{1u} \equiv 0$ also.

We shall look for a transition solution near $S = S_U$, periodic in τ, in which standard boundary layer scalings dictate the use of the inner variable

$$S = S_U(1 + \epsilon x),$$

and correspondingly we write $W(S,t',\tau) = w(x,t',\tau)$ (and we shall omit the argument t' when it is not ambiguous to do so), and $\overline{W}(S,t',\tau) = \overline{w}(x,\tau)$. For the matching to come, we first write the outer expansion (17) in terms of the inner variable x and expand to $O(\epsilon^2)$ so that for $x > 0$ the two-term inner expansion of the two-term outer expansion is

$$\begin{aligned} W(S_U(1+\epsilon x),t',\tau) \sim\ & \mathcal{W}_{0u} + \epsilon x S_U \mathcal{W}_{0u,S} + \frac{1}{2}\epsilon^2 x^2 S_U^2 \mathcal{W}_{0u,SS} \\ & + \epsilon \mathcal{W}_{1u} + \epsilon^2 x S_U \mathcal{W}_{1u,S} \\ & + \epsilon^2 \mathcal{W}_{2u} \\ & + O(\epsilon^3), \end{aligned}$$

and for $x < 0$ it is

$$\begin{aligned}\overline{W}(S_U(1+\epsilon x), t', \tau) \sim \overline{\mathcal{W}}_{0u} + \epsilon x S_U \overline{\mathcal{W}}_{0u,S} + \frac{1}{2}\epsilon^2 x^2 S_U^2 \overline{\mathcal{W}}_{0u,SS} \\ + \epsilon \overline{\mathcal{W}}_{1u} + \epsilon^2 x S_U \overline{\mathcal{W}}_{1u,S} \\ + \epsilon^2 \left(\overline{\mathcal{W}}_{2u} - \kappa\tau\right) \\ + O(\epsilon^3).\end{aligned}$$

Here (for example) $\mathcal{W}_{2u,S}$ means $\partial W_{2u}/\partial S\,(S_U, t')$. Expanding

$$\begin{aligned}w(x,t',\tau) &\sim w_0(x,t',\tau) + \epsilon w_1(x,t',\tau) + \epsilon^2 w_2(x,t',\tau) + \cdots\,, \\ \overline{w}(x,t',\tau) &\sim \overline{w}_0(x,t',\tau) + \epsilon \overline{w}_1(x,t',\tau) + \epsilon^2 \overline{w}_2(x,t',\tau) + \cdots\,,\end{aligned}$$

matching dictates that

$$\begin{aligned}w_0(x,t',\tau) &\sim \mathcal{W}_{0u}(t'), \\ w_1(x,t',\tau) &\sim \mathcal{W}_{1u}(t') + x S_U \mathcal{W}_{0u,S}(t'), \\ w_2(x,t',\tau) &\sim \mathcal{W}_{2u}(t') + x S_U \mathcal{W}_{1u,S}(t') + \frac{1}{2} x^2 S_U^2 \mathcal{W}_{0u,SS}(t')\end{aligned}$$

as $x \to \infty$, and

$$\begin{aligned}\overline{w}_0(x,t',\tau) &\sim \overline{\mathcal{W}}_{0u}(t'), \\ \overline{w}_1(x,t',\tau) &\sim \overline{\mathcal{W}}_{1u}(t') + x S_U \overline{\mathcal{W}}_{0u,S}(t'), \\ \overline{w}_2(x,t',\tau) &\sim \overline{\mathcal{W}}_{2u}(t') - \kappa\tau + x S_U \overline{\mathcal{W}}_{1u,S}(t') + \frac{1}{2} x^2 S_U^2 \overline{\mathcal{W}}_{0u,SS}(t')\end{aligned}$$

as $x \to -\infty$.

The PDE gives

$$\frac{1}{\epsilon^2}\frac{\partial}{\partial \tau}\begin{pmatrix} w \\ \overline{w}\end{pmatrix} = \frac{1}{2}\frac{(1+\epsilon x)^2}{\epsilon^2}\frac{\partial^2}{\partial x^2}\begin{pmatrix} w \\ \overline{w}\end{pmatrix} + \rho(1+\epsilon x)\frac{\partial}{\partial x}\begin{pmatrix} w \\ \overline{w}\end{pmatrix} - \rho\begin{pmatrix} w \\ \overline{w}\end{pmatrix}\frac{\partial}{\partial t'}\begin{pmatrix} w \\ \overline{w}\end{pmatrix} + \begin{pmatrix} 0 \\ -\kappa\end{pmatrix}$$

in their respective domains, and the final piece of the set-up is that both $w(x,t',\tau)$ and $\overline{w}(x,t',\tau)$ are periodic functions of τ, and $\partial w/\partial x = \partial \overline{w}/\partial x$ at $x = 0$.

At leading order, $w_0(x,t',\tau)$ and $\overline{w}_0(x,t',\tau)$ both satisfy the heat equation in x and τ and are asymptotic to $\mathcal{W}_{0u}(\tau)$ and $\overline{\mathcal{W}}_{0u}(\tau)$ respectively as $x \to \pm\infty$. Continuity of the functions and their x derivatives at $x = 0$ means that they are the restrictions to $-\infty < x < 0$, $0 < x < \infty$ respectively of a periodic solution of the heat equation on $-\infty < x < \infty$ with bounded asymptotic behaviour, and the only such is a constant. Hence, $\mathcal{W}_{0u}(t') = \overline{\mathcal{W}}_{0u}(t')$ and $\partial w_0/\partial x = 0$, $\partial \overline{w}_0/\partial x = 0$. The same argument can now be applied to w_1 and $\overline{w}_1$, with a minor modification to accommodate the linear asymptotic behaviour, and leads to the conclusion that they are the restrictions of an affine function of x with τ-independent coefficients. Thus, matching gives that $\mathcal{W}_{0u,S}(t') = \overline{\mathcal{W}}_{0u,S}(t')$ (from the coefficient of x) and $\mathcal{W}_{1u}(t') = \overline{\mathcal{W}}_{1u}(t')$ (from the $O(1)$ terms), and $w_1(x,t',\tau) = S_U \mathcal{W}_{0u,S}(t') x + \mathcal{W}_{1u}(t')$.

We now return to the outer region and conclude from continuity of the value functions and their derivatives at $S = S_U$ that W_{0u} and $\overline{W}_{0u}$ are the restrictions of a solution of the BSPDE in $0 < S < \infty$ with zero terminal data, which therefore vanishes. Hence, these leading-order correction terms are absent (and the first order inner solution w_1 (resp.$\overline{w}_1$) is a function of t' only). If we can show continuity of the S derivative of the $O(\epsilon)$ outer correction (we already have continuity of the correction itself), that function too must vanish.

To this end, we consider the $O(\epsilon^2)$ terms in the inner expansion. Incorporating what we already know about the expansion, we have

$$\frac{\partial}{\partial \tau}\begin{pmatrix} w_2 \\ \overline{w}_2 \end{pmatrix} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\begin{pmatrix} w_2 \\ \overline{w}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\kappa \end{pmatrix}$$

with

$$w_2 \sim xS_U\mathcal{W}_{1u,S}(t') + \mathcal{W}_{2u}(t'), \qquad \overline{w}_2 \sim xS_U\overline{\mathcal{W}}_{1u,S}(t') + \overline{\mathcal{W}}_{2u}(t') - \kappa\tau,$$

as $x \to \pm\infty$ respectively; the periodicity condition is $w_2(x,t',1-) = w_2(x,t',0+)$ and $\overline{w}_2(x,t',1-) = \overline{w}_2(x,t',0+) - \kappa$. Just as above, the terms linear in x tell us that $\mathcal{W}_{1u,S}(t') = \overline{\mathcal{W}}_{1u,S}(t')$, from which we deduce that $W_{1u}(S,t') \equiv 0$ and $\overline{W}_{1u}(S,t') \equiv 0$. Then, the functions $w_2(x,t',\tau) - \mathcal{W}_{2u}(t')$ and $\overline{w}_2(x,t',\tau) - \overline{\mathcal{W}}_{2u}(t') + \kappa\tau$ are properly periodic in τ (no jump at $\tau = 1$) and decay at $x = \pm\infty$; so they have eigenfunction expansions of the form

$$\sum_{m\neq 0} a_m^{\pm} \mathrm{e}^{2\pi \mathrm{i} m\tau} f_m^{\pm}(x),$$

where $\pm$ refers to $x \gtrless 0$ respectively. The coefficients are found from continuity of the value function and its x derivative at $x = 0$; the absence of a term corresponding to $m = 0$, necessary for decay at $x = \pm\infty$, leads to the solvability condition (which can also be derived by integrating x times the solution over the solution domain and using the divergence theorem)

$$\int_0^1 \kappa\tau - \overline{\mathcal{W}}_{2u}(t')\,\mathrm{d}\tau = \int_0^1 -\mathcal{W}_{2u}(t')\,\mathrm{d}\tau,$$

and so

$$\mathcal{W}_{2u}(t') - \overline{\mathcal{W}}_{2u}(t') = -\tfrac{1}{2}\kappa = -\tfrac{1}{2}c/\sigma^2.$$

This jump condition at $S = S_U$ for the $O(\epsilon^2)$ correction is supplemented with continuity of its S derivative there, which follows from consideration of the terms linear in x at $O(\epsilon^3)$ in the inner expansion, a calculation (omitted here) very similar to those above.

It remains to find the outer correction; here we revert to the original, unscaled time variable t. Apart from the prefactor ϵ^2 $(= \sigma^2\,\delta t)$ which cancels some denominators σ^2 below, this has the form of a solution of the BSPDE, $W_{2u}(S,t)$, for $S > S_U$, and a solution of the BSPDE, $\overline{W}_{2u}(S,t)$ plus a sawtooth function for $S < S_U$. At $S = S_U$, $W_{2u}(S_U+,t') - \overline{W}_{2u}(S_U-,t') = -\frac{1}{2}c/\sigma^2$ (the jump is of course smoothed off on the inner scale) and $\partial W_{2u}/\partial S(S_U+,t') = \partial\overline{W}_{2u}/\partial(S_U-,t')$. Both solutions of the BSPDE decay at $S = 0$ and $S = \infty$ respectively, and both vanish at expiry $t = T$.

The first step is to subtract a suitable static solution, here the function

$$W_{2u0}(S) = \begin{cases} A_- S/S_U, & S < S_U, \\ A_+ (S/S_u)^{-2r/\sigma^2} & S > S_U, \end{cases}$$
$$= A_- S/S_U \mathbb{I}_{S<S_U} + A_+ (S/S_u)^{-2r/\sigma^2} \mathbb{I}_{S>S_U},$$

where
$$A_- = \frac{rc}{\sigma^2(\sigma^2+2r)}, \qquad A_+ = -\frac{c}{2(\sigma^2+2r)},$$

which removes the discontinuity at the expense of introducing a nonzero payoff. Then, we price this payoff, which is equivalent to

- A short position in A_- gap options with payoff $(S/S_U)\mathbb{I}_{(0,S_U)}$;
- a long position in A_+ power gap options with payoff $(S/S_U)^{-2r/\sigma^2}\mathbb{I}_{S>S_U}$. This payoff can also be written
$$\left(\frac{S}{S_U}\right)^{1-2r/\sigma^2} \frac{1}{(S/S_U)} \mathbb{I}_{S_U^2/S<S_U}$$
which reveals it as the reflection (in the sense of the method of images applied to the BSPDE) of the payoff of the first gap option. Hence the value of this power gap option before expiry is also the reflection of the value of the first power gap option.

The first gap option has time-t value

$$V^{\mathrm{gap}}(S,t) = \frac{S}{S_U} - \frac{1}{S_U} C^{\mathrm{v}}(S,t;S_U) - C^{\mathrm{b}}(S,t;S_U),$$

where $C^{\mathrm{v/b}}(S,t;S_U)$ is the value of a vanilla/binary call with strike S_U. Hence the value of the power gap option is

$$V^{\mathrm{Pgap}}(S,t) = \left(\frac{S}{S_U}\right)^{1-2r/\sigma^2} V^{\mathrm{gap}}(S_U/S,t),$$

and the value of the whole second-order outer correction to the range accrual note is

$$A_- \left((S/S_U)\mathbb{I}_{S<S_U} - V^{\mathrm{gap}}(S,t)\right) + A_+(S/S_u)^{1-2r/\sigma^2}\left(\frac{S_u}{S}\mathbb{I}_{S>S_U} - V^{\mathrm{gap}}(S_U/S,t)\right).$$