Derived Algebraic Geometry

Lecture 3 of 14: 2-categories

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Summer Term 2022

These slides available at
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Plan of talk:

3.1. The definition of 2-categories

A 2-category $\mathscr{C}$ has *objects* $X, Y, \ldots$, 1-*morphisms* $f, g : X \to Y$ (morphisms), and 2-*morphisms* $\eta : f \Rightarrow g$ (morphisms between morphisms). Here are some examples to bear in mind:

Example 3.1

(a) The 2-category $\mathfrak{Cat}$ has objects categories $\mathscr{C}, \mathscr{D}, \ldots$, 1-morphisms functors $F, G : \mathscr{C} \to \mathscr{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$.

(b) The 2-category $\mathbf{Top}^{\mathbf{ho}}$ of *topological spaces up to homotopy* has objects topological spaces $X, Y, \ldots$, 1-morphisms continuous maps $f, g : X \to Y$, and 2-morphisms isotopy classes $[H] : f \Rightarrow g$ of homotopies H from f to g. That is, $H : X \times [0,1] \to Y$ is continuous with $H(x,0) = f(x)$, $H(x,1) = g(x)$, and $H, H' : X \times [0,1] \to Y$ are isotopic if there exists continuous $I : X \times [0,1]^2 \to Y$ with $I(x,s,0) = H(x,s)$, $I(s,x,1) = H'(x,s)$, $I(x,0,t) = f(x)$, $I(x,1,t) = g(x)$.

Definition

A (*strict*) *2-category* $\mathscr{C}$ consists of a proper class of *objects* $\mathrm{Obj}(\mathscr{C})$, for all $X, Y \in \mathrm{Obj}(\mathscr{C})$ a category $\mathrm{Hom}(X, Y)$, for all X in $\mathrm{Obj}(\mathscr{C})$ an object id_X in $\mathrm{Hom}(X, X)$ called the *identity 1-morphism*, and for all X, Y, Z in $\mathrm{Obj}(\mathscr{C})$ a functor $\mu_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$. These must satisfy the *identity property*, that

$$\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}(X,Y)} \tag{3.1}$$

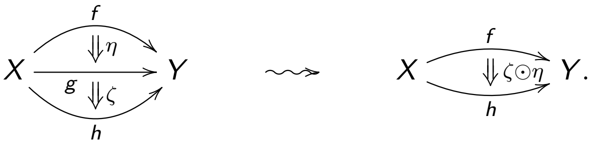
as functors $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y)$, and the *associativity property*, that

$$\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) = \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z}) \tag{3.2}$$

as functors $\mathrm{Hom}(W, X) \times \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(W, X)$. Objects f of $\mathrm{Hom}(X, Y)$ are called 1-*morphisms*, written $f : X \to Y$. For 1-morphisms $f, g : X \to Y$, morphisms $\eta \in \mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f, g)$ are called 2-*morphisms*, written $\eta : f \Rightarrow g$.

I'm explaining 2-categories partly to give you some intuition for ∞-categories at the end of the lecture and later.

There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms then $\mu_{X,Y,Z}(f,g)$ is the *horizontal composition of 1-morphisms*, written $g \circ f : X \to Z$. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms then composition of η, ζ in $\mathrm{Hom}(X, Y)$ gives the *vertical composition of 2-morphisms* of η, ζ, written $\zeta \odot \eta : f \Rightarrow h$, as a diagram



(3.3)

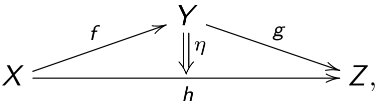
And if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \tilde{f}$, $\zeta : g \Rightarrow \tilde{g}$ are 2-morphisms then $\mu_{X,Y,Z}(\eta, \zeta)$ is the *horizontal composition of 2-morphisms*, written $\zeta * \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f}$, as a diagram

$$X \underset{\tilde{f}}{\overset{f}{\rightrightarrows}} \!\!\Downarrow\eta\;\, Y \underset{\tilde{g}}{\overset{g}{\rightrightarrows}} \!\!\Downarrow\zeta\;\, Z \quad \rightsquigarrow \quad X \underset{\tilde{g}\circ\tilde{f}}{\overset{g\circ f}{\rightrightarrows}} \!\!\Downarrow\zeta*\eta\;\, Z. \tag{3.4}$$

There are also two kinds of identity: *identity 1-morphisms* $\mathrm{id}_X : X \to X$ and *identity 2-morphisms* $\mathrm{id}_f : f \Rightarrow f$.
A 2-morphism is a 2-*isomorphism* if it is invertible under vertical composition. A 2-category is called a (2,1)-*category* if all 2-morphisms are 2-isomorphisms. They are arguably the nicest kind of 2-category. For example, stacks in algebraic geometry form a (2,1)-category.

In a 2-category $\mathfrak{C}$, there are three notions of when objects X, Y in $\mathfrak{C}$ are 'the same': *equality* $X = Y$, and *isomorphism*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and *equivalence*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Usually equivalence is the correct notion.

Commutative diagrams in 2-categories should in general only commute *up to (specified) 2-isomorphisms*, rather than strictly. A simple example of a commutative diagram in a 2-category $\mathfrak{C}$ is



which means that X, Y, Z are objects of $\mathfrak{C}$, $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\mathfrak{C}$, and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

3.2. Fibre products in categories and 2-categories

Definition (Fibre products in ordinary categories, as in §2.3.)

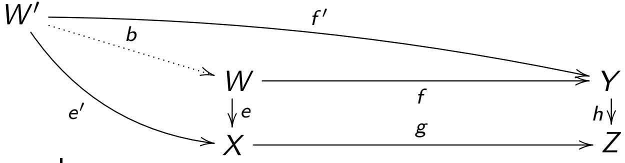
Let $\mathscr{C}$ be a category, and $g : X \to Z$, $h : Y \to Z$ be morphisms in $\mathscr{C}$. A *fibre product* (W, e, f) for g, h in $\mathscr{C}$ consists of an object W and morphisms $e : W \to X$, $f : W \to Y$ in $\mathscr{C}$ with $g \circ e = h \circ f$, with the *universal property* that if $e' : W' \to X$, $f' : W' \to Y$ are morphisms in $\mathscr{C}$ with $g \circ e' = h \circ f'$, then there is a unique morphism $b : W' \to W$ with $e' = e \circ b$ and $f' = f \circ b$. We write $W = X \times_{g,Z,h} Y$ or $W = X \times_Z Y$. The commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\quad f \quad} & Y \\ \downarrow{\scriptstyle e} & & {\scriptstyle h}\downarrow \\ X & \xrightarrow{\quad g \quad} & Z \end{array}$$

is called a *Cartesian square*.

In general, fibre products may or may not exist. If a fibre product exists, it is unique up to canonical isomorphism.

The universal property may be summarized by the diagram



Some examples:

- All fibre products exist in $\mathbf{Sch}_{\mathbb{K}}$.
- All fibre products $W = X \times_{g,Z,h} Y$ exist in **Top**. We can take $W = \{(x, y) \in X \times Y : g(x) = h(y)\}$, with the subspace topology.
- Not all fibre products exist in **Man**. If $g : X \to Z$, $h : Y \to Z$ are *transverse* then a fibre product $W = X \times_{g,Z,h} Y$ exists in **Man** with $\dim W = \dim X + \dim Y - \dim Z$.
- Intersections of subschemes, or submanifolds, are examples of fibre products. If $C, D \subseteq S$ are $\mathbb{K}$-subschemes of a $\mathbb{K}$-scheme S, then by the $\mathbb{K}$-subscheme $C \cap D$, we actually mean the fibre product $C \times_{i,S,j} D$ in $\mathbf{Sch}_{\mathbb{K}}$, with $i : C \hookrightarrow S$, $j : D \hookrightarrow S$ the inclusions.

Definition (Fibre products in 2-categories)

Let $\mathscr{C}$ be a strict 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\mathscr{C}$. A *fibre product* $X \times_Z Y$ in $\mathscr{C}$ is an object W, 1-morphisms $e : W \to X$ and $f : W \to Y$ and a 2-isomorphism $\eta : g \circ e \Rightarrow h \circ f$ in $\mathscr{C}$ with the following universal property: suppose $e' : W' \to X$ and $f' : W' \to Y$ are 1-morphisms and $\eta' : g \circ e' \Rightarrow h \circ f'$ is a 2-isomorphism in $\mathscr{C}$. Then there exists a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta : e \circ b \Rightarrow e'$, $\lambda : f \circ b \Rightarrow f'$ such that the following diagram commutes:

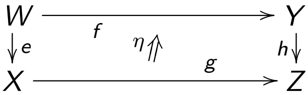
$$\begin{array}{ccc} g \circ e \circ b & \xRightarrow{\eta * \mathrm{id}_b} & h \circ f \circ b \\ {\scriptstyle \mathrm{id}_g * \zeta} \Downarrow & & \Downarrow {\scriptstyle \mathrm{id}_h * \lambda} \\ g \circ e' & \xRightarrow{\eta'} & h \circ f'. \end{array}$$

Furthermore, if $\tilde{b}, \tilde{\zeta}, \tilde{\lambda}$ are alternative choices of b, ζ, λ then there should exist a unique 2-isomorphism $\theta : \tilde{b} \Rightarrow b$ with

$$\tilde{\zeta} = \zeta \odot (\mathrm{id}_e * \theta) \quad \text{and} \quad \tilde{\lambda} = \lambda \odot (\mathrm{id}_f * \theta).$$

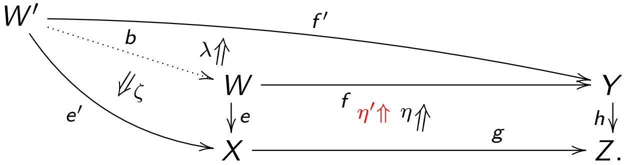
If a fibre product $X \times_Z Y$ exists, it is unique up to equivalence.

The 2-commutative diagram



is called a 2-*Cartesian square*.

The universal property may be summarized by the diagram



The composition of 2-morphisms η, ζ, λ across the diagram is η'.

Example: classical and derived affine schemes

Let $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)$ be polynomials over a field $\mathbb{K}$, and let $f=(f_1,\ldots,f_m):\mathbb{A}^n\rightarrow\mathbb{A}^m$ be the associated morphism. Write $X=f^{-1}(0)\subset\mathbb{A}^n$ for the zero locus of f, as a $\mathbb{K}$-subscheme. Then X is a fibre product $*\times_{0,\mathbb{A}^m,f}\mathbb{A}^n$ in $\mathbf{Sch}_{\mathbb{K}}$, in a Cartesian square

$$\begin{array}{ccc} X & \longrightarrow & *=\operatorname{Spec}\mathbb{K} \\ \downarrow & & \downarrow 0 \\ \mathbb{A}^n & \xrightarrow{f} & \mathbb{A}^m. \end{array}$$

If some f_i is zero, or if $f_i=f_j$, we can omit f_i without changing X. This is always true for fibre products in an ordinary category of spaces. But for fibre products in the ∞-category $\mathbf{dSch}_{\mathbb{K}}$ of derived schemes, X is changed by omitting f_i. That is, for a derived zero locus, setting $f=0$ twice is different to setting $f=0$ once. This works because the universal property of fibre products in $\mathbf{dSch}_{\mathbb{K}}$ involves 2- (and higher) morphisms, which can see the difference.

3.3. More about 2-categories; ∞-categories
The homotopy category of a 2-category

Let $\mathfrak{C}$ be a 2-category. The *homotopy category* of $\mathfrak{C}$ is the ordinary category $\mathrm{Ho}(\mathfrak{C})$ whose objects $X, Y, \ldots$ are the objects of $\mathfrak{C}$, and whose morphisms $[f] : X \to Y$ are 2-isomorphism classes $[f]$ of 1-morphisms $f : X \to Y$. Objects X, Y in $\mathfrak{C}$ are equivalent in $\mathfrak{C}$ if and only if they are isomorphic in $\mathrm{Ho}(\mathfrak{C})$.

Given a fibre product $X \times_{g,Z,h} Y$ in a 2-category $\mathfrak{C}$, one could also consider the fibre product $X \times_{[g],Z,[h]} Y$ in the homotopy category $\mathrm{Ho}(\mathfrak{C})$. These can be different (if they both exist). The fibre product in $\mathfrak{C}$ is usually the 'correct' one to consider.

The homotopy category $\mathrm{Ho}(\mathfrak{C})$ is a truncation of $\mathfrak{C}$. Passing to the homotopy category of a 2-category (or an ∞-category) forgets information stored in the 2- (and higher) morphisms. Sometimes you need this information, and should work with the higher category. For example, derived categories $D^b \operatorname{coh}(X)$ are the homotopy categories $\mathrm{Ho}(\mathbb{D}^b \operatorname{coh}(X))$ of an underlying stable ∞-category $\mathbb{D}^b \operatorname{coh}(X)$. Issues such as 'nonfunctoriality of the cone' in $D^b \operatorname{coh}(X)$ are a sign that the homotopy category $\mathrm{Ho}(\mathbb{D}^b \operatorname{coh}(X))$ is not enough, and you need the ∞-category $\mathbb{D}^b \operatorname{coh}(X)$.

Weak 2-categories

A *weak 2-category*, or *bicategory*, is like a strict 2-category, except that the equations of functors (3.1), (3.2) are required to hold not up to equality, but up to specified natural isomorphisms. That is, a weak 2-category $\mathscr{C}$ consists of data $\mathrm{Obj}(\mathscr{C}), \mathrm{Hom}(X, Y), \mu_{X,Y,Z}$, id_X as above, but in place of (3.1), a natural isomorphism

$$\alpha : \mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) \Longrightarrow \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z}),$$

and in place of (3.2), natural isomorphisms

$$\beta : \mu_{X,X,Y}(\mathrm{id}_X, -) \Longrightarrow \mathrm{id}, \quad \gamma : \mu_{X,Y,Y}(-, \mathrm{id}_Y) \Longrightarrow \mathrm{id},$$

satisfying some identities. That is, composition of 1-morphisms is associative *only up to specified 2-isomorphisms*, so for 1-morphisms $e : W \to X$, $f : X \to Y$, $g : Y \to Z$ we have a 2-isomorphism

$$\alpha_{g,f,e} : (g \circ f) \circ e \Longrightarrow g \circ (f \circ e).$$

Similarly identities $\mathrm{id}_X, \mathrm{id}_Y$ work up to 2-isomorphism, so for each $f : X \to Y$ we have 2-isomorphisms

$$\beta_f : f \circ \mathrm{id}_X \Longrightarrow f, \qquad \gamma_f : \mathrm{id}_Y \circ f \Longrightarrow f.$$

2-categories 'in nature' are often weak 2-categories.

The 3-category of 2-categories

The most basic example of a 2-category is the 2-category $\mathfrak{Cat}$ of categories, with objects categories $\mathscr{C},\mathscr{D}$, 1-morphisms functors $F,G:\mathscr{C}\rightarrow\mathscr{D}$, and 2-morphisms natural transformations $\eta:F\Rightarrow G$. Similarly, the family of all 2-categories 2-$\mathfrak{Cat}$ is actually a 3-*category*.

- *Objects* of 2-$\mathfrak{Cat}$ are 2-categories $\mathfrak{C},\mathfrak{D}$.
- 1-*morphisms* of 2-$\mathfrak{Cat}$ are 2-*functors* $F,G:\mathfrak{C}\rightarrow\mathfrak{D}$. Here F maps objects $X\in\mathfrak{C}$ to objects $F(X)\in\mathfrak{D}$, and 1-morphisms $f:X\rightarrow Y$ in $\mathfrak{C}$ to 1-morphisms $F(f):F(X)\rightarrow F(Y)$ in $\mathfrak{D}$, and 2-morphisms $\eta:f\Rightarrow g$ in $\mathfrak{C}$ to 2-morphisms $F(\eta):F(f)\Rightarrow F(g)$ in $\mathfrak{D}$.
- 2-*morphisms* of 2-$\mathfrak{Cat}$ are 2-*natural transformations* $\Xi,\Theta:F\Rightarrow G$. Here Ξ maps objects $X\in\mathfrak{C}$ to 1-morphisms $\Xi(X):F(X)\rightarrow G(X)$ in $\mathfrak{D}$, and 1-morphisms $f:X\rightarrow Y$ in $\mathfrak{C}$ to 2-morphisms $\Xi(f):F(f)\Rightarrow G(f)$ in $\mathfrak{D}$.
- 3-*morphisms* of 2-$\mathfrak{Cat}$ are called *modifications* $\aleph:\Xi\Rrightarrow\Theta$. Here $\aleph$ maps objects $X\in\mathfrak{C}$ to 2-morphisms $\aleph(X):\Xi(X)\Rightarrow\Theta(X)$ in $\mathfrak{D}$.

Thr rough idea is a 3-category is like a 2-category but with added 3-morphisms $\aleph : \Xi \Rrightarrow \Theta$ between the 2-morphisms $\Xi, \Theta : f \Rightarrow g$.
The actual axioms and details are horribly complicated.
If $\mathfrak{C}, \mathfrak{D}$ are fixed 2-categories then $\mathrm{Fun}(\mathfrak{C}, \mathfrak{D})$ is a 2-category, with objects 2-functors $F : \mathfrak{C} \to \mathfrak{D}$, and 1-morphisms 2-natural transformations of such F, and 2-morphisms modifications.
We can define *equivalences* of 2-categories: 2-functors $F : \mathfrak{C} \to \mathfrak{D}$ which are invertible up to 2-natural isomorphism. We consider 2-categories $\mathfrak{C}, \mathfrak{D}$ to be 'the same' if they are equivalent.
All this also works for *weak 2-categories*. It is a useful fact that every weak 2-category is equivalent to a strict 2-category.

Outlook on higher categories

The theories of ordinary categories, and 2-categories, are well understood and well behaved. However, there is a *problem* with n-categories for $n \geqslant 3$. For 2-categories we saw that:

- There are theories of strict 2-categories and weak 2-categories.
- 2-categories that arise 'in nature' are often weak 2-categories.
- Every weak 2-category is equivalent to a strict 2-category, so the difference between strict and weak doesn't really matter.

For $n \geqslant 3$, it turns out that there is an essentially unique definition of strict n-category, with nice properties. However, being strict is too restrictive, as it excludes many important examples. There are many different, non-equivalent definitions of weak n-category. Weak n-categories need not be equivalent to strict n-categories. Working with enormous diagrams of k-morphisms for $k = 1, 2, \ldots, n$ gets really long and confusing.

First rough idea of ∞-categories

It turns out that life is much nicer if you skip n-categories for $2 < n < \infty$ and go straight to ∞-categories; especially if you are only interested in $(\infty, 1)$-categories, that is, ∞-categories in which the n-morphisms are all invertible (modulo $n+1$-morphisms) for all $n \geqslant 2$. We will only care about $(\infty, 1)$-categories in this course.

There are several theories of $(\infty, 1)$-categories, which are essentially equivalent. We will discuss them in more detail later in the course. For today, I would like to try and give the rough idea in terms of *categories enriched in topological spaces*.

Categories enriched in topological spaces

Our first model for an $(\infty, 1)$-category is a category $\mathscr{C}$ such that for all objects X, Y in $\mathscr{C}$, the set $\mathrm{Hom}(X, Y)$ of morphisms $f : X \to Y$ is given the structure of a topological space (generally a nice topological space, e.g. Hausdorff, ..., and homotopy equivalent to a CW complex), and for objects X, Y, Z the composition $\mu_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is a continuous map.

We care about the Hom spaces $\mathrm{Hom}(X, Y)$ only *up to homotopy equivalence*. There is no need to require composition $\mu_{X,Y,Z}$ to be strictly associative: for objects W, X, Y, Z we need a homotopy

$$\mathrm{M}_{W,X,Y,Z} : \mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) \to \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z})$$

between the two continuous maps

$$\mathrm{Hom}(W, X) \times \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(W, Z),$$

and then higher homotopies for 5,6,... objects.

We can now define 1-morphisms, 2-morphisms, ... in $\mathscr{C}$:

- A 1-*morphism* $f : X \to Y$ is a point of $\mathrm{Hom}(X, Y)$.
- If $f, g : X \to Y$ are 1-morphisms, a 2-*morphism* $\eta : f \Rightarrow g$ is a continuous path $\eta : [0,1] \to \mathrm{Hom}(X, Y)$ with $\eta(0) = f$ and $\eta(1) = g$. Note that η is invertible with $\eta^{-1}(s) = \eta(1-s)$.
- If $\eta, \zeta : f \Rightarrow g$ are 2-morphisms, a 3-*morphism* $\aleph : \eta \Rrightarrow \zeta$ is a continuous map $\aleph : [0,1]^2 \to \mathrm{Hom}(X, Y)$ such that for $s, t \in [0,1]$
$$\aleph(0,t) = f, \quad \aleph(1,t) = g, \quad \aleph(s,0) = \eta, \quad \aleph(s,1) = \zeta.$$
- *n-morphisms* are continuous maps $[0,1]^{n-1} \to \mathrm{Hom}(X, Y)$ with prescribed boundary conditions on $\partial([0,1]^{n-1})$.
- If $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms, the *vertical composition* $\zeta \odot \eta : f \Rightarrow h$ is $(\zeta \odot \eta)(s) = \eta(2s)$ if $s \in [0, \frac{1}{2}]$ and $(\zeta \odot \eta)(s) = \zeta(2s-1)$ if $s \in [\frac{1}{2}, 1]$. This is not associative, but is associative up to homotopy, i.e. up to 3-isomorphism. Other kinds of composition can be defined in a similar way.

The homotopy category $\mathrm{Ho}(\mathscr{C})$ has the same objects as $\mathscr{C}$, and morphisms $\mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(X, Y) = \pi_0(\mathrm{Hom}_{\mathscr{C}}(X, Y))$, that is, the set of connected components of $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.
A lot of the theory of ∞-categories gets modelled on homotopy theory of topological spaces. For example, the Whitehead Theorem says that if X, Y are connected topological spaces homotopy equivalent to CW complexes and $f : X \to Y$ is continous with $\pi_n(f) : \pi_n(X) \to \pi_n(Y)$ an isomorphism for all $n \geqslant 1$ then f is a homotopy equivalence. Because we only care about $\mathrm{Hom}(X, Y)$ up to homotopy equivalence, the homotopy groups $\pi_n(\mathrm{Hom}(X, Y)_i)$ for each connected component $\mathrm{Hom}(X, Y)_i$ of $\mathrm{Hom}(X, Y)$ (which basically parametrize $(n + 1)$-morphisms in $\mathscr{C}$ modulo $(n + 2)$-morphisms) contain a lot of the important information.

Derived Algebraic Geometry

Lecture 4 of 14: Global quotient stacks and orbifolds

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Summer Term 2022

These slides available at
http://people.maths.ox.ac.uk/$\sim$joyce/

Plan of talk:

Introduction

Our next topic is classical stacks: Deligne–Mumford stacks $\mathbf{DMSta}_{\mathbb{K}}$ and Artin stacks $\mathbf{Art}_{\mathbb{K}}$, which are 2-categories of spaces generalizing schemes $\mathbf{Sch}_{\mathbb{K}}$. There is also a differential-geometric analogue: the 2-category of orbifolds **Orb**, which is to the category of manifolds **Man** as $\mathbf{DMSta}_{\mathbb{K}}$ is to $\mathbf{Sch}_{\mathbb{K}}$.

We may define $\mathbf{DMSta}_{\mathbb{K}}$ and $\mathbf{Art}_{\mathbb{K}}$ as full 2-subcategories of the functor 2-category $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Groupoids})$. That is, a stack X is a functor $X : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$ satisfying a load of complicated conditions. But (I think) this definition is difficult to get your head round, and doesn't give much geometric intuition.

I am going to talk about examples and properties of stacks for a while before giving the formal definition, so that when I do give the definition it may be a bit more motivated. Also I will discuss orbifolds, for which more geometric definitions are available. Today I will talk about *global quotient stacks* $[U/\Gamma]$.

4.1. Global quotient stacks

We will discuss stacks and orbifolds which are *global quotients* $X = [U/\Gamma]$. Here:

- If X is a Deligne–Mumford stack over a field $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$) then U is a $\mathbb{K}$-scheme and Γ is a finite group acting on U.
- If X is an Artin stack over $\mathbb{K}$ then U is a $\mathbb{K}$-scheme and Γ is an algebraic $\mathbb{K}$-group (basically a Lie group over $\mathbb{K}$, e.g. $\mathrm{GL}(n, \mathbb{K})$) acting on U.
- If X is an orbifold then U is a smooth manifold and Γ is a finite group acting smoothly on U.

Roughly, one should expect that a general (nice) stack or orbifold X has an open cover $\{X_i : i \in I\}$ with $X_i \simeq [U_i/\Gamma_i]$, so global quotients are local models for general stacks. This is an over-simplification in the Artin case, but holds for large classes of interesting stacks, e.g. moduli stacks of coherent sheaves.

1- and 2-morphisms of global quotients: first guess

Let $[U/\Gamma],[V/\Delta]$ be global quotients. How should we define 1-morphisms $\phi,\psi:[U/\Gamma]\rightarrow[V/\Delta]$ and 2-morphisms $\eta:\phi\Rightarrow\psi$? Clearly ϕ should induce a map of the orbit spaces $U/\Gamma\rightarrow V/\Delta$. The natural guess is:

Definition 4.1 (Quotient 1- and 2-morphisms)

Suppose $\rho:\Gamma\rightarrow\Delta$ is an (algebraic) group morphism and $f:U\rightarrow V$ is a morphism of schemes or manifolds which is equivariant under $\rho:\Gamma\rightarrow\Delta$, that is, $f(\gamma\, u)=\rho(\gamma)f(u)$ for all $\gamma\in\Gamma$ and $u\in U$. Then $[f/\rho]:[U/\Gamma]\rightarrow[V/\Delta]$ is a 1-*morphism*. Suppose $[f/\rho],[g/\sigma]:[U/\Gamma]\rightarrow[V/\Delta]$ are 1-morphisms. A 2-*morphism* $[\delta]:[f/\rho]\Rightarrow[g/\sigma]$ is $\delta\in\Delta$ such that $g(u)=\delta\, f(u)$ for all $u\in U$ and $\sigma(\gamma)=\delta\rho(\gamma)\delta^{-1}$ for all $\gamma\in\Gamma$. Then

$$g(\gamma\, u)=\delta f(\gamma\, u)=\delta\rho(\gamma)f(u)=\delta\rho(\gamma)\delta^{-1}\,\delta f(u)=\sigma(\gamma)g(u).$$

We define 2-morphisms like this as if such δ exists then the maps of orbit spaces $f/\rho,g/\sigma:U/\Gamma\rightarrow V/\Delta$ are equal.

It is easy to define the various kinds of compositions of 1- and 2-morphisms. For example, given 1-morphisms $[f/\rho]:[U/\Gamma]\to[V/\Delta]$, $[g/\sigma]:[V/\Delta]\to[W/K]$, we define $[g/\sigma]\circ[f/\rho]=[g\circ f/\sigma\circ\rho]:[U/\Gamma]\to[W/K]$. Vertical composition of 2-morphisms is multiplication in the group Δ.

Then we get a *strict 2-category* of quotient stacks $[W/\Gamma]$. All 2-morphisms are 2-isomorphisms, so it is a (2,1)-*category*.

If Δ acts freely on V (basically, if $[V/\Delta]$ is a scheme/manifold rather than a stack/orbifold) and $U\neq\emptyset$ then the condition $g(u)=\delta\, f(u)$ determines δ uniquely if it exists, so there is at most one 2-morphism $[\delta]:[f/\rho]\Rightarrow[g/\sigma]$. In this case, the 2-category behaves like an ordinary category. We only get genuine stacky, 2-categorical behaviour (non-identity 2-morphisms $\delta:[f/\rho]\Rightarrow[f/\rho]$) if Δ does not act freely on V, that is, if points in V have nontrivial stabilizer groups in Δ.

We consider $\mathbb{K}$-schemes (and manifolds) U as examples of global quotients by writing $U=[U/\{1\}]$.

1- and 2-morphisms of global quotients: the right answer

In fact Definition 4.1 is too restrictive: there are more general notions of 1- and 2-morphisms of global quotients

Definition 4.2 (1-morphisms of global quotients)

Let $[U/\Gamma], [W/\Delta]$ be global quotients. A 1-*morphism* $(P, \pi, f) : [U/\Gamma] \to [W/\Delta]$ is a triple (P, π, f) where

- P is a scheme/manifold with an action of $\Gamma \times \Delta$.
- $\pi : P \to U$ is a Γ-equivariant, Δ-invariant smooth map making P into a principal Δ-bundle over U.
- $f : P \to W$ is a smooth Δ-equivariant and Γ-invariant map.

If $[g/\sigma] : [U/\Gamma] \to [W/\Delta]$ is as in Definition 4.1 then we define $P = U \times \Delta$, with $\Gamma \times \Delta$-action $(\gamma, \delta) : (u, \delta') \mapsto (\gamma u, \delta\delta'\sigma(\gamma)^{-1})$, and π, f given by $\pi(u, \delta') = u$, $f(u, \delta') = \delta' g(u)$. Then $(P, \pi, f) : [U/\Gamma] \to [W/\Delta]$ is a 1-morphism as above. As principal bundles are locally trivial, all 1-morphisms are locally of the naïve form in Definition 4.1.

Definition (Continued.)

If $(P,\pi,f),(Q,\pi,g):[U/\Gamma]\rightarrow[V/\Delta]$ are 1-morphisms, a 2-*morphism* $\iota:(P,\pi,f)\Rightarrow(Q,\pi,g)$ is an isomorphism $\iota:P\rightarrow Q$ which preserves $\Gamma\times\Delta$-actions and projections π to U, and has $f=g\circ\iota$.

If $[\delta]:[f,\rho]\Rightarrow[g,\sigma]$ is as in Definition 4.1 and $(P,\pi,f),(Q,\pi,g)$ are defined from $[f,\rho],[g,\sigma]$ as above with $P=Q=U\times\Delta$, we define $\iota:(P,\pi,f)\Rightarrow(Q,\pi,g)$ by $\iota(u,\delta')=(u,\delta'\delta^{-1})$. Then ι is a 2-morphism as above.

It is easy to define the various kinds of compositions of 1- and 2-morphisms. If $(P,\pi,f):[U/\Gamma]\rightarrow[W/\Delta]$ $(Q,\pi,g):[V/\Delta]\rightarrow[W/K]$ are 1-morphisms then composition of 1-morphisms is

$$(Q,\pi,g)\circ(P,\pi,f)=\bigl((P\times_{f,V,\pi}Q)/\Delta,\pi\circ\pi_P,g\circ\pi_Q\bigr).$$

Vertical composition of 2-morphisms is composition of isomorphisms ι. This gives a (2,1)-category of quotients $[U/\Gamma]$.

Points and isotropy groups

If X is a $\mathbb{K}$-scheme then ($\mathbb{K}$-)points x of X are equivalent to morphisms $x : * = \operatorname{Spec}\mathbb{K} \to X$. Similarly, if X is a $\mathbb{K}$-stack then ($\mathbb{K}$-)points $[x]$ of X are 2-isomorphism classes $[x]$ of 1-morphisms $x : * \to X$. For quotient stacks $[U/\Gamma]$, 2-isomorphism classes of 1-morphisms $x : * = [*/\{1\}] \to [U/\Gamma]$ correspond to Γ-orbits $u\Gamma$ in U, for $u : * \to U$. The set of points of X is a topological space, which is the quotient topological space U/Γ for a quotient $[U/\Gamma]$. For stacks (and orbifolds) X, there is an extra piece of data associated to each point $[x]$ in X, the *isotropy group* (or *orbifold group*) $\mathrm{Iso}([x])$, the group of 2-morphisms $\eta : x \Rightarrow x$. This is a finite group for Deligne–Mumford stacks and orbifolds, and an algebraic $\mathbb{K}$-group for Artin stacks. For a Γ-orbit $u\Gamma$ we have $\mathrm{Iso}(u\Gamma) = \mathrm{Stab}_\Gamma(u)$, the stabilizer group of u.

Conclusion: considered as geometric spaces, stacks and orbifolds have an extra geometric structure not present in schemes and manifolds, each point $[x]$ has an *isotropy group* $\mathrm{Iso}([x])$.

4.2. Examples. Classifying stacks of principal bundles

To understand the job that isotropy groups do, consider the quotient $[*/\Gamma]$, which is a single point with isotropy group Γ. Let U be a $\mathbb{K}$-scheme, considered as a trivial quotient stack $[U/\{1\}]$. Then

- 1-morphisms $f : U \to [*/\Gamma]$ are equivalent to principal Γ-bundles $\pi : P \to U$.
- 2-morphisms $\eta : f \Rightarrow g$ of 1-morphisms $f, g : U \to [*/\Gamma]$ are equivalent to isomorphisms $\iota : P \to Q$ of principal Γ-bundles $P, Q \to U$.

Thus $[*/\Gamma]$ is the *classifying stack for principal* Γ*-bundles*, and is sometimes written $B\Gamma$. In the homotopy category $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$, morphisms $U \to [*/\Gamma]$ are equivalent to *isomorphism classes* $[P]$ of principal Γ-bundles $P \to U$, but the homotopy category forgets $\mathrm{Aut}(P)$, which is encoded in the 2-morphisms in $\mathbf{Art}_{\mathbb{K}}$. For the multiplicative group $\mathbb{G}_m$, 1-morphisms $U \to [*/\mathbb{G}_m]$ are equivalent to line bundles $L \to U$.

Example: fibre products of global quotients

All fibre products of Deligne–Mumford and Artin stacks exist; transverse fibre products of orbifolds exist. Consider a fibre product of global quotient stacks with morphisms of the form Definition 4.1. We may write the 2-Cartesian square explicitly as

$$\begin{array}{ccc} \left[\frac{\{(u,v,k)\in U\times V\times K: f(u)=k\,g(v)\}}{\Gamma\times\Delta}\right] & \xrightarrow[{[\pi_V/\pi_\Delta]}]{} & [V/\Delta] \\ \Big\downarrow {[\pi_U/\pi_\Gamma]} & \eta \Uparrow & {[g/\sigma]} \Big\downarrow \\ [U/\Gamma] & \xrightarrow{[f/\rho]} & [W/K]. \end{array}$$

Here in the top left $\Gamma \times \Delta$ acts by $(\gamma, \delta) : (u, v, k) \mapsto (\gamma u, \delta v, \rho(\gamma) k \sigma(\delta)^{-1})$. The 2-morphism η is not of the form in Definition 4.1, and is induced by the morphism $(u, v, k) \mapsto k$.

As a simple example, we have a 2-Cartesian square

$$\begin{array}{ccc} \Gamma & \longrightarrow & * \\ \downarrow & \eta \Uparrow & \downarrow \\ * & \longrightarrow & [*/\Gamma]. \end{array} \qquad (4.1)$$

The corresponding fibre product in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$ is $*$, so fibre products in $\mathrm{Ho}(\mathbf{Art}_{\mathbb{K}})$ and in $\mathbf{Art}_{\mathbb{K}}$ need not coincide.
Given a scheme morphism $f : U \to V$ and a point $v \in V$, the fibre of f over v is $* \times_{v,V,f} U$. Equation (4.1) shows that the fibre of $* \to [*/\Gamma]$ is Γ. In fact $* \to [*/\Gamma]$ is a principal bundle in stacks. If $\pi : P \to U$ is a principal Γ-bundle and $f_P : U \to [*/\Gamma]$ is the associated 1-morphism, we have a 2-Cartesian square

$$\begin{array}{ccc} P & \longrightarrow & * \\ \downarrow{\scriptstyle \pi} & \eta \Uparrow & \downarrow \\ U & \xrightarrow{f_P} & [*/\Gamma]. \end{array}$$

Thus $P \to U$ is the pullback along f_P of the tautological principal γ-bundle $* \to [*/\Gamma]$.

Smooth Artin stacks and dimension

There is a good notion of when an Artin stack X is *smooth*, and smooth stacks have a *dimension* $\dim X$ in $\mathbb{Z}$ (locally constant on each connected component). A quotient stack $[U/\Gamma]$ is smooth if and only if U is a smooth $\mathbb{K}$-scheme, and then the dimension is $\dim[U/\Gamma] = \dim U - \dim \Gamma$. Note in particular that smooth Artin stacks can have negative dimension, e.g. $\dim[*/\mathrm{GL}(n,\mathbb{K})] = -n^2$.

There is a good notion of when a stack 1-morphism $f : X \to Y$ is *smooth*. To be smooth roughly means that the fibres $* \times_{y,Y,f} X$ are smooth stacks for all points y in Y.

Every quotient stack $[U/\Gamma]$ has a natural surjective 1-morphism $U = [U/\{1\}] \to [U/\Gamma]$ with U a scheme. Its fibres are G, which is a smooth scheme, so $U \to [U/\Gamma]$ is smooth.

An important part of the definition of an Artin stack X is that there exists a smooth surjective 1-morphism $U \to X$ (called an *atlas*) with U a scheme. For quotients $[U/\Gamma]$, $U \to [U/\Gamma]$ is an atlas.

Example (Weighted projective spaces)

Let n and $a_0, \ldots, a_n$ be positive integers, with $\mathrm{hcf}(a_0, \ldots, a_n) = 1$. Define the *weighted projective space* $\mathbb{CP}^n_{a_0,\ldots,a_n} = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$\lambda : (z_0, z_1, \ldots, z_n) \longmapsto (\lambda^{a_0} z_0, \ldots, \lambda^{a_n} z_n).$$

Then $\mathbb{CP}^n_{a_0,\ldots,a_n}$ is a Deligne–Mumford stack, or compact complex orbifold. Near $[z_0, \ldots, z_n]$ it is modelled on $\mathbb{C}^n/\mathbb{Z}_k$, where k is the highest common factor of those a_i for $i = 0, \ldots, n$ with $z_i \neq 0$.

Note that although we can write $\mathbb{CP}^n_{a_0,\ldots,a_n}$ as a global quotient $[U/\Gamma] = [(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*]$ in the world of Artin stacks, we generally cannot do so as a Deligne–Mumford stack or orbifold, which requires Γ to be finite. In general the best we can do is to cover $\mathbb{CP}^n_{a_0,\ldots,a_n}$ by $n+1$ open substacks equivalent to $[\mathbb{C}^n/\mathbb{Z}_k]$.

4.3. Sheaves and stacks on topological spaces

Sheaves are a central idea in algebraic geometry.

Definition 4.3

Let X be a topological space. A *presheaf of sets* $\mathcal{E}$ on X consists of a set $\mathcal{E}(U)$ for each open $U \subseteq X$, and a restriction map $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ for all open $V \subseteq U \subseteq X$, such that:

(i) $\mathcal{E}(\emptyset) = *$ is one point;

(ii) $\rho_{UU} = \mathrm{id}_{\mathcal{E}(U)}$ for all open $U \subseteq X$; and

(iii) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for all open $W \subseteq V \subseteq U \subseteq X$.

We call $\mathcal{E}$ a *sheaf* if also whenever $U \subseteq X$ is open and $\{V_i : i \in I\}$ is an open cover of U, then:

(iv) If $s, t \in \mathcal{E}(U)$ with $\rho_{UV_i}(s) = \rho_{UV_i}(t)$ for all $i \in I$, then $s = t$;

(v) If $s_i \in \mathcal{E}(V_i)$ for all $i \in I$ with $\rho_{V_i(V_i \cap V_j)}(s_i) = \rho_{V_j(V_i \cap V_j)}(s_j)$ in $\mathcal{E}(V_i \cap V_j)$ for all $i, j \in I$, then there exists $s \in \mathcal{E}(U)$ with $\rho_{UV_i}(s) = s_i$ for all $i \in I$. This s is unique by (iv).

Definition (Continued.)

Let $\mathcal{E}, \mathcal{F}$ be (pre)sheaves on X. A *morphism* $\phi : \mathcal{E} \to \mathcal{F}$ consists of a map $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ for all open $U \subseteq X$, such that $\rho_{UV} \circ \phi(U) = \phi(V) \circ \rho_{UV} : \mathcal{E}(U) \to \mathcal{F}(V)$ for all open $V \subseteq U \subseteq X$. Then sheaves form a category.

If $\mathscr{C}$ is any category in which direct limits exist, such as the categories of sets, rings, vector spaces, $\mathbb{K}$-algebras, ..., then we can define (pre)sheaves $\mathcal{E}$ of objects in $\mathscr{C}$ on X in the obvious way, and morphisms $\phi : \mathcal{E} \to \mathcal{F}$ by taking $\mathcal{E}(U)$ to be an object in $\mathscr{C}$, and $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$, $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ to be morphisms in $\mathscr{C}$, and $\mathcal{E}(\emptyset)$ to be a terminal object in $\mathscr{C}$ (e.g. the zero ring). Almost any class of functions on X, or sections of a bundle on X, will form a sheaf on X. To be a sheaf means to be 'local on X', determined by its behaviour on any cover of small open sets.

Example: non-locality of morphisms in $\mathrm{Ho}(\mathbf{Orb})$

Example

Let $X = \mathcal{S}^1 \subset \mathbb{R}^2$ and $G = \mathbb{Z}_2$. Then $X = U \cup V$ for $U = \mathcal{S}^1 \setminus \{(1,0)\}$, $V = \mathcal{S}^1 \setminus \{(-1,0)\}$. There are two principal $\mathbb{Z}_2$-bundles on $\mathcal{S}^1$ up to isomorphism, with monodromy 1 and -1 around $\mathcal{S}^1$. But on $U \cong \mathbb{R} \cong V$ there are only one principal $\mathbb{Z}_2$-bundle (the trivial bundle) up to isomorphism.
Therefore morphisms $[\mathfrak{f}] : \mathcal{S}^1 \to [*/\mathbb{Z}_2]$ in $\mathrm{Ho}(\mathbf{Orb})$ are not determined by their restrictions $[\mathfrak{f}]|_U, [\mathfrak{f}]|_V$ for the open cover $\{U, V\}$ of $\mathcal{S}^1$, so such $[\mathfrak{f}]$ do not form a sheaf on $\mathcal{S}^1$.

Regarding X as a quotient $[\mathcal{S}^1/\{1\}]$, this example also shows that morphisms $\mathfrak{f} : [U/\Gamma] \to [V/\Delta]$ in **Orb** or $\mathrm{Ho}(\mathbf{Orb})$ are *not* globally determined by a smooth map $f' : U \to V$ and morphism $\rho : \Gamma \to \Delta$, as $f' : \mathcal{S}^1 \to *$, $\rho : \{1\} \to \mathbb{Z}_2$ are unique, but $\mathfrak{f}$ is not.

Stacks on topological spaces

Definition 4.3 defined *sheaves of sets* $\mathcal{E}$ on a topological space X. There is a parallel notion of 'sheaves of groupoids' on X, which is called a *stack* (or 2-*sheaf*) on X. As sets form a category **Sets**, but groupoids form a 2-category **Groupoids** (in fact, a (2,1)-category), stacks on X are a (2,1)-category generalization of sheaves.

The connection with stacks in algebraic geometry is that both are examples of 'stacks on a site', where here we mean the site of open sets in X, and in algebraic geometry we use the site of $\mathbb{K}$-algebras $\mathbf{Alg}_{\mathbb{K}}$ (or $\mathbf{Sch}_{\mathbb{K}}$), regarded as a kind of generalized topological space.

As for sheaves, we define *prestacks* and *stacks*. Sheaves are presheaves which satisfy a gluing property on open covers $\{V_i : i \in I\}$, involving data on V_i and conditions on double overlaps $V_i \cap V_j$. For the 2-category generalization we need data on $V_i, V_i \cap V_j$ and conditions on triple overlaps $V_i \cap V_j \cap V_k$.

Definition 4.4

Let X be a topological space. A *prestack* (or *prestack in groupoids*, or 2-*presheaf*) $\mathcal{E}$ on X, consists of the data of a groupoid $\mathcal{E}(S)$ for every open set $S \subseteq X$, and a functor $\rho_{ST} : \mathcal{E}(S) \to \mathcal{E}(T)$ called the *restriction map* for every inclusion $T \subseteq S \subseteq X$ of open sets, and a natural isomorphism of functors $\eta_{STU} : \rho_{TU} \circ \rho_{ST} \Rightarrow \rho_{SU}$ for all inclusions $U \subseteq T \subseteq S \subseteq X$ of open sets, satisfying the conditions that:

(i) $\rho_{SS} = \mathrm{id}_{\mathcal{E}(S)} : \mathcal{E}(S) \to \mathcal{E}(S)$ for all open $S \subseteq X$, and $\eta_{SST} = \eta_{STT} = \mathrm{id}_{\rho_{ST}}$ for all open $T \subseteq S \subseteq X$; and

(ii) $\eta_{SUV} \odot (\mathrm{id}_{\rho_{UV}} * \eta_{STU}) = \eta_{STV} \odot (\eta_{TUV} * \mathrm{id}_{\rho_{ST}}) : \rho_{UV} \circ \rho_{TU} \circ \rho_{ST} \Longrightarrow \rho_{SV}$ for all open $V \subseteq U \subseteq T \subseteq S \subseteq X$.

Definition (Continued)

A prestack $\mathcal{E}$ on X is called a *stack* (or *stack in groupoids*, or *2-sheaf*) on X if whenever $S \subseteq X$ is open and $\{T_i : i \in I\}$ is an open cover of S, then (iii)–(v) hold, where:

(iii) If $\alpha, \beta : A \to B$ are morphisms in $\mathcal{E}(S)$ and $\rho_{ST_i}(\alpha) = \rho_{ST_i}(\beta) : \rho_{ST_i}(A) \to \rho_{ST_i}(B)$ in $\mathcal{E}(T_i)$ for all $i \in I$, then $\alpha = \beta$.

(iv) If A, B are objects of $\mathcal{E}(S)$ and $\alpha_i : \rho_{ST_i}(A) \to \rho_{ST_i}(B)$ are morphisms in $\mathcal{E}(T_i)$ for all $i \in I$ with

$$\eta_{ST_i(T_i\cap T_j)}(B) \circ \rho_{T_i(T_i\cap T_j)}(\alpha_i) \circ \eta_{ST_i(T_i\cap T_j)}(A)^{-1}$$
$$= \eta_{ST_j(T_i\cap T_j)}(B) \circ \rho_{T_j(T_i\cap T_j)}(\alpha_j) \circ \eta_{ST_j(T_i\cap T_j)}(A)^{-1}$$

in $\mathcal{E}(T_i \cap T_j)$ for all $i, j \in I$, then there exists $\alpha : A \to B$ in $\mathcal{E}(S)$ (unique by (iii)) with $\rho_{ST_i}(\alpha) = \alpha_i$ for all $i \in I$.

Definition (Continued)

(v) If $A_i \in \mathcal{E}(T_i)$ for $i \in I$ and $\alpha_{ij} : \rho_{T_i(T_i\cap T_j)}(A_i) \to \rho_{T_j(T_i\cap T_j)}(A_j)$ are morphisms in $\mathcal{E}(T_i \cap T_j)$ for $i,j \in I$ with

$$\eta_{T_k(T_j\cap T_k)(T_i\cap T_j\cap T_k)}(A_k)\circ\rho_{(T_j\cap T_k)(T_i\cap T_j\cap T_k)}(\alpha_{jk})\circ\eta_{T_j(T_j\cap T_k)(T_i\cap T_j\cap T_k)}(A_j)^{-1}$$
$$\circ\,\eta_{T_j(T_i\cap T_j)(T_i\cap T_j\cap T_k)}(A_j)\circ\rho_{(T_i\cap T_j)(T_i\cap T_j\cap T_k)}(\alpha_{ij})\circ\eta_{T_i(T_i\cap T_j)(T_i\cap T_j\cap T_k)}(A_i)^{-1}$$
$$=\eta_{T_k(T_i\cap T_k)(T_i\cap T_j\cap T_k)}(A_k)\circ\rho_{(T_i\cap T_k)(T_i\cap T_j\cap T_k)}(\alpha_{ik})\circ\eta_{T_i(T_i\cap T_k)(T_i\cap T_j\cap T_k)}(A_i)^{-1}$$

for all $i,j,k \in I$, then there exist an object A in $\mathcal{E}(S)$ and morphisms $\beta_i : A_i \to \rho_{ST_i}(A)$ for $i \in I$ such that for all $i,j \in I$ we have

$$\eta_{ST_i(T_i\cap T_j)}(A)\circ\rho_{T_i(T_i\cap T_j)}(\beta_i) = \eta_{ST_j(T_i\cap T_j)}(A)\circ\rho_{T_j(T_i\cap T_j)}(\beta_j)\circ\alpha_{ij}.$$

If we wanted to define stacks or orbifolds as ‘topological space with geometric structure’ (like defining schemes as topological spaces with sheaves of rings) then the geometric structure needs to be written in terms of stacks on your topological space.