Nick Kuhn — Shifted symplectic pushforwards

(Hyeonjun Park).

Recall: $\begin{array}{c} X \\ \downarrow \\ B \end{array}$ geometric morphism of derived stacks, characteristic zero, locally finitely presented, etc.

here: X, B n-geometric

This is **d-shifted symplectic** if we have a non-degenerate d-shifted closed 2-form

$$\omega \in \mathcal{A}^{2,cl}(X/B, d)$$

$$\Theta_\omega : \mathbb{T}_{X/B} \longrightarrow \mathbb{L}_{X/B}(d).$$

Given $p : U \longrightarrow B$ can pull back: $(p^*X, p^*\omega)$ d-shifted symplectic / U.

Recall: A d-Lagrangian in X is $L \xrightarrow{\ell} X$, $L \to B$, $X \to B$

and a path $\gamma_\ell : \ell^*\omega_X \xrightarrow{\sim} 0$ in $\mathcal{A}^{2,cl}(L/B, d)$ ← space of closed 2-forms such that

$$\begin{tikzcd}
\Pi_{L/D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swArrow \gamma_e"] & e^* \Pi_{X/D} \simeq e^* \mathbb{L}_{X/D}(d) \arrow[d] \\
0 \arrow[r] & \mathbb{L}_{L/D}(d)
\end{tikzcd}$$

is Cartesian.

$$\left(\Rightarrow \Pi_{L/D} \simeq \mathbb{L}_{L/X}(d-1). \right)$$
$$\Vert$$
$$\nu_{L/X}(d)$$

A d-Lagrangian correspondence

$$\begin{tikzcd}
& C \arrow[dl, "a"'] \arrow[dr, "b"] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& D &
\end{tikzcd}$$

$+ \gamma_c : a^* \omega_X \to b^* \omega_Y$

s.t. $\omega_X \oplus - \omega_Y$

$$\begin{tikzcd}
C \arrow[r] \arrow[dr] & X \times Y \arrow[d] \\
& D
\end{tikzcd}$$

is Lagrangian

Have $(\infty,1)$-category

$$\mathrm{Symp}_{B,d} = \left(\begin{array}{l} d\text{-shifted} \\ \text{sympl. der.} \\ \text{stacks } /B \end{array} , \begin{array}{l} d\text{-lag} \\ \text{correspondences} \end{array} \right)$$

Composition:

$$\begin{tikzcd}
 & & C \times_Y D \arrow[dl] \arrow[dr] & \\
 & C \arrow[dl] \arrow[dr] & & D \arrow[dl] \arrow[d] \\
X \arrow[drr] & & Y \arrow[d] & Z \arrow[dl] \\
 & & B &
\end{tikzcd}$$

Pullback gives $p^*: \mathrm{Symp}_{D,d} \to \mathrm{Symp}_{C,d}$

$p: C \to D$

Question: is there a push forward?

Example: Symplectic quotients

X 0-symplectic space with free symplectic G-action

would like to produce $X//G$. symplectic space

$X \to X/G$

$$\begin{tikzcd} \mathfrak{g} \arrow[d] & \\ T_x \arrow[r, "\sim"] \arrow[d] & \Omega_x \\ T_{X/G} & \end{tikzcd}$$

Need a moment map

$$X \xrightarrow{\mu} \mathfrak{g}^*$$

(G-equiv.)

$\forall\, \xi \in \mathfrak{g}$:

$d(\langle \mu, \xi \rangle) = \iota_{\rho(\xi)} \omega$

$$X/\!/G = \mu^{-1}(0)/G.$$

sympl + G-equiv

$$\begin{tikzcd} & \mu^{-1}(0) \arrow[dl, hook'] \arrow[dr] & \\ X & & X/\!/G. \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\text{symplectic}"] \arrow[d, "f"] & * \arrow[d] \\ (X/G) \arrow[r, "\text{sympl.}"] \arrow[d, dotted] & (*/G) \arrow[d, "\text{pushforward}"] \\ (X/\!/G) \arrow[r] & * \end{tikzcd}$$

So: pushforward along $(X/G) \to *$ is like defining symplectic quotients.

Symplectic pushforward via moment maps

Given $U \xrightarrow{p} B$

Consider shifted cotangent bundle $E = T^*_{U/B}(d+1)$

$(d+1)$ shifted symplectic $/B$.

Moreover

$$\begin{tikzcd} T^*_{U/B}(d+1) \arrow[r, "\pi"] \arrow[d] & U \arrow[dl] \\ B & \end{tikzcd}$$

is a Lagrangian fibration

(really, fibers of π are Lagrangians in E.)

$\iff$ $E \to E \times_B U$ is $(d+1)$-Lagrangian

Let $N \xrightarrow{n} E$ be $(d+1)$-Lagrangian

Claim: $N \xrightarrow{\pi \circ n} U$ is d-shifted symplectic

"Proof":

$$\begin{tikzcd} & N \arrow[dl, hook'] \arrow[dr] & \\ N \times_B U \arrow[dr, "\mathrm{Lag}"'] & & E \arrow[dl, "\mathrm{Lag}"] \\ & E \times_B U & \end{tikzcd}$$

$\perp$

n

– Fiber product of $(d+1)$-Lagrangians is d-symplectic over the base.

Conversely, some symplectic things, $N \longrightarrow U$ might admit a lift

$$\begin{tikzcd} & E \arrow[d] \\ N \arrow[ur, "n"] \arrow[r] & U \end{tikzcd}$$

and N is like a moment map.

Get map:

$$\left(\begin{array}{c} (d+1) \text{ Lag} \\ N \to T^*_{U|B}(d+1) \\ \text{over } B \end{array} \right) \longmapsto \left(\begin{array}{c} d\text{-symplectic} \\ \text{str. } \\ \text{on } U \end{array} \right)$$

Definition A moment map μ of

$$N \to U \in \mathrm{Sympl}_{U,d}$$

is a lift of N to a $(d+1)$-Lagrangian

$$N \xrightarrow{\mu} T^*_{U|B}(d+1)$$

Definition The shifted symplectic pushforward of N w.r.t. μ is:

$$\begin{tikzcd} N \times_{\mu} U \arrow[r] \arrow[d] & U \arrow[d, "0"] \\ N \arrow[r, "\mu"] & T^*_{U|B}(d+1) \arrow[dr] & \\ & & 0 \end{tikzcd}$$

$N \times_{\mu} U \in \mathrm{Symp}_{B,d}$

— Using shifted symplectic geometry

see for $(*/G) \longrightarrow *$
$\ddot{U}$ $\quad$ $\ddot{B}$

$$T^*_{U/B}(1) = (\mathfrak{g}^*/G)$$ ← ordinary vector bundle in degree 0.

$$\begin{array}{c} (\mathfrak{g}^*/G) \\ \downarrow \\ (*/G) \end{array}$$

Moment map:

$$\begin{tikzcd} & & (\mathfrak{g}^*/G) \arrow[d] \\ (X/G) \arrow[urr, dotted, "\text{lift}"] \arrow[rr] & & (*/G) \end{tikzcd}$$

= equivariant map $X \to \mathfrak{g}^*$ + condition.

Park: There is a variant of

$\mathrm{Symp}_{B,d}$ which allows pushforward

$\begin{array}{c} M \\ \downarrow \\ B \end{array}$ d-symplectic. $\quad \omega \in A^{2,cl}(M/B, d)$

$\overline{\omega} \in \Lambda^2 \mathbb{L}_{M/B}(d)$

+ "extra data" "$d_{dR}\omega \sim 0$"

New viewpoint:

Deformation to the normal sheaf

For $M \longrightarrow B$ a regular closed immersion, have deformation to normal bundle.

$$\begin{tikzcd}
M \times \{0\} \arrow[r] \arrow[d, "f_0"] & M \times \mathbb{A}^1 \arrow[d] & M \times \mathbb{G}_m \arrow[l] \arrow[d, "j \times \mathrm{id}"] \\
\mathbb{T}[1]_{M/B} = N_{M/B} \arrow[r] \arrow[d] & D_{M/B} \arrow[d] & B \times \mathbb{G}_m \arrow[l] \arrow[d] \\
B \times \{0\} \arrow[r] & B \times \mathbb{A}^1 & B \times \mathbb{G}_m \arrow[l]
\end{tikzcd}$$

Everything: $\mathbb{G}_m$-equivariant, $N_{M/B}$ of weight -1, $\mathbb{A}^1$ of weight 1.

In derived algebraic geometry, this works for any finitely presented geometric map $j : M \to B$.

$$D_{M/B} := \mathrm{Map}_{B \times \mathbb{A}^1}(B \times \{0\}, M \times \mathbb{A}^1)$$

Over $B_\tau \mathbb{G}_m$: $\underline{\mathrm{Map}}_{B_\tau \mathbb{G}_m}(\phi, M \times \mathbb{G}_m) = B_\tau \mathbb{G}_m$

Over $B_\tau\{0\}$: $\mathrm{Map}_B\Big(B_\tau \underbrace{\left(\begin{smallmatrix} x & \longrightarrow & b \\ \downarrow & 0 & \downarrow \\ b & \longrightarrow & \mathbb{A}^1 \end{smallmatrix}\right)}_{\mathrm{Spec}\, k(q)}, M\Big) = T_{M/B}[1]$

$|q| = -1.$

Definition

Theorem. A d-shifted ~~closed~~ locked p-form is the same as a $(d+p)$-shifted $\mathbb{G}_m$-equivariant ~~fund~~ function on $D_{M/B}$ along the fibre on $B_\tau\{0\}$

i.e. a function on the formal completion of $D_{M/B}$ at the fibre over $B_\tau\{0\}$

of $\mathbb{G}_m$ weight p

Remark The underlying p-form is obtained by restriction to $T_{M/B}[1]$

(A p-form is like a homogeneous degree p polynomial on $T_{m|B}(1)$.)

— (1) from Λ^p into Sym^p.

So: it's clear that p-forms are like functions on $\Pi T_{m|B}(1)$.

— Peak tells us that dual p-forms are functions on $\Pi T_{m|B}(1)$ which extend to a formal neighbourhood of $\Pi T_{m|B}(1)$ in $D_{m|B}$.

— He defines local p-forms as functions on $\Pi T_{m|B}(1)$ to things which actually extend to all of $D_{m|B}$

We can extract locked forms:

$$
\begin{tikzcd}
 & \mathbb{A}^{p,c}(M/B, d) \arrow[d] \\
\mathbb{A}^{p,lc}(M/B, d) \arrow[ur, "\text{extract to formal nbhd of } \mathbb{T}_{M/B}(1)"] \arrow[d, "\text{extract to } \frac{p+q=1}{q}"'] \arrow[r] & \mathbb{A}^{p}(M/B, d) \\
\mathbb{A}^{0}(B, p+d) & \text{extract to } \mathbb{T}_{M/B}(1). \arrow[u]
\end{tikzcd}
$$

Every locked form has an underlying function

$$B \to \mathbb{A}^{1}(p+d).$$

Remark

$$
\begin{tikzcd}
\mathbb{A}^{p,ex}(B, d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
\mathbb{A}^{p,lc}(M/B, d) \arrow[r] & \mathbb{A}^{0}(B, p+d)
\end{tikzcd}
$$

(In negative shifted world: if you want to vary things in families, locked forms are much closer to closed forms than to exact forms, so locked forms are a useful idea.)

Relation to PTVV:

$$A^{p,cl}(M|B, d) := \mathrm{Map}_{QGr}(\mathcal{O}_B, F^p \widehat{DR}_{M/B}(p+d))$$

$M \to B \Rightarrow$ complete filtered complex

$$\widehat{DR}_{M/B} = F^0 \widehat{DR}_{M/B} \supset F^1 \widehat{DR}_{M/B} \supset F^2 \widehat{DR}_{M/B}$$

$\exists$ uncompleted version $DR_{M/B}$.

? locked forms are for uncompleted version

$(\omega_0, \omega_1, \omega_2, \ldots)$

locked if series eventually zero.

Exact forms: Why

$$\begin{tikzcd}
\mathcal{A}^{p,ex}(M/B, d) \arrow[r] \arrow[d] & \mathcal{A}^{p,cl}(M/B, d) \arrow[r] \arrow[d] & \mathcal{A}^{p,cl}(M/B, d) \arrow[d] \\
X \arrow[r, "0"] & \mathcal{A}^0(B, p+d) \arrow[r] & \mathcal{A}^{DR}(M/B, p+d)
\end{tikzcd}$$

Reminder: de Rham cohomology.

Linked forms: generalized version of exact forms, particularly for families over a base.

Q: * What are linked forms good for?
* How different from closed forms?

Fix $\omega : B \to \mathcal{A}^1(d+2)$ shifted function.

$$\mathrm{Symp}^{\omega}_{B,d} = \left\{ \begin{array}{c} \omega\text{-linked symplectic} \\ \text{structures} \end{array} \right\}$$

$$= \left(\begin{array}{c} M \\ \big\downarrow f \\ B \end{array} j \quad \omega \in A^{2,cl}(M/B, d) \;\Big|\; j \ \ \omega\text{-closed} \ \ d\text{-Lag} \ \ \text{compactness} \right)$$

modulo ij
function in ω

<u>Main</u>
<u>Theorem</u> for any $U \xrightarrow{p} B$ the pullback map

$$p^* : \mathrm{Symp}^{\omega}_{B,d} \longrightarrow \mathrm{Symp}^{\omega/n}_{U,d}$$

admits a right adjoint.

<u>Key point:</u> lower form exists a moment map (in a functorial way).

<u>Main application:</u> BBJ local structure theorem for negative shifted <u>local</u> symplectic fibrations $M \longrightarrow B$

— Now works in families. Really means local structure: M is derived 1-Artin, with reductive stabiliser grps.