

Nick Kuhn

Shifted symplectic pushforwards

(Hyunjin Park).

Recall:

hee:
 X/B
 n -genetic



geometric map of derived stacks,
characteristic zero, locally finitely
presented, etc.

This is d -shifted symplectic if we have
a non-degenerate d -shifted closed 2-form

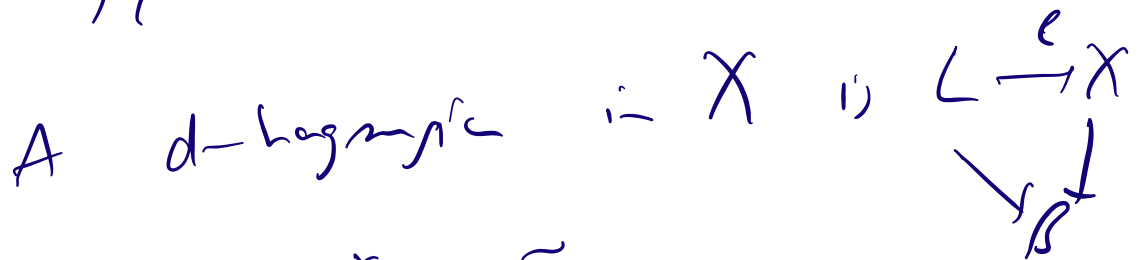
we $A^{2,cl}(X/B, d)$

$$\Theta_\omega: T_{X/B} \rightarrow L_{X/B}(d).$$

Given $p: U \rightarrow B$ can pull back:

$(p^*X, p^*\omega)$ d -shifted symplectic / \mathcal{Y} .

Recall:



and a point $x \in L$: $\ell^* \omega_x \xrightarrow{\sim} 0$

is $A^{2,cl}(L/B, d)$

← space of our 2-forms

such that

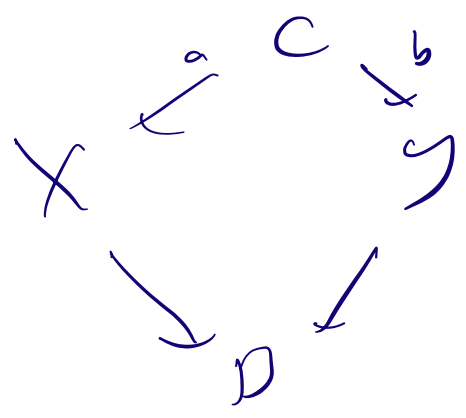
$$\begin{array}{ccc} \mathbb{A}_{L/B} & \longrightarrow & e^* \mathbb{A}_{X/D} \cong e^* \mathbb{A}_{X/B}^{(d)} \\ \downarrow & \searrow \gamma_e & \downarrow \\ 0 & \longrightarrow & \mathbb{A}_{L/D}^{(d)} \end{array}$$

is Cartier.

$$\left(\Rightarrow \mathbb{A}_{L/D} \cong \mathbb{A}_{L/X}^{(d-1)} \right)$$

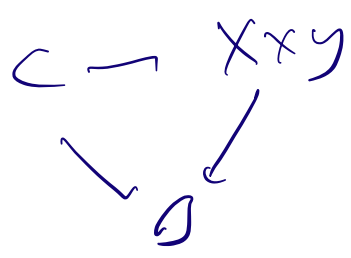
"
 $\mathbb{A}_{L/X}^{(d)}$

A d-Lagrange wreath product



$$\begin{aligned} \exists \gamma_c: a^* \omega_X &\rightarrow \\ &b^* \omega_Y \end{aligned}$$

s.t. $\omega_X \oplus \omega_Y$

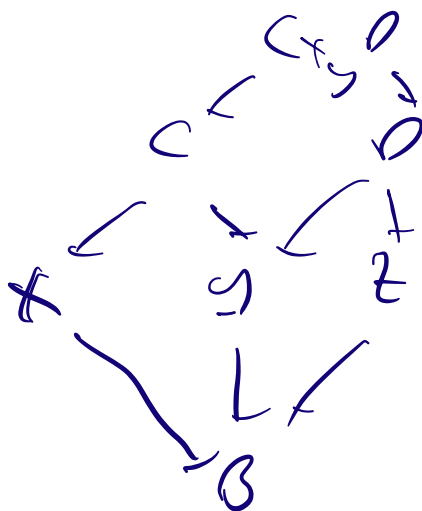


ii) $\mathbb{A}_{L/D}$

Have $(\infty, 1)$ -category

$$\text{Symp}_{\beta, d} = \left(\begin{array}{l} d\text{-shift} \\ \text{symp. l. der.} \\ \text{stack} / \beta \end{array} \right), \quad \left(\begin{array}{l} d\text{-hgr} \\ \text{comparability} \end{array} \right)$$

Compositions



pullback over p^* : $\text{Symp}_{\beta, d} \rightarrow \text{Symp}_{\alpha, d}$
 $p: \alpha \rightarrow \beta$

Question: is this a pullback?

Example: Symplectic groups

X O -symplectic slice with free G -action
 well like to produce X/G .
 symplectic slice

$$X \rightarrow X/G$$

Need a map

$$X \xrightarrow{N} \mathbb{C}^*$$

(G-equiv.)

\mathbb{C}^*

$$\int_{T_x} \rightarrow \int_X$$

$$\int_{T_x/G}$$

$\forall \rho \in \mathfrak{g}$:

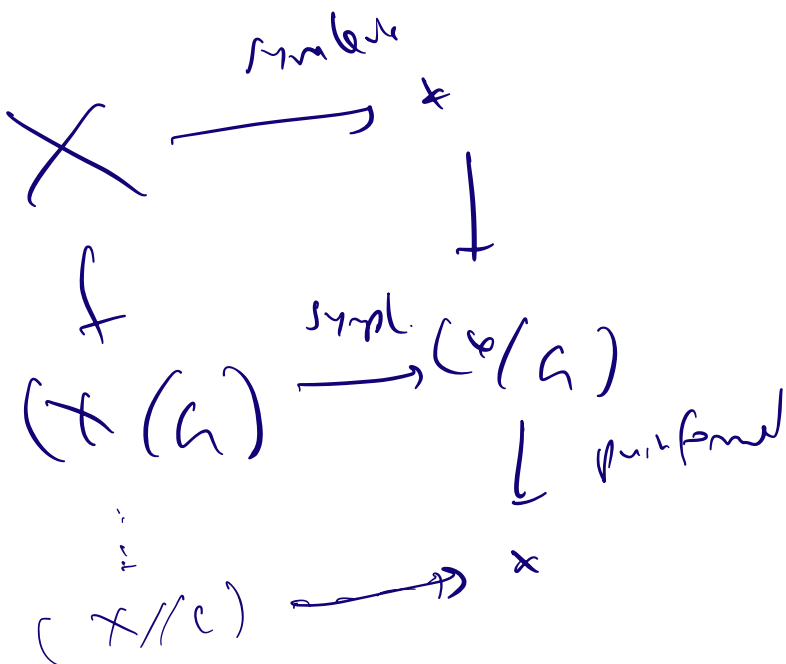
$$d(N, \rho) =$$

$$c_{\rho}(s) \cdot \omega$$

$$X//G = N^{-1}(0)/G$$

simpl. + G-equiv

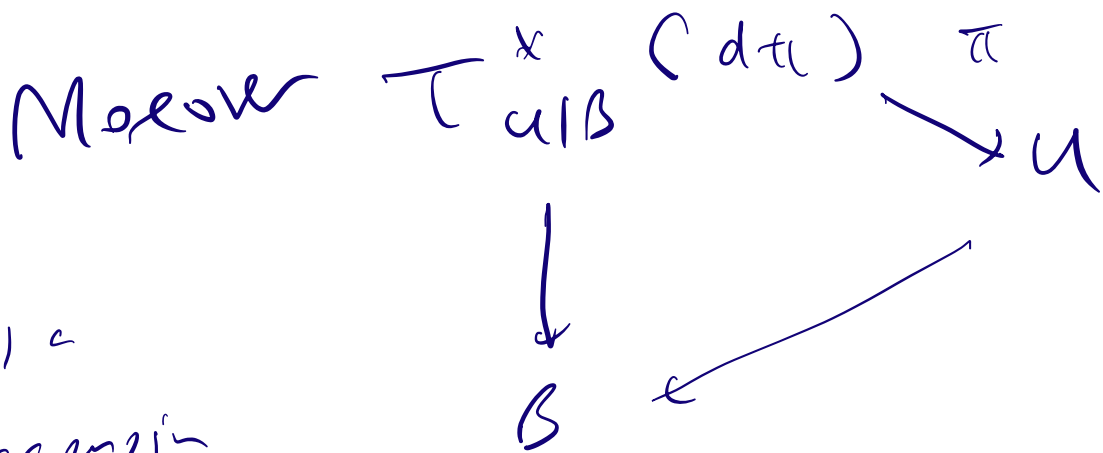
$$X \xrightarrow{N^{-1}(0)} X//G$$



So: pushforward
 der $(X/G) \rightarrow X^*$
 is like
 de Rham
 cohomology
 of X/G .

Symplectic pushforward via moment maps

Given $U \xrightarrow{r} B$
 Given shift σ on U bundle $E = T^*_{U/B} (d\pi)$
 $(d\pi)$ shifted symplectic / B .



i) ω
 Lagrangian
 fibration

($\omega|_U$), fibers of π are Lagrangian in E .)

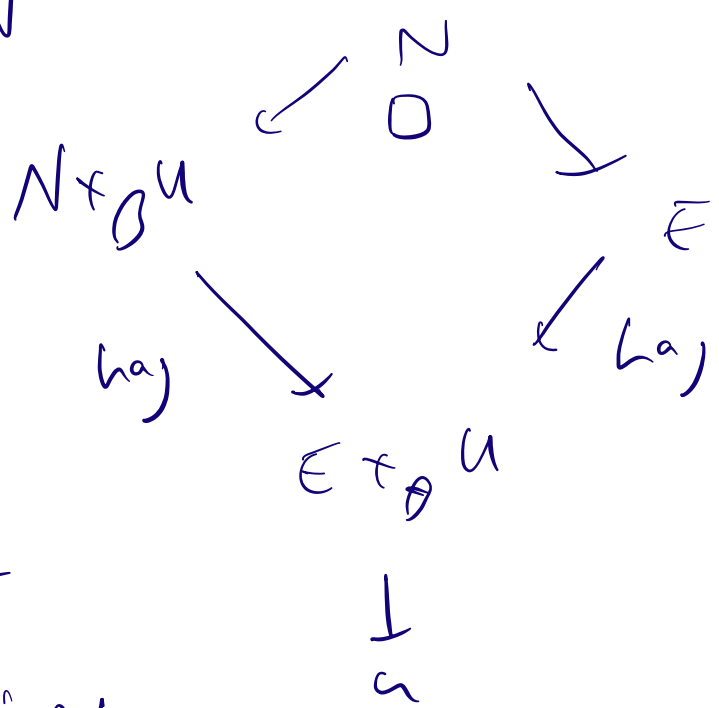
$$\Rightarrow E \rightarrow E \times_B U$$

i) $(d\pi)$ -Lagrangian

Let $N \xrightarrow{\pi} E$ be (d+1)-Lagrangian

Claim: $N \xrightarrow{\pi \circ \alpha} U$ is d-shifted symplectic

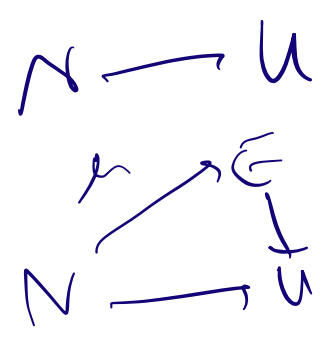
"Proof":



— Fiber product of
 (d+1)-Lagrangian,
 is d-symplectic over
 the base.

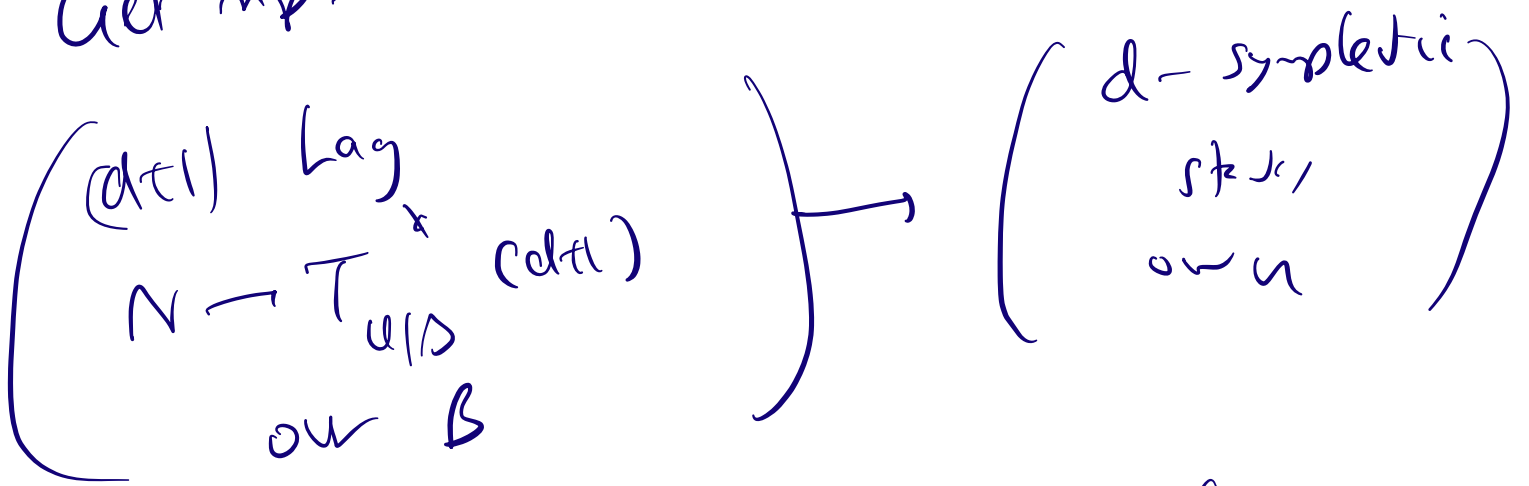
Conversely, For symplectic (U, ω) ,

a lift



might admit
 as N is like a
 moment map.

Get map:



Definition A moment map μ is

$$N \xrightarrow{\hat{\mu}} u \in \text{Symp}_{u,d}$$

i) a lift ϕ of N to a $(d\pi)$ -Lagrangian

$$N \xrightarrow{\mu} T_{u/D}^* (d\pi)$$

Definition The shifted symplectic pullback

$$\begin{array}{ccc} \text{of } N \text{ wr.t. } \mu & & \text{(i):} \\ N \times_{\mathbb{D}} u & \xrightarrow{\quad} & u \\ \downarrow & & \downarrow 0 \\ N & \xrightarrow{\mu} & T_{u/D}^* (d\pi) \\ & & \downarrow 0 \end{array}$$

$$N \times_{\mathbb{D}} u \in \text{Symp}_{\mathbb{D},d}$$

— Using shifted symplectic geometry

$$\text{we have } \begin{array}{ccc} (X/G) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \tilde{U} & & B \end{array}$$

$$T^*_{U/B}(1) = (S^1/G) \downarrow (X/G)$$

← ordinary vector bundle
∴ degree 0.

$$\text{Morse map: } \begin{array}{ccc} \text{left } \dots \rightarrow (S^1/G) & & \\ \downarrow & & \\ (X/G) & \longrightarrow & (X/G) \end{array}$$

= equivalent map $X \rightarrow \mathfrak{g}^A + \text{condition}$.

Park: There is a variant of

$\text{Sym}_{B,d}$ which allows pushforward

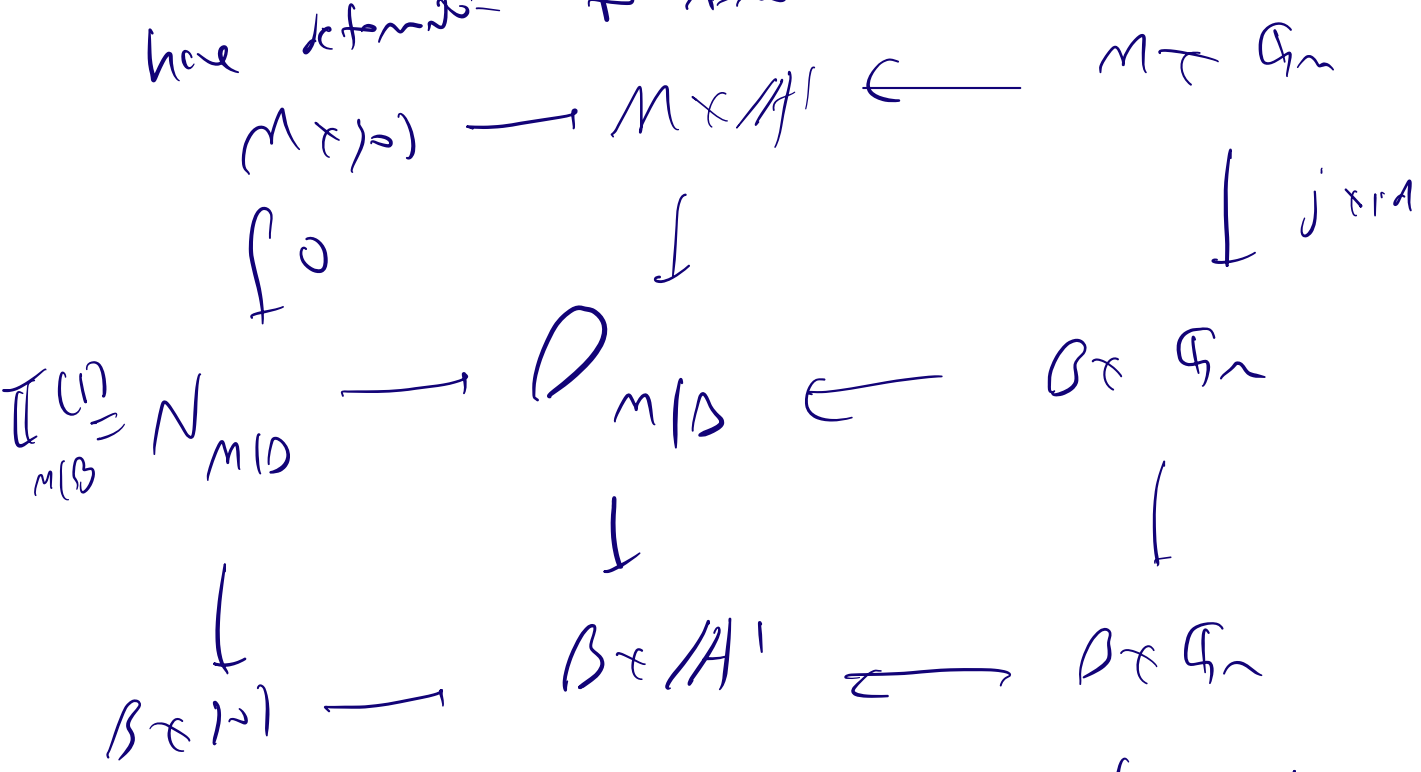
$$\begin{array}{c} M \text{ d-symplectic} \\ \downarrow \\ B \end{array}$$

$$\text{we have } A^{2, \text{ccr}}(M/B, d) \\ \bar{\omega} \in A^2 \text{LMS}(d) \\ + \text{"extra data"} \quad "d_{d/2} \bar{\omega} \sim 0"$$

New viewpoint:

Deformation to the normal sheaf

For $M \rightarrow B$ a regular local immersion,
have deformation to normal bundle.



Equival: \mathbb{A}^n -equiv $N_{M/B}$ of weight -1
 \mathbb{A}^1 of weight 1 .

In derived algebraic geometry, this works for
 any Artin local no $j: M \rightarrow B$.
 geometric

$$\mathcal{O}_{M/B} := \text{Map}_{\mathcal{O}_B \times \mathbb{A}^1}(\mathcal{O}_B \times \mathbb{A}^1, M \times \mathbb{A}^1)$$

Our $B \times G_m$. $\frac{M \vee}{B \times G_m} (p, M \times G_m) = B \times G_m$

Our $B \times \mathbb{A}^1$: $\text{Map}_D \left(B \times \left(\begin{array}{c|c} x & b \\ \hline \downarrow & \downarrow \\ \mathbb{A}^1 & \mathbb{A}^1 \end{array} \right), M \right)$

$\text{Spec } k(\epsilon) = \mathbb{A}^1_{M/B}(\epsilon)$
 $|\epsilon| = -1.$

Definition

Theorem. A d -shifted closed p -form is the local G_m -equivariant ~~form~~ function

same as $(d+p)$ -shifted

on $D_{M/B}$ along the fibre on $B \times \mathbb{A}^1$
 i.e. a function on the formal completion of $D_{M/B}$ at the fibre on $B \times \mathbb{A}^1$

of G_m weight p

Remark The underlying p -form ω obtained by restriction to $T_{M/B}(1)$

(A p-form is like a homogeneous degree
 p polynomial $\approx T_{m/B}(1)$.)
 - (1) has Prime Smp.

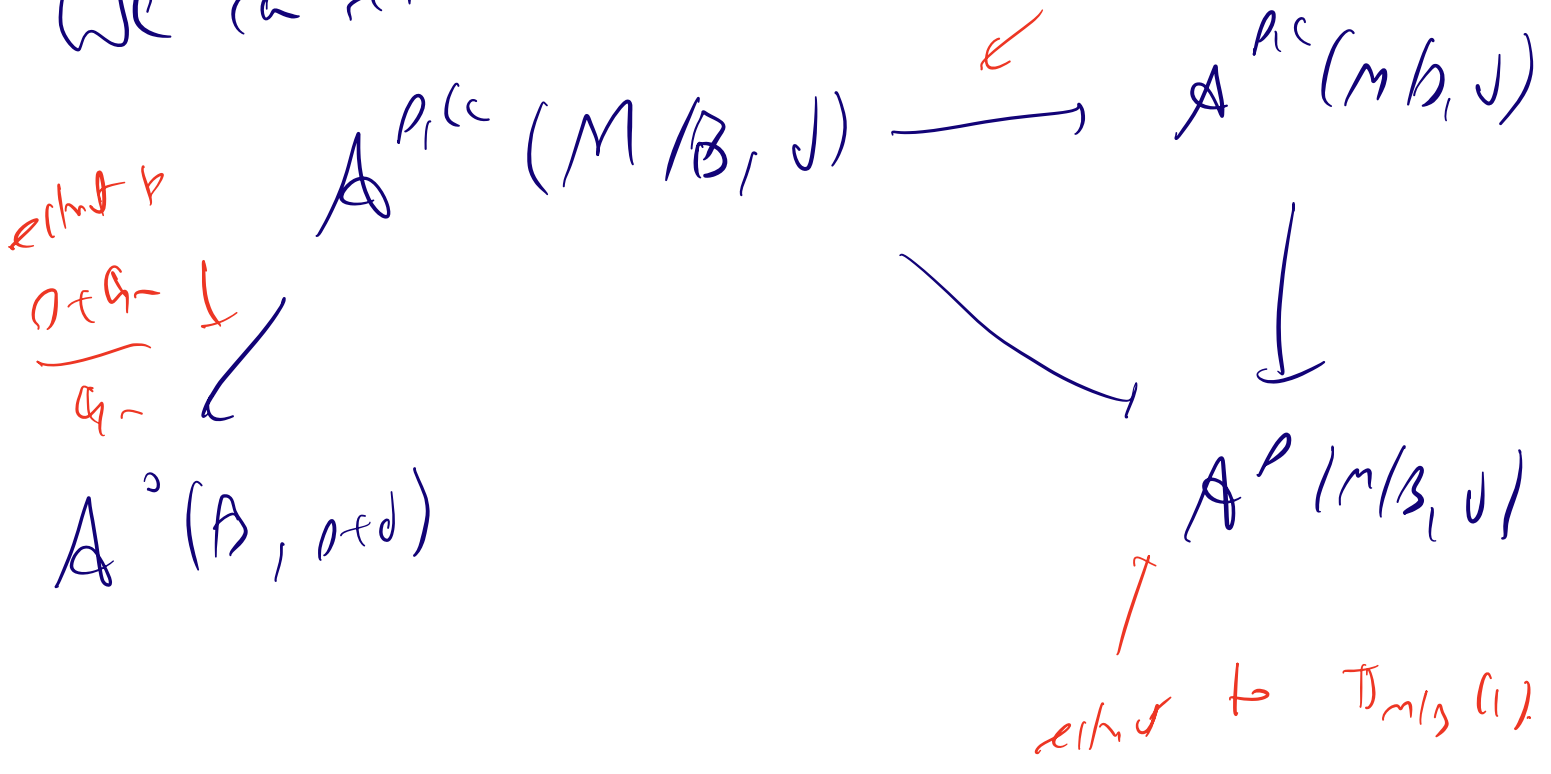
So: it's clear that p-forms
 are like factors $\approx T_{m/B}(1)$.
 - Each form in the down p-form
 are factors $\approx T_{m/B}(1)$ which extend
 to a full neighborhood of
 $T_{m/B}(1)$ is $D_{m/B}$.

- The down local p-form \approx
 factors $\approx T_{m/B}(1)$ to things
 which actually extend to all of $D_{m/B}$

We can reflect back form:

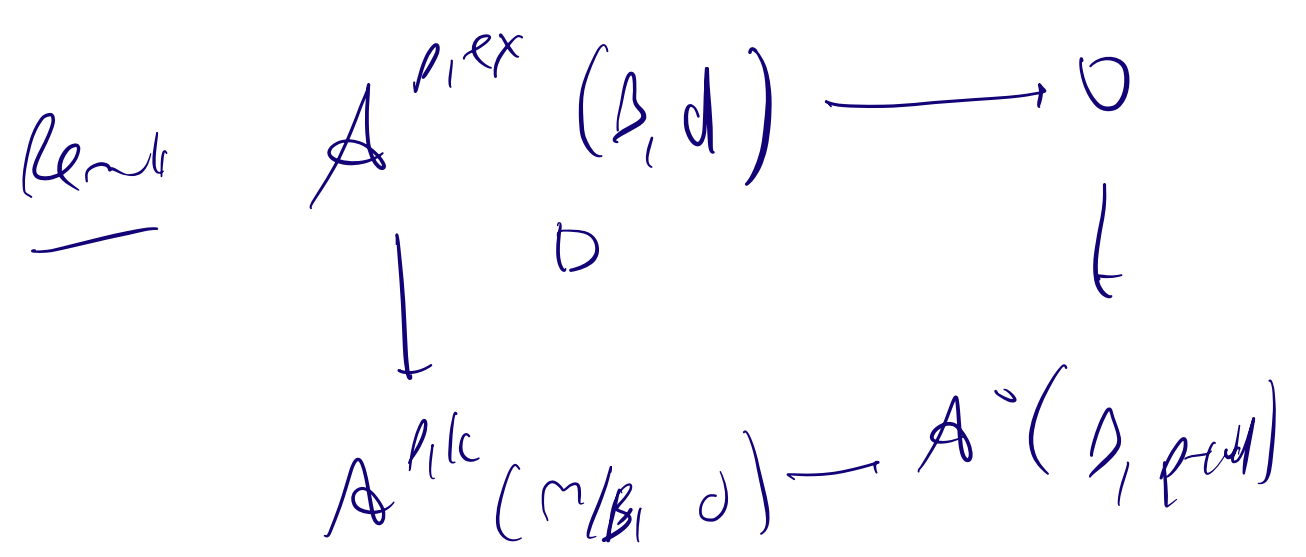
reflect to $\frac{\sigma \in \mathcal{G}_n}{\mathcal{G}_n}$

reflect to $\text{fund. obj. of } \mathbb{D}_{m/d}(1)$



Every label for has an underlying function

$$B \rightarrow A^1(\rho \in \mathcal{G}_n).$$



(In register shifted world: if you want to
 vary things in families, look for more
 much less to cover families than to exact
 form, so look for more a useful idea.)

Relation to PTVV:

$$A^{PCC}(MID, d) := \text{Map}_{\mathbb{Q}GL}(\mathbb{O}_B, F^P \widehat{DR}_{m/b}^{(PCC)})$$

$M \rightarrow D \Rightarrow$ work filtered over

$$\widehat{DR}_{m/b} = F^0 \widehat{DR}_{m/b} \supseteq F^1 \widehat{DR}_{m/b} \subseteq \dots \subseteq F^l \widehat{DR}_{m/b}$$

\exists maximal series $DR_{m/b}$

? Looked for
 one for
 unbounded series

(w_0, v_1, v_2, \dots)

look if series
 eventually res.

Exact form: U, W, Y

$$\begin{array}{ccccc}
 A^{D, \text{ex}}(M/D, d) & \longrightarrow & A^{D, \text{cc}}(M/D, d) & \longrightarrow & A^{D, \text{cc}}(M/D, d) \\
 \downarrow \times & & \downarrow & & \downarrow \\
 & \xrightarrow{\circ} & A^0(D, \text{cc}) & \longrightarrow & A^{D, \text{cc}}(M/B, D, d)
 \end{array}$$

(cc) = de Rham cohomology.

local form: general version of exact

form, particularly for families over a base.

Q: * What are local form, good for?

* How different from local form?

Fix $\omega: B \rightarrow A^1(d+2)$ shift factor.

$$\text{Symp}_{B, d}^\omega = \left\{ \begin{array}{l} \omega\text{-local symplectic} \\ \text{ideals} \end{array} \right\}$$

$$= \left(\begin{array}{c} M \\ f_j \end{array} \omega \in A^{2lc} (M/B, d) \begin{array}{c} \uparrow \\ j \end{array} \begin{array}{l} \omega = \omega(d) \\ d\text{-Lag} \\ \text{congruence} \end{array} \right)$$

$\omega \in A^{2lc}$
 $\omega = \omega(d)$
 $d\text{-Lag}$
 congruence

Main
Theorem for any $U \xrightarrow{p} B$ the pullback

$$\text{map } p^*: \text{Symp}_{S, d}^{\omega} \longrightarrow \text{Symp}_{u, d}^{\omega|_u}$$

admits a right adjoint.

key point: better for ex. a more
 map (in a functorial way).

Main application: BBT ω structure
 theorem for regular shifted labeled

symplectic fibration $M \rightarrow B$

— Now ω is in families. $\text{Ker} \omega$ needs

labeled structure. M is ω and $\mathbb{1}$ -A fib. with reductive additive sps.