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Virtual classes for -2-shifted symplectic derived schemes

1. Differential-geometric motivation

(Borisov–Joyce, Cao–Leung)

Let (X, J, g, Ω) be a compact Calabi–Yau 4-fold and $\mathcal{M}^{ss}_\alpha(\tau)$ a moduli space of Gieseker stable algebraic vector bundles on X in Chern character α, supposed proper. We would like to form a virtual class $[\mathcal{M}^{ss}_\alpha(\tau)]_{virt}$ for $\mathcal{M}^{ss}_\alpha(\tau)$.

The natural obstruction theory on $\mathcal{M}^{ss}_\alpha(\tau)$ is perfect in $[-2, 0]$, with cohomology $\mathrm{Ext}^{1-i}_0(E,E)^*$ at $E \in \mathcal{M}^{ss}_\alpha(\tau)$ for $i = 0, -1, -2$.

The usual Behrend–Fantechi construction of virtual classes does not apply, it needs perfect in $[-1, 0]$.

By the Hitchin–Kobayashi correspondence, each vector bundle E in $\mathcal{M}^{ll}_\alpha(c)$ has a Hermitian–Einstein $U(r)$-connection ∇, $r = \operatorname{rank} E$, whose curvature satisfies, $F_\nabla^{0,2} = 0$ and $F_\nabla^{1,1} \wedge \omega^3 = \lambda \, \mathrm{id}_E \, \omega^4$ $\lambda \in i\mathbb{R}$ fixed.

As $\Lambda^{0,2} X \cong \mathbb{C}^6$, this is $(12+1) r^2$ equations on $8r^2$ functions in the connection. Dividing by gauge subtracts another r^2 functions. As $8r^2 - r^2 - 13r^2 < 0$, the problem is overdetermined elliptic. So we don't expect a good moduli space theory.

There is a second way to do gauge theory on Calabi–Yau 4-folds. By the inclusion $SU(4) \hookrightarrow Spin(7)$, we regard X as a $Spin(7)$ manifold. This induces a splitting $\Lambda^2 T^*X = \Lambda^2_7 \oplus \Lambda^2_{21}$. A Spin(7) instanton is a connection ∇ on a complex vector bundle

$E \to X$ with $\pi^2_7(F_\nabla) = 0$ (or $\pi^2_7(F_\nabla) = \lambda \, \mathrm{id}_E \, \omega$)

As $8r^2 - 7r^2 - r^2 = 0$, this is <u>determined</u> elliptic. So on general grounds, we expect moduli spaces of Spin(7) instantons to be well-behaved — to be smooth manifolds in a generic case — and to have virtual classes, if they are compact and oriented.

On the CY 4-fold X, there is a <u>real</u> vector bundle splitting $\underbrace{(\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}}}_{\mathbb{R}^{12}}$

$$= \underbrace{\Lambda^+}_{\mathbb{R}^6} \oplus \underbrace{\Lambda^-}_{\mathbb{R}^6}, \quad \text{and} \quad \Lambda^2_7 = \langle \omega \rangle \oplus \Lambda^+.$$

Thus the Spin(7) instanton equations are

$$\pi^+(F_\nabla^{0,2}) = 0, \quad F_\nabla^{1,1} \wedge \omega^3 = \lambda \, \mathrm{id}_E \, \omega^4.$$

That is, we require only a <u>real half</u> of the holomorphic curvature $F_\nabla^{0,2}$ to vanish. The Spin(7) instanton equations are <u>weaker</u> than the HE equations.

Now a miracle occurs: there is an identity writing $\|\pi_+(F_D^{0,2})\|_{L^2}^2 - \|\pi_-(F_D^{0,2})\|_{L^2}^2$ in terms of $ch\,E$ and $[\mathrm{Re}\,\Omega] \in H^4(X,\mathbb{R})$.

A consequence is that if E has the Chern character of a holomorphic vector bundle, then

$$\pi_+(F_D^{0,2}) = 0 \Rightarrow \pi_-(F_D^{0,2}) = 0$$
$$\Rightarrow F_D^{0,2} = 0.$$

That is, the Spin(7) equations imply the Hermitian-Einstein equations.

This is only true pointwise.

In (C^∞-) schemes we have

$$\mathcal{M}_{HE} \subset \mathcal{M}_{Spin(7)},$$

and they agree as reduced schemes, but not as non-reduced schemes.

<u>Moral:</u> we should try to define virtual classes for CY 4-fold moduli spaces, by regarding them as moduli spaces of Spin(7) instantons.

PTW: M is a -2-shifted symplectic derived scheme.

BBJ Darboux Theorem: The local model for

M is:
$$\begin{array}{c} E, Q \\ \downarrow \uparrow s \\ V \end{array}$$

V: smooth scheme.
E vector bundle.
Q nondegenerate quadratic form on E
s section with $Q(s,s)=0$.

Then $M \sim s^{-1}(0)$

$$\mathbb{L}_M = \Big[\underset{-2}{TV|_{s^{-1}(0)}} \xrightarrow{ds} \underset{-1}{E|_{s^{-1}(0)} \overset{Q}{\simeq} E^*|_{s^{-1}(0)}} \xrightarrow{(ds)^*} \underset{0}{T^*V|_{s^{-1}(0)}} \Big)$$

Differentiating $Q(s,s)=0$ twice shows this is a complex.

$$\mathrm{vdim}_{\mathbb{C}} M = 2\dim_{\mathbb{C}} V - \mathrm{rank}_{\mathbb{C}} E.$$

Main idea of Borisov-Joyce:

Choose a real splitting $E \cong E^+ \oplus E^-$

$E^- = iE^+$ Q positive definite on E^+, neg def on E^-, defines norms on $E_\pm$

$s = s_+ \oplus s_-$

$\operatorname{Re} Q(s,s) = |s_+|^2 - |s_-|^2 = 0.$

Thus $|s_+| = |s_-|$, so $s_+ = 0$

implies $s_- = 0$ set theoretically.

In (C^∞-) schemes we have

$\{s = 0\} \subset \{s_+ = 0\}$,

$\{s_+ = s_- = 0\}$ agree as reduced schemes, but not as non reduced schemes.

Borisov–Joyce 2015 build a <u>real derived manifold</u> M^{C^∞} with the same underlying topological space as M. The local model for M^{C^∞} is $s_+^{-1}(0)$ in

$$\begin{array}{c} E_+ \\ \downarrow \uparrow s_+ \\ V \end{array}.$$

The real virtual dimension is $2\dim_{\mathbb{C}} V - \operatorname{rk}_{\mathbb{C}} E$.

Note this is <u>half</u> the real virtual dimension of M as a derived $\mathbb{C}$-scheme.

If M^{C^∞} is compact and <u>oriented</u>, it has a virtual class $[M^{(\infty)}]_{\mathrm{virt}} \in H_*(M, \mathbb{Z})$.

<u>Interpretation</u>: $\mathcal{M} \longrightarrow M^{C\infty}$

$\uparrow$ −2 shifted symplectic derived $\mathbb{C}$-scheme

$\uparrow$ real derived manifold

is a <u>−2-Lagrangian fibration</u>. The Lagrangian fibers are points with a nontrivial derived structure. We have $\mathrm{vdim}_{\mathbb{R}} M^{C\infty} = \frac{1}{2} \mathrm{vdim}_{\mathbb{R}} \mathcal{M}$ as the dimension of the base of a Lagrangian fibration of a symplectic manifold is half the dimension of the total space.

<u>How to define orientations</u>: by PTVV

$$\mathbb{T}_M \simeq \mathbb{L}_M(2).$$

So

$$\det \mathbb{T}_M \simeq \det \mathbb{L}_M, \quad \text{and}$$

$$\det \mathbb{T}_M = (\det \mathbb{L}_M)^* \qquad (\det \mathbb{L}_M)^{\otimes 2} \xrightarrow[\Phi]{\simeq} \mathcal{O}_M.$$

An <u>orientation</u> on M is an isomorphism $\omega : \det \mathbb{L}_M \longrightarrow \mathcal{O}_M$ with $\omega^{\otimes 2} = \Phi$.

Need to find an orientation on M to define a virtual class.

Oh–Thomas 2020: Give an algebraic geometry construction of the B–J virtual class in Chow homology. Note that the B–J virtual class may potentially have odd real dimension, which would not make sense in Chow homology.

OT show that if $\mathrm{vdim}_{\mathbb{C}} M$ is odd then the B–J virtual class is 2-torsion, that is, it is zero in $H_{odd}(M, \mathbb{Z}[\frac{1}{2}])$.

So suppose $\mathrm{vdim}_{\mathbb{C}} M$ is even, and M is oriented. OT construct a virtual class in Chow homology

$$A_{\frac{1}{2}\mathrm{vdim}_{\mathbb{C}} M}(M)[\tfrac{1}{2}],$$

whose image in $H_*(M, \mathbb{Z}[\frac{1}{2}])$ agrees with the virtual class.

Given the usual local model $\begin{array}{c}(E, Q)\\ \downarrow s\\ V\end{array}$, where now rank E is even as $\dim_{\mathbb{C}} M$ is even, Oh–Thomas locally choose a splitting

$$0 \to \Lambda \to E \to \Lambda^* \to 0,$$

where Λ is a maximal isotropic subbundle of (E, Q).

OT show that $\pm c_{top}(\Lambda)$ can be localized, using intersection theory and cosection localization, to $s^{-1}(0)$, giving a class in $A_{\frac{1}{2}\mathrm{vdim} M}(s^{-1}(0))$. They can make this independent of Λ by working in $A_*(s^{-1}(0))[\frac{1}{2}]$. When M is projective, these local models glue to give a class in $A_*(M)[\frac{1}{2}]$.

Oh–Thomas also define a K-theoretic analogue of the CY4 virtual class, in the K theory of complexes of coherent sheaves on M. To see the point of this, consider the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a nice (locally finitely presented) derived scheme $\mathcal{M}$. (Étale) locally, can write $\mathcal{M}$ as $\operatorname{Spec} A^\bullet = \operatorname{Spec}(\mathbb{C}(y_i^j), d)$ for a graded polynomial algebra $\mathbb{C}[y_i^j]$ on finitely many variables y_i^j, $i=1,\dots,d_j$, $j=0,-1,-2,\dots$ with $\deg y_i^j = j$, with a differential $d: \mathbb{C}[y_i^j] \to \mathbb{C}[y_i^j]$ of degree 1. The cotangent complex of $\mathcal{M}$ is $A^\bullet[dy_i^j, d_{ij}]$.

Then $\mathcal{M}$ is <u>quasi-smooth</u> if $d_j = 0$ for $j < -1$.

If $\mathcal{M}$ is quasi-smooth then $A^\bullet$ is of <u>finite rank over A^0</u>, as take the exterior algebra on the variables y_i^{-1}. But if $A^\bullet$

has variables y_j^{-2} then get terms in all negative degrees $-2n$, and A is of infinite rank over A^0.

Thus, if $\mathcal{M}$ is quasi-smooth then the derived structure sheaf $\mathcal{O}_{\mathcal{M}}$ is a perfect complex on the classical scheme $M = t_0(\mathcal{M})$, and can be regarded as a <u>K-theoretic virtual class</u> $[\mathcal{O}_{\mathcal{M}}] \in K(\mathrm{Perf}_M)$.

This is a kind of enumerative invariant. For example, if $\mathcal{M}$ is proper, pushing forward along $\pi: \mathcal{M} \longrightarrow *$ gives the virtual holomorphic Euler characteristic $\chi_{hol}(\mathcal{M})$.

If $\mathcal{M}$ is an oriented -2 shifted symplectic derived $\mathbb{C}$-scheme of even virtual dimension, Oh–Thomas define a "twisted virtual structure sheaf" $[\hat{\mathcal{O}}_{\mathcal{M}}]_{vir} \in K(\mathrm{Perf}_M)[\tfrac{1}{2}]$.

The local model, if $\mathcal{M}$ is locally modelled on

$$\begin{tikzcd} (E, Q) \arrow[d, "\pi"] \\ V \end{tikzcd}$$

with

$$0 \to \Lambda \to E \overset{Q}{\cong} E^* \to \Lambda^* \to 0$$

for Λ an isotropic bundle, is

$$[\Lambda^* \Lambda \otimes (\det \Lambda)^{-1/2}] \in K(\mathrm{Perf}_{s^{-1}(0)}).$$

Again, we make $\mathcal{M}$ behave like a quasi-smooth derived scheme of half the virtual dimension.

Oh–Thomas' results have been improved and extended in several ways, notably by Hyeonjin Park.

Oh–Thomas originally only proved their results for a projective moduli scheme of coherent sheaves on a CY 4-fold.

Kiem – Park arXiv: 2012.13167 improve this to −2-shifted symplectic Deligne – Mumford stack, $\mathcal{M}$.

Kiem – Park also show that you can define a virtual class from the following data:

* $\mathcal{M}$ classical $\mathbb{C}$-scheme or Deligne-Mumford $\mathbb{C}$-stack,
* $\phi: \mathcal{E}^\bullet \longrightarrow \mathbb{L}_{\mathcal{M}}$,

  $\omega: (\mathcal{E}^\bullet)^\vee \xrightarrow{\sim} \mathcal{E}^\bullet(2)$ a symmetric obstruction theory, perfect in $[-2,0]$,
* an orientation of $\mathcal{E}^\bullet, \omega$,

such that $\phi, \mathcal{E}^\bullet, \omega$ satisfy an "isotropic cone condition".

– This holds if the data is defined by truncation from a −2-shifted symplectic $\mathbb{C}$-scheme.

Oh–Thomas prove a $\mathbb{G}_m$-localization result for their CY4 virtual classes, provided the $\mathbb{G}_m$ action preserves the CY volume form and shifted symplectic forms and does not just scale it.