Rapid inviscid flows

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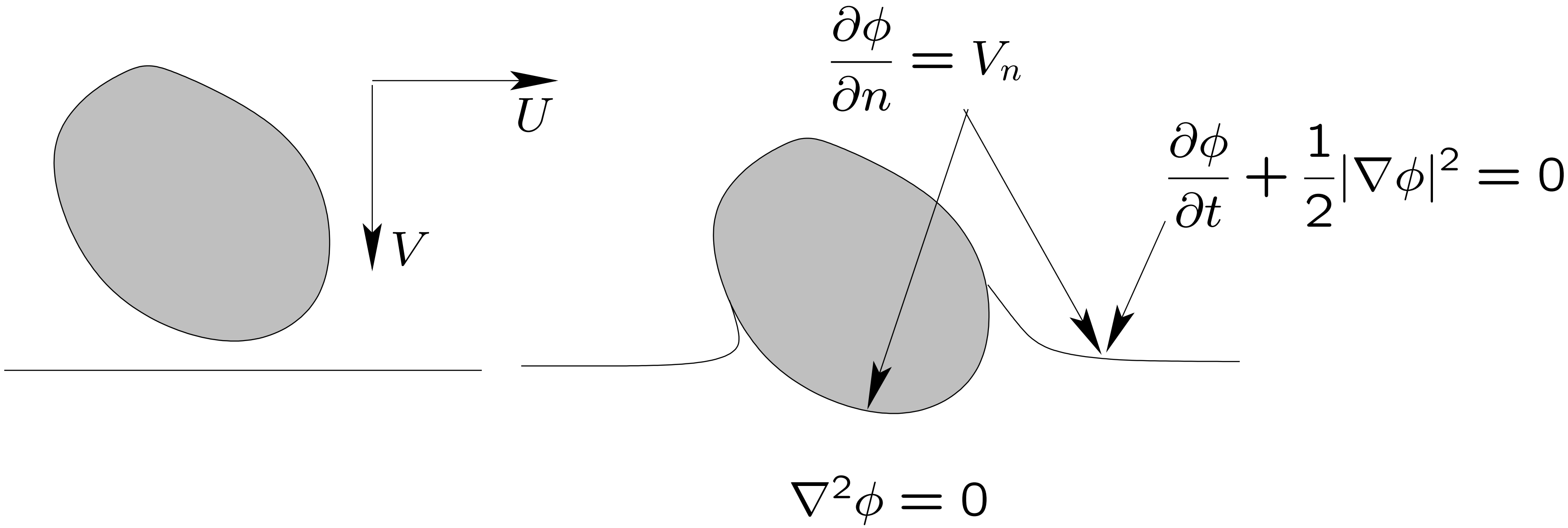
Generic impact problems



Simplest possible inviscid free surface model.

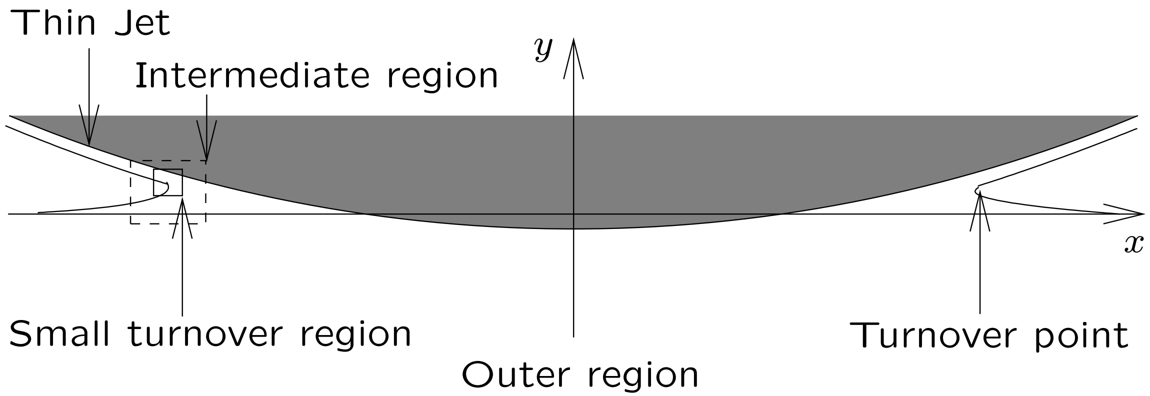
Basiliscus Basiliscus: the Jesus lizard


17
Sherwin Williams
Powder
Chemetall
Oakite
30

Classical Wagner theory for blunt (nearly flat) bodies

This deals with the rapid impact of a blunt body with small 'deadrise angle' on a half-space of liquid. Gravity, surface tension, viscosity and compressibility are all neglected. When the deadrise angle is small an asymptotic approach is possible: the flow is decomposed into regions which are linked by the technique of matched asymptotic expansions.

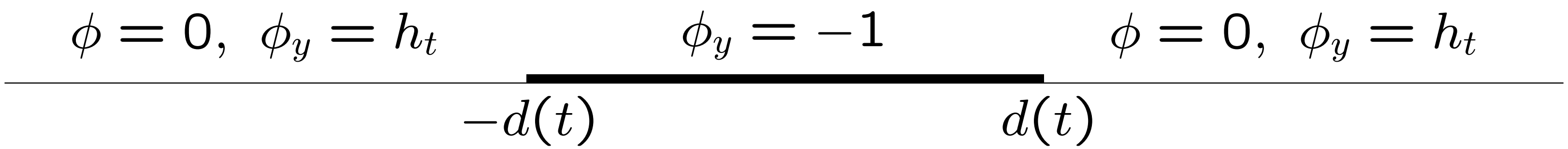
Original application was to seaplane floats (Wagner 1932).



The classic example is that of impact of a wedge. This is the only problem for which rigorous theory exists, and it confirms the asymptotic results.

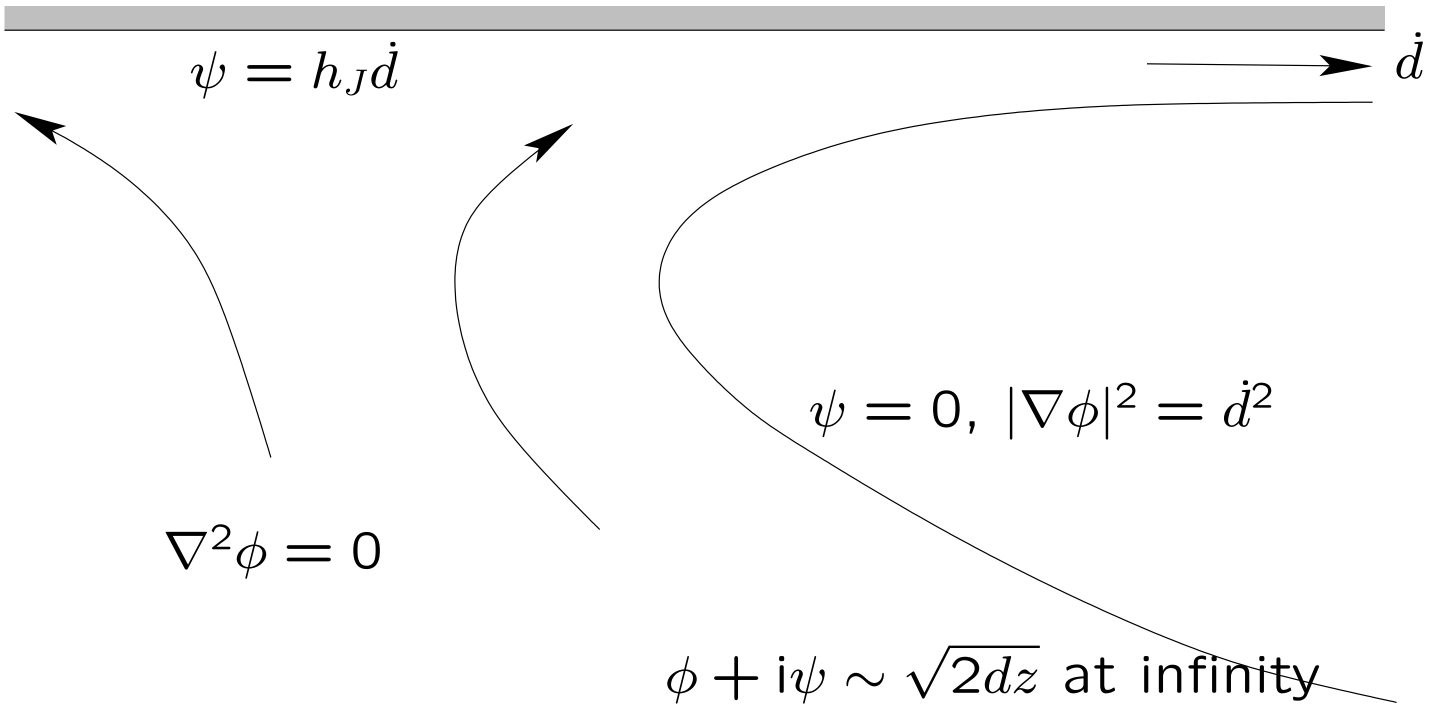
The outer region sees the impact of an 'effective flat plate' extending between the turnover points. This is a linearised problem in which the kinematic and dynamic conditions are applied in linearised form on the undisturbed water level. The points corresponding to turnover are unknown.

In dimensionless variables:



$$\phi_{xx} + \phi_{yy} = 0$$

The inner (turnover region) is a standard Kelvin-Helmholtz flow. The pressure scale is $O(\rho V^2/\epsilon^2)$ where V is the impact speed (but the force on the body is dominated by the pressure in the outer region).

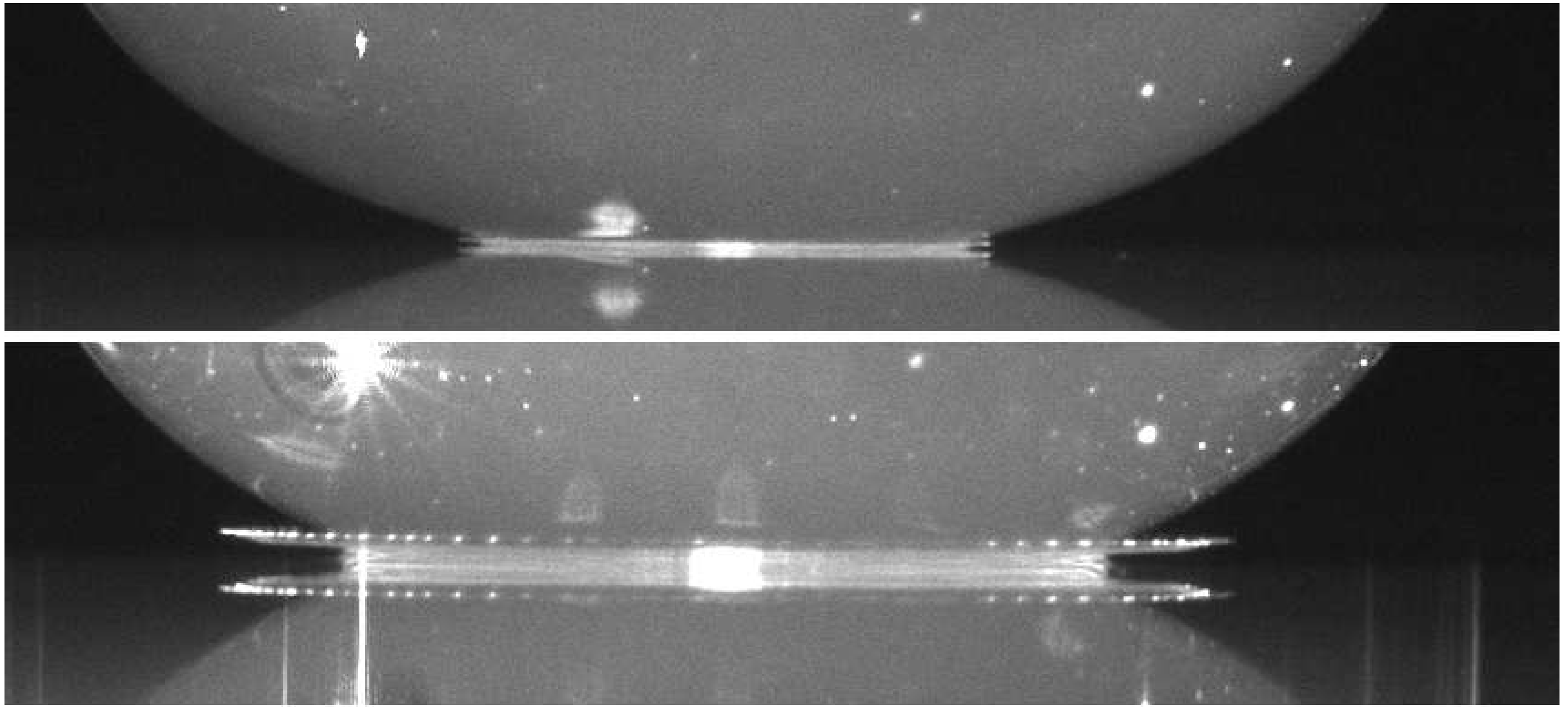


The two are joined up by asymptotic matching, leading to the *Wagner condition*, that the free surface comes up to meet the impacting body at the turnover points. This is because the size of the turnover region is asymptotically small compared with the surface displacement. This leads to an integral equation for $d(t)$

$$f(d(t)) - t = \int_0^t \frac{d\tau}{\sqrt{d^2(t) - d^2(\tau)}}$$

from which the 'law of motion' of the free points can be found by the substitution $d(\tau) = \xi$, $d(t) = x$.

Wagner flows can be extended easily to symmetric impact of liquid bodies (the line of symmetry is like a rigid boundary): see eg experiments by Thorodssen (JFM 2002).

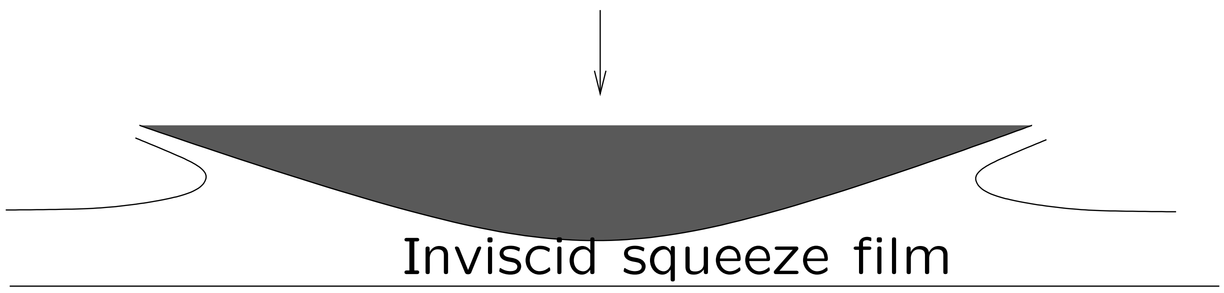


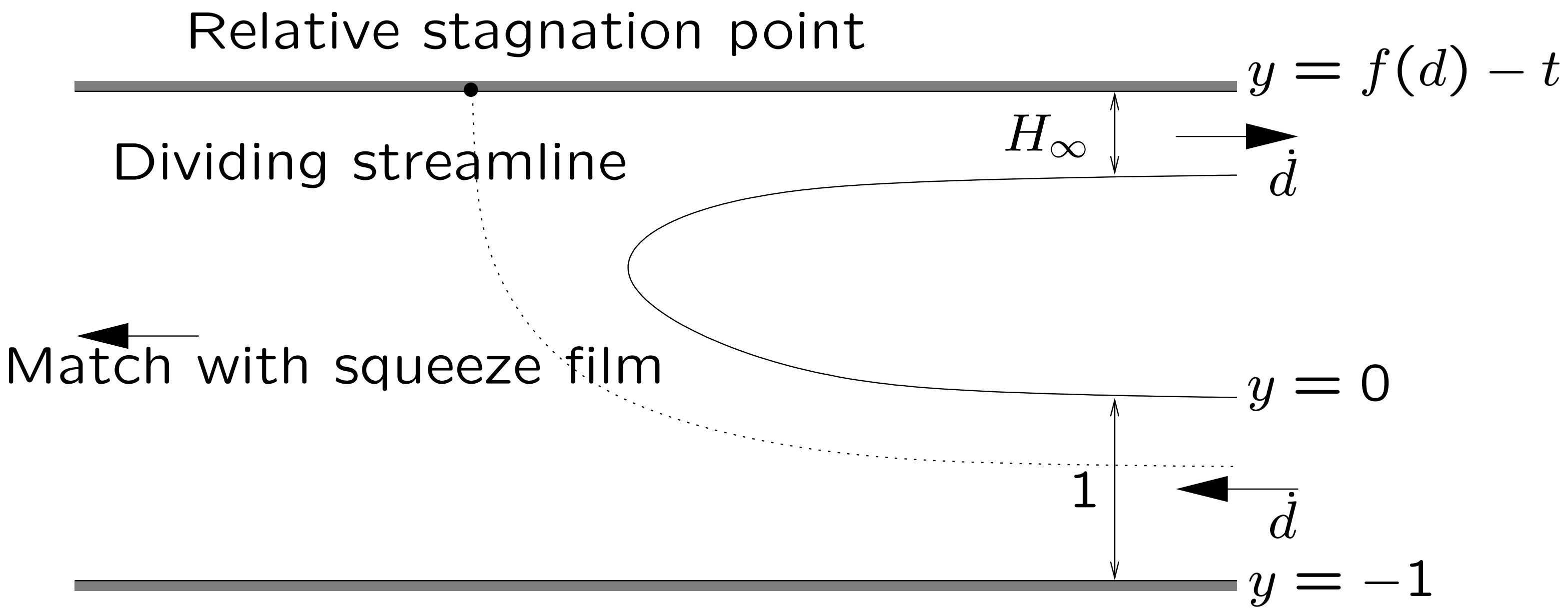
Korobkin theory

This deals with the impact of a blunt body $y = f(x) - t$ on a thin liquid layer above a substrate $y = -1$. The turnover regions are now of the same size as the layer depth and the flow under the body is approximated by an ‘inviscid squeeze film in which the velocity is approximately $u(x,t)\mathbf{i}$ and

$$\frac{\partial}{\partial t}\left(f(x) - t + 1\right) + \frac{\partial}{\partial x}\left(u(x,t)\left(f(x) - t + 1\right)\right) = 0.$$

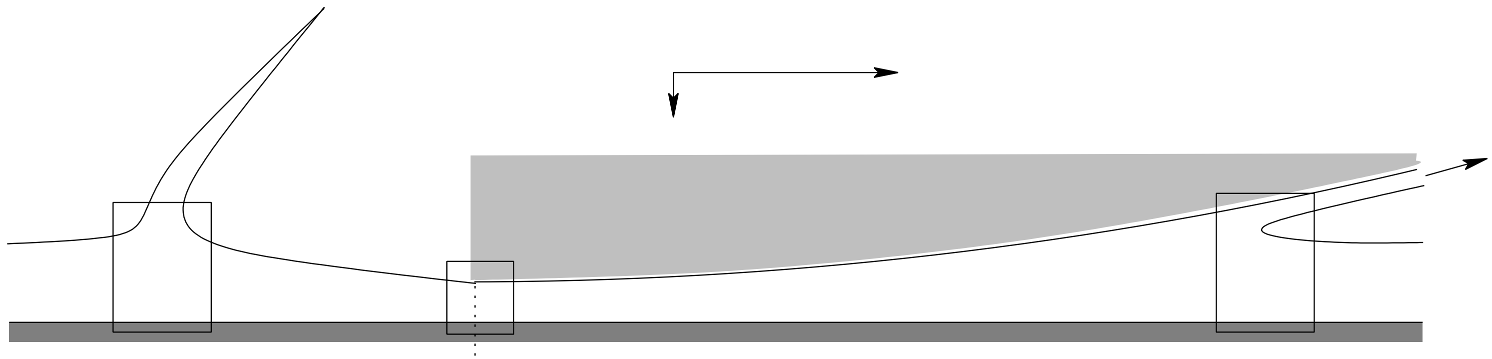
Korobkin theory can be obtained as a limit of Wagner theory with a base.


Inviscid squeeze film


Relative stagnation point
$y = f(d) - t$
H_∞
Dividing streamline
$\dot{d}$
Match with squeeze film
$y = 0$
1
$\dot{d}$
$y = -1$

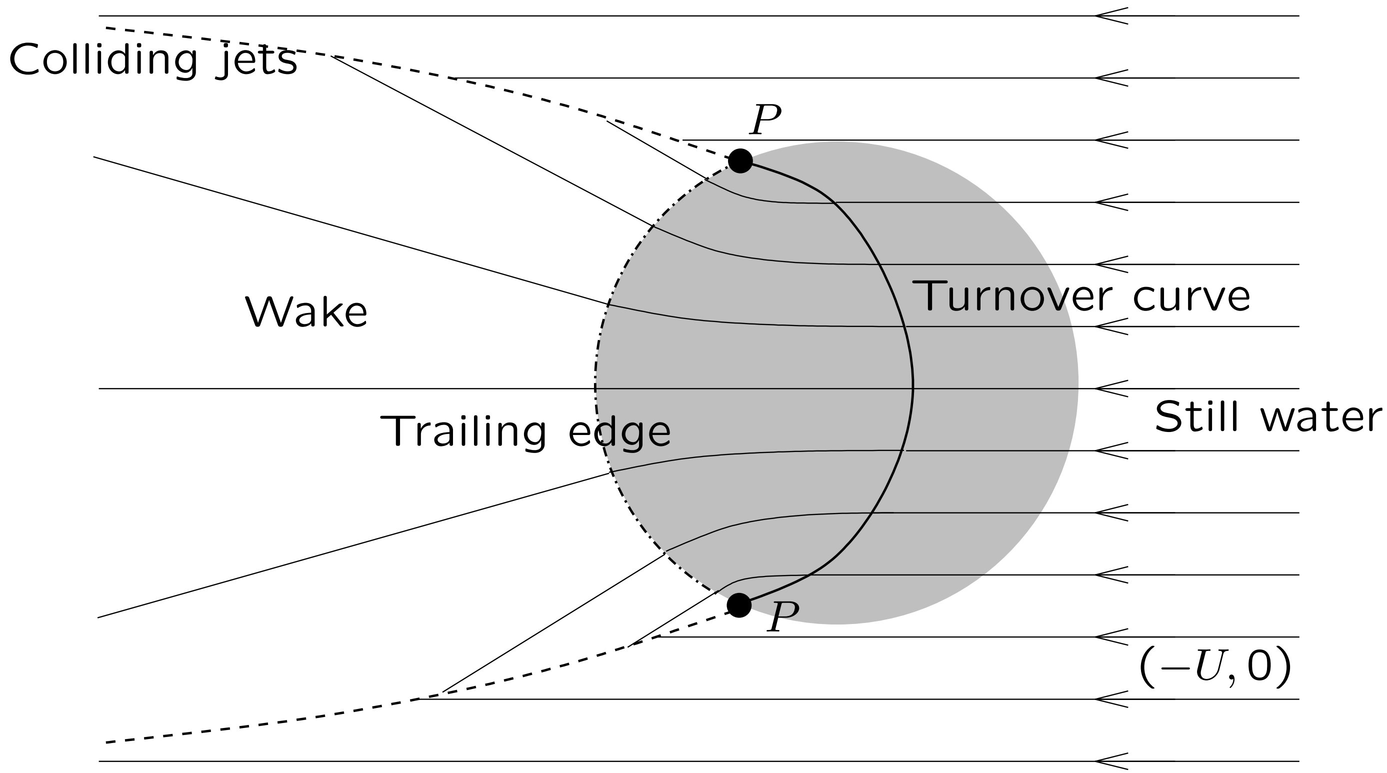
Oblique impact

Oblique impact, with a horizontal impactor velocity as well as a vertical one, can also be handled in Wagner and Korobkin theory. If the horizontal velocity is similar to the normal velocity, its effect is $O(\epsilon)$ and the splash is nearly symmetrical. If it is large the theory is complicated but works well until we have effective exit at the trailing edge. Then it breaks down and we only consider sharp separation from the trailing edge.



The load such an impactor can support is exploited by surf skimmers:

Ernie Tuck has a 1-D model; here is the 2-D generalisation:



The model is $|\nabla\phi|^2 = U^2$ (the eikonal equation) in the wake and still water; a squeeze film under the board; plus a turnover condition. The water in the splash jet is ignored.

High Froude number shallow flows on substrates

On a horizontal base the shallow water equations for flow with typical speed U_0 and depth h_0 are

$$h_t + (hu)_x = 0, \qquad u_t + (\tfrac{1}{2}u^2)_x = -\frac{h_x}{F^2},$$

where $F^2 = U_0^2/gh_0$ is the Froude number. What happens as $F \to \infty$, as the system becomes degenerate?

The standard approach is to derive Rankine–Hugoniot conditions for a weak solution in which u and h have jump discontinuities at a shock. Using distributions, we put jumps in the mass (h), momentum (uh) and energy ($\frac{1}{2}u^2h$) densities:

$$h(x,t) = h_l(x,t) + (h_r(x,t) - h_l(x,t))\,\mathcal{H}\,(x - x_s(t))$$

and similarly for uh and $\frac{1}{2}u^2h$:

$$uh = u_l h_l + \Big[uh\Big]_l^r \mathcal{H}\,(x - x_s(t))\,, \quad \frac{1}{2}u^2h = \frac{1}{2}u_l^2 h_l + \Big[\frac{1}{2}u^2h\Big]_l^r \mathcal{H}\,(x - x_s(t))\,;$$

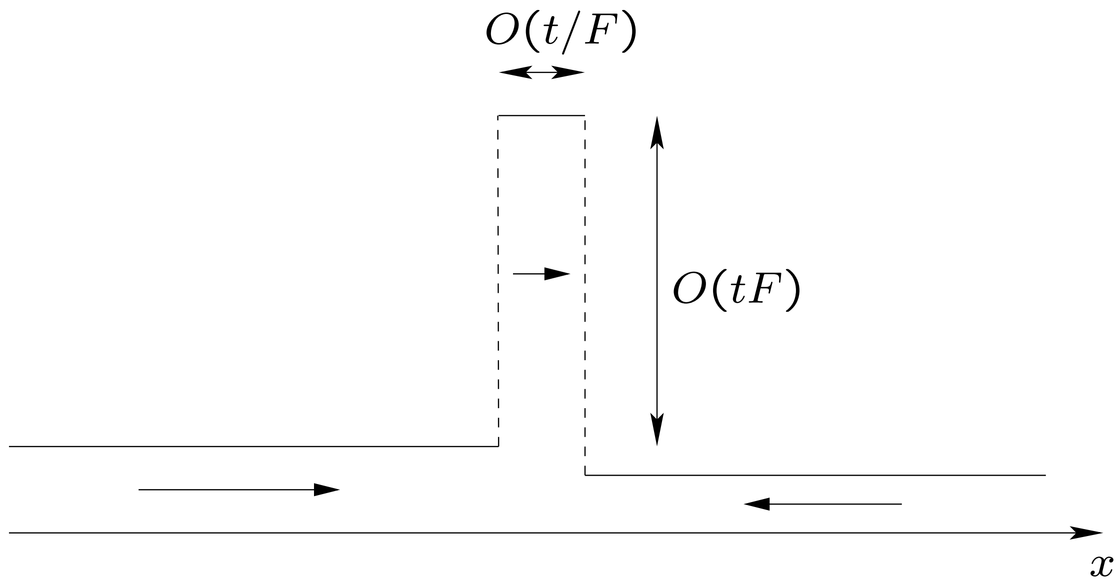
here $\mathcal{H}(\cdot)$ is the Heaviside function. Then substitution into the shallow water equations gives

$$\Big[h\Big]_l^r \delta(x - x_s)\frac{\mathrm{d}x_s}{\mathrm{d}t} + \Big[uh\Big]_l^r \delta(x - x_s) + \text{smoother terms} = 0,$$

so equating coefficients of $\delta(x - x_s)$ gives the first R–H relation, similarly for the other.

Shallow water model: collision of two horizontal jets

The collision of two flat jets (Riemann problem) has a similarity solution with two shocks moving slowly [speed $O(1/F)$] bounding a deep column [height $O(tF)$] of width $O(t/F)$. Rankine–Hugoniot gives shock speeds.



We can model this as a *delta shock* (cf Keyfitz) in which we allow delta functions in the mass (h), momentum (uh) and energy ($\frac{1}{2}u^2h$) densities, as well as the jump discontinuities:

$$h(x,t) = h_l(x,t) + \Big[h(x,t)\Big]_l^r \mathcal{H}\left(x - x_s(t)\right) + h_\delta(t)\delta\left(x - x_s(t)\right)$$

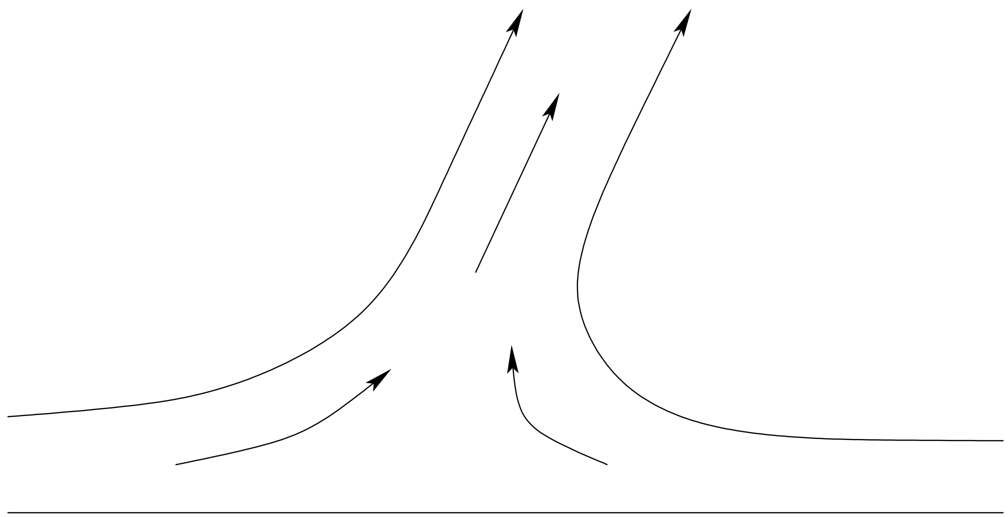
and similarly for uh. $\frac{1}{2}u^2h$.

So $h_\delta(t)$ is the mass absorbed by the delta shock, and equating coefficients of the singular terms

$$\delta'\left(x - x_s(t)\right), \quad \delta(x - x_s(t))$$

gives o.d.e.s for h_δ and the corresponding momentum and energy, to replace R–H.

This isn't the whole picture though because in the early stages of such a collision we could model the flow by a steady 2-D flow leaving the horizontal plane. at the root of a jet (which may translate quasisteadily).



Mass and momentum sinks

In the jet root model above, local conservation of mass and momentum give the jet angle and the speed of its root:

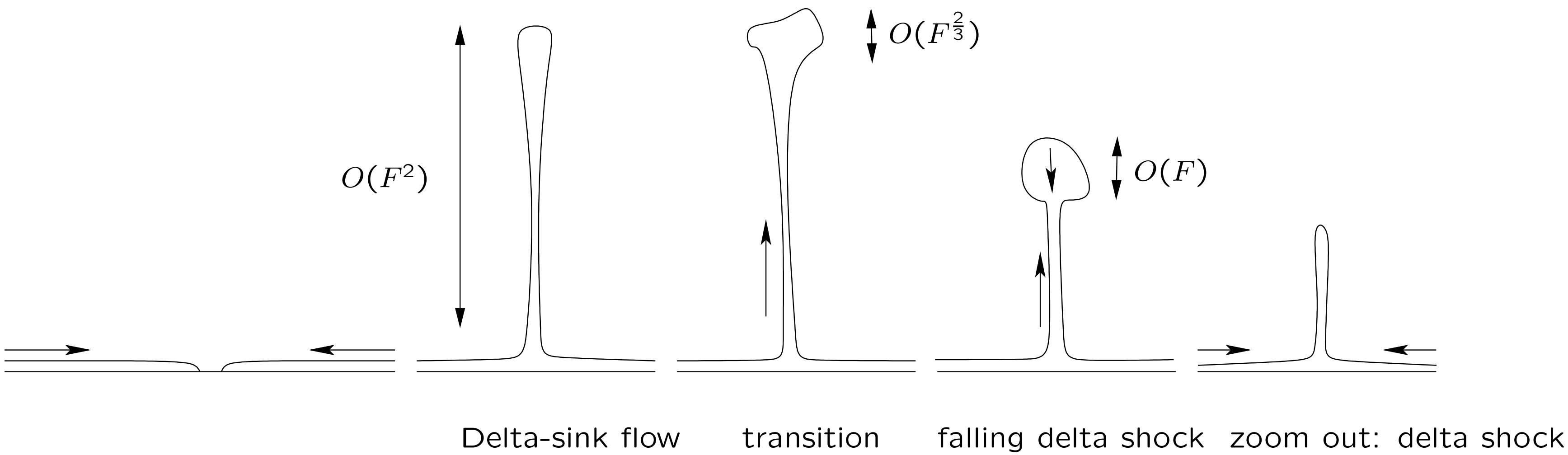
$$\cos\beta = \frac{h^- - h^+}{h^- + h^+}, \quad U_J = \frac{u_- - u_+}{2}.$$

Looking from further away, we can put mass and momentum delta-sink terms in the shallow water model to give

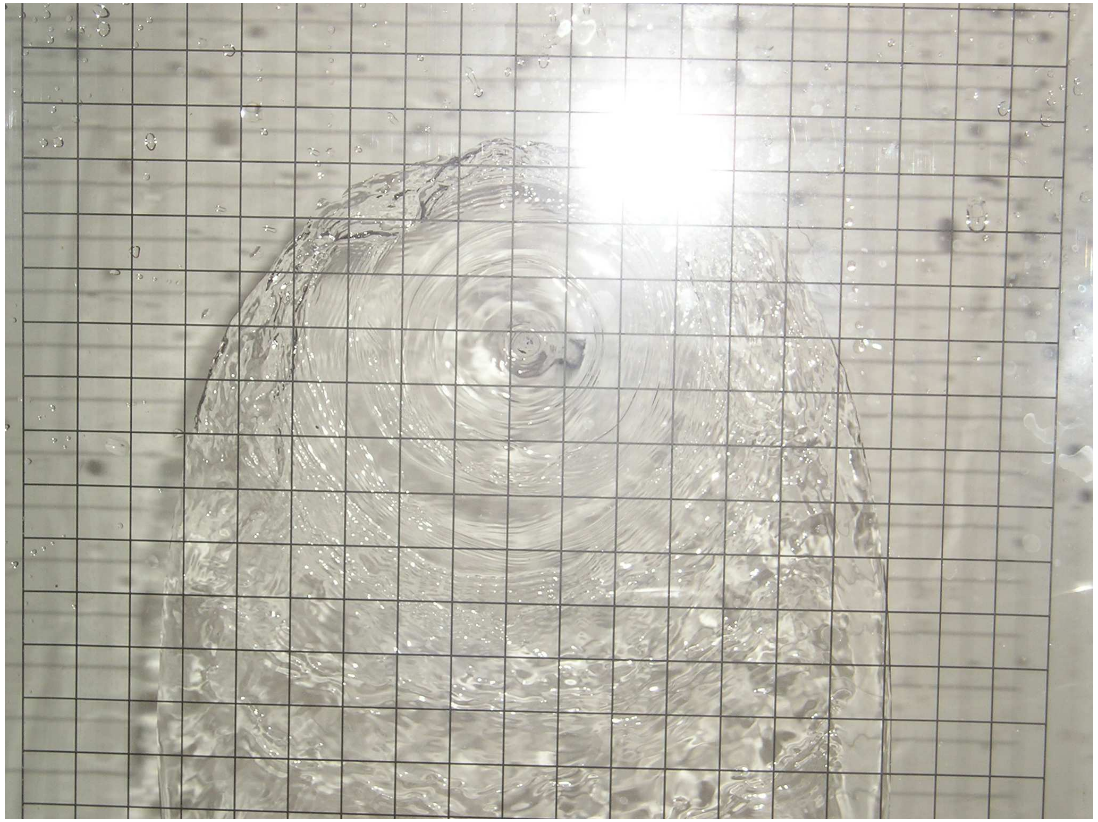
$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = -h_{\mathsf{loss}}\delta(x - x_J(t)),$$
$$\frac{\partial uh}{\partial t} + \frac{\partial (\frac{1}{2}u^2 h)}{\partial x} = -M_{\mathsf{loss}}\delta(x - x_J(t)),$$

and with the local closure conditions above we find $x_J(t)$.

Birth of a delta shock



Steady 2-d flows on a sloping base



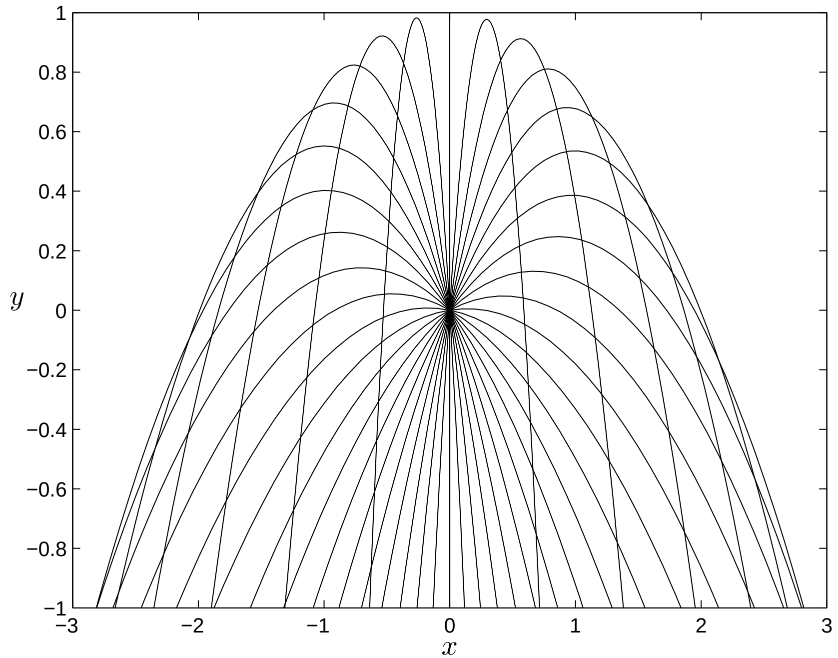
Can now scale lengths so $F^2 = 1$. Simple model for steady flow: in-plane velocity $\mathbf{u} = (u, v)$ with v upwards:

$$\nabla \cdot (h\mathbf{u}) = 0, \quad \nabla \cdot (hu\mathbf{u}) = 0 \quad \nabla \cdot (hv\mathbf{u}) = -h,$$

equivalent to

$$\frac{1}{2}|\nabla\phi|^2 + y = 0 \quad \text{(Bernoulli)}, \qquad \nabla \cdot (h\nabla\phi) = 0$$

Solve by Charpit.

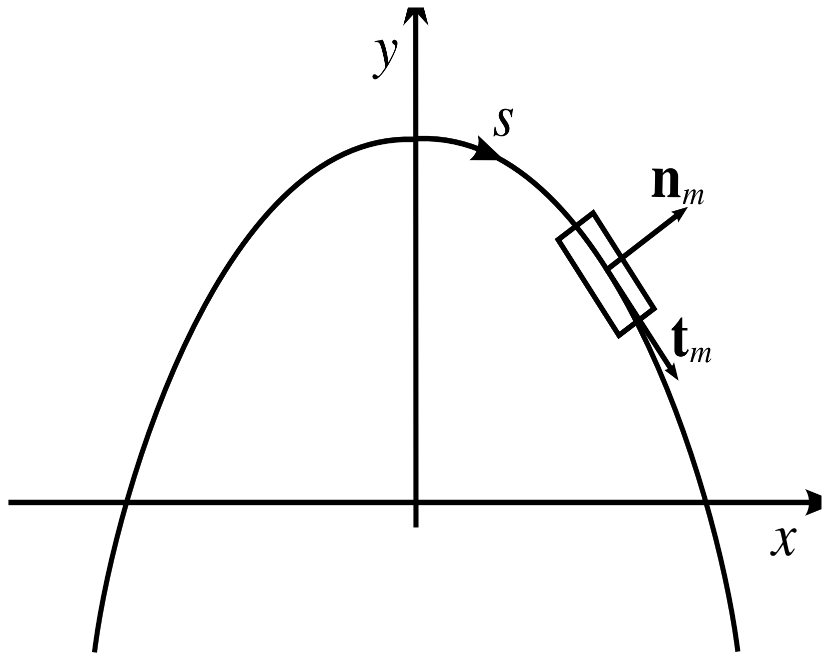


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
y
−3
−2
−1
0
1
2
3
x

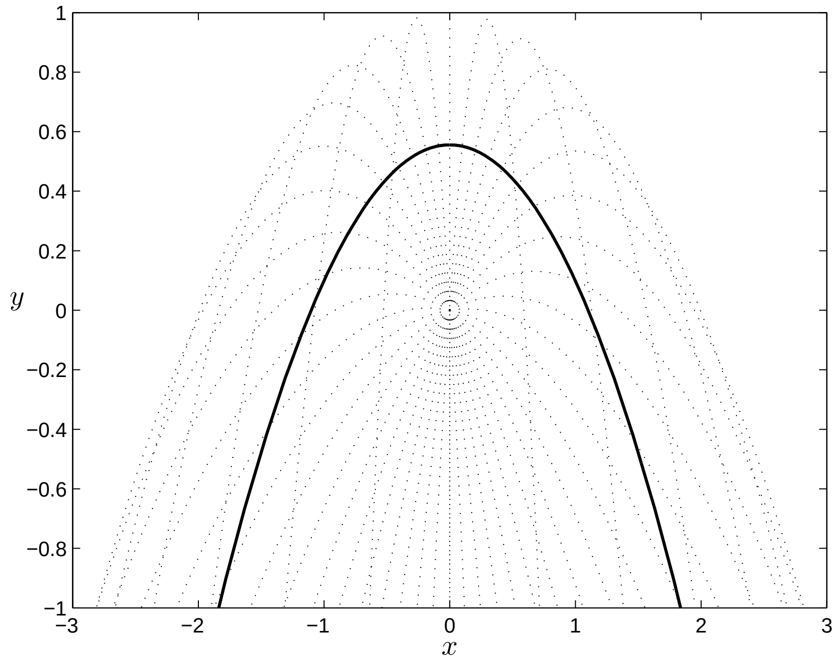
Resolution of caustic is to introduce a ‘mass tube’ (cf GI Taylor, Yarin/Weiss), i.e a delta shock in the mass and momentum fluxes. Eg

$$h\mathbf{u} = h\mathbf{u}_{\mathsf{sheet}} + Q_{\mathsf{m}}\delta_{\mathsf{m}}\mathbf{t}_{\mathsf{m}}.$$

Then the delta shock conditions conserve mass and momentum along and normal to the tube (with fluxes in from the thin sheet).



y
s
n_m
t_m
x


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
y
−3
−2
−1
0
1
2
3
x

Open questions: eg impact with air layers. See experiments of Xu & Nagel 2004.

Above: in vacuum, below: in air.