Derived Algebraic Geometry

Lecture 7 of 14: Triangulated categories and derived categories I

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Helpful references for this lecture:

R.P. Thomas, *'Derived categories for the working mathematician'*, math.AG/0001045.

S.I. Gelfand and Y.I. Manin, *Methods of Homological Algebra*, 2003.

These slides available at

`http://people.maths.ox.ac.uk/~joyce/`

Plan of talk:

Given an abelian category $\mathcal{A}$, one can define its (*bounded*) *derived category* $D^b\mathcal{A}$, which is a *triangulated category*, a class of categories satisfying axioms like an abelian category, but with a different notion of exact sequence. The objects of $D^b\mathcal{A}$ are complexes in $\mathcal{A}$ with cohomology in bounded degrees, but the morphisms in $D^b\mathcal{A}$ are not morphisms of complexes: they are obtained from morphisms of complexes by inverting 'quasi-isomorphisms'. The *unbounded derived category* $D\mathcal{A}$ allows complexes in $\mathcal{A}$ with cohomology in all degrees.

Derived categories of coherent sheaves $D^b\,\mathrm{coh}(X)$ are very important. They are better behaved than $\mathrm{coh}(X)$ in some ways (functors on $D^b\,\mathrm{coh}(X)$ are often exact, when the corresponding functor on $\mathrm{coh}(X)$ is only left or right exact). They are also central to Derived Algebraic Geometry. Many objects in DAG live in categories obtained by inverting quasi-isomorphisms. For example, if $\boldsymbol{X}$ is a derived stack the tangent complex $\mathbb{T}_{\boldsymbol{X}}$ and cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ (the analogues of TX and T^*X for X a manifold) lie in derived categories, essentially $D\,\mathrm{coh}(\boldsymbol{X})$.

7.1. Definition and motivation of derived categories. The category of complexes

Definition

Let $\mathcal{A}$ be an abelian category, for example $\mathcal{A} = \mathrm{coh}(X)$ for X a smooth projective $\mathbb{K}$-scheme. A *complex* $E^\bullet = (E^*, \mathrm{d})$ in $\mathcal{A}$ is a family $(E^k)_{k\in\mathbb{Z}}$ of objects in $\mathcal{A}$, and morphisms $\mathrm{d} = \mathrm{d}^k : E^k \to E^{k+1}$ in $\mathcal{A}$ for $k \in \mathbb{Z}$ such that $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0 : E^k \to E^{k+2}$ for all k.

If $E^\bullet, F^\bullet$ are complexes, a *morphism of complexes* $\phi : E^\bullet \to F^\bullet$ is morphisms $\phi^k : E^k \to F^k$ for all $k \in \mathbb{Z}$ such that $\mathrm{d}^k \circ \phi^k = \phi^{k+1} \circ \mathrm{d}^k : E^k \to F^{k+1}$ for all k.

Write $\mathrm{Com}(\mathcal{A})$ for the (abelian) category of complexes in $\mathcal{A}$.

There is an inclusion $\mathcal{A} \hookrightarrow \mathrm{Com}(\mathcal{A})$ mapping $E \in \mathcal{A}$ to the complex $E^\bullet$ with $E^0 = E$ and $E^k = 0$ for $k \neq 0$. This identifies $\mathcal{A}$ with a full subcategory of $\mathrm{Com}(\mathcal{A})$.

Actually, this definition of morphism of complexes is wrong for some purposes. Let $\phi, \tilde{\phi} : E^\bullet \to F^\bullet$ be morphisms as above. We say that $\phi, \tilde{\phi}$ are *equivalent*, written $\phi \sim \tilde{\phi}$, if there exist $\psi^k : E^k \to F^{k-1}$ for $k \in \mathbb{Z}$ with $\tilde{\phi}^k = \phi^k + \mathrm{d}^{k-1} \circ \psi^k + \psi^{k+1} \circ \mathrm{d}^k$ for all k. Write $[\phi]$ for the $\sim$-equivalence class of ϕ. The *homotopy category* $\mathrm{Ho}(\mathrm{Com}(\mathcal{A}))$ has the same objects as $\mathrm{Com}(\mathcal{A})$ and morphisms $\sim$-equivalence classes $[\phi] : \mathcal{E}^\bullet \to F^\bullet$. There is an obvious functor $\mathrm{Com}(\mathcal{A}) \to \mathrm{Ho}(\mathrm{Com}(\mathcal{A}))$ mapping $E^\bullet \mapsto E^\bullet$ and $\phi \mapsto [\phi]$.

In fact $\mathrm{Ho}(\mathrm{Com}(\mathcal{A}))$ is already a triangulated category, but it is not as interesting as the derived category $D(\mathcal{A})$.

Write $\mathrm{Com}^b(\mathcal{A})$ for the full subcategory of $\mathrm{Com}(\mathcal{A})$ of *bounded complexes* $E^\bullet$ such that $E^k = 0$ for $|k| \gg 0$, that is, $E^k \neq 0$ for only finitely many k. Also write $\mathrm{Com}^+(\mathcal{A})$ for $E^\bullet$ with $E^k = 0$ for $k \ll 0$, and $\mathrm{Com}^-(\mathcal{A})$ for $E^\bullet$ with $E^k = 0$ for $k \gg 0$.

Definition

Let $E^\bullet$ be a complex in $\mathcal{A}$, and $k \in \mathbb{Z}$. Form a commutative diagram in the abelian category $\mathcal{A}$, with the top row exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}\,\mathrm{d}^{k-1} & \longrightarrow & \mathrm{Ker}\,\mathrm{d}^k & \longrightarrow & H^k(E^\bullet) & \longrightarrow & 0 \\ & & \uparrow & \searrow\!\!\!\nearrow & \downarrow & & & & \\ \cdots & \xrightarrow[\mathrm{d}^{k-2}]{} & E^{k-1} & \xrightarrow[\mathrm{d}^{k-1}]{} & E^k & \xrightarrow[\mathrm{d}^{k}]{} & E^{k+1} & \xrightarrow[\mathrm{d}^{k+1}]{} & \cdots . \end{array}$$

Here $\mathrm{Im}\,\mathrm{d}^{k-1} = \mathrm{Ker}(E^k \to \mathrm{Coker}\,\mathrm{d}^{k-1})$. Then d^{k-1} factors uniquely via $\mathrm{Im}\,\mathrm{d}^{k-1}$. As $\mathrm{d}^k \circ \mathrm{d}^{k-1} = 0$, the universal property of $\mathrm{Ker}\,\mathrm{d}^k$ shows d^{k-1} and $\mathrm{Im}\,\mathrm{d}^{k-1} \to E^k$ factor uniquely via $\mathrm{Ker}\,\mathrm{d}^k$. The *cohomology* $H^k(E^\bullet)$ is the cokernel of $\mathrm{Im}\,\mathrm{d}^{k-1} \to \mathrm{Ker}\,\mathrm{d}^k$. It is an object in $\mathcal{A}$, unique up to canonical isomorphism.

First definition of derived categories

Definition

If $\phi : E^\bullet \to F^\bullet$ is a morphism in $\mathrm{Com}(\mathcal{A})$, there are natural morphisms $H^k(\phi) : H^k(E^\bullet) \to H^k(F^\bullet)$ for $k \in \mathbb{Z}$. We call ϕ a *quasi-isomorphism* if $H^k(\phi)$ is an isomorphism for all $k \in \mathbb{Z}$. Write $\mathcal{Q}$ for the family of all quasi-isomorphisms in $\mathrm{Com}(\mathcal{A})$.

Definition 7.1

The *derived category* $D(\mathcal{A}) = \mathrm{Com}(\mathcal{A})[\mathcal{Q}^{-1}]$ is the localization of $\mathrm{Com}(\mathcal{A})$ at the quasi-isomorphisms $\mathcal{Q}$. That is, $D(\mathcal{A})$ is a category with a functor $\Pi : \mathrm{Com}(\mathcal{A}) \to D(\mathcal{A})$ such that if $\phi \in \mathcal{Q}$ then $\Pi(\phi)$ is an isomorphism in $D(\mathcal{A})$, and $D(\mathcal{A})$ has the universal property that if $\Pi' : \mathrm{Com}(\mathcal{A}) \to \mathscr{C}$ is a functor such that if $\phi \in \mathcal{Q}$ then $\Pi'(\phi)$ is an isomorphism in $\mathscr{C}$, then there is a functor $F : D(\mathcal{A}) \to \mathscr{C}$ and a natural isomorphism $\eta : \Pi' \Rightarrow F \circ \Pi$. Similarly, $D^b(\mathcal{A}) = \mathrm{Com}^b(\mathcal{A})[\mathcal{Q}^{-1}]$ and $D^\pm(\mathcal{A}) = \mathrm{Com}^\pm(\mathcal{A})[\mathcal{Q}^{-1}]$.

One can show that the localization $D(\mathcal{A}) = \mathrm{Com}(\mathcal{A})[\mathcal{Q}^{-1}]$ exists and is a triangulated category. We can and do take $D(\mathcal{A})$ to have the same objects as $\mathrm{Com}(\mathcal{A})$. Also $D(\mathcal{A}) \simeq \mathrm{Ho}(\mathrm{Com}(\mathcal{A}))[\mathcal{Q}^{-1}]$.

Problem

Definition 7.1 tells us almost nothing useful about what the morphism sets $\mathrm{Hom}_{D(\mathcal{A})}(E^\bullet, F^\bullet)$ actually are.

In principle, morphisms $\tilde{\phi} : E^\bullet \to F^\bullet$ in $D(\mathcal{A})$ can be constructed as equivalence classes of diagrams in $\mathrm{Com}(\mathcal{A})$:

$$\begin{array}{ccccccccccc} & B_1^\bullet & & B_2^\bullet & & \cdots & & B_k^\bullet & \\ {\scriptstyle q_1}\swarrow\nearrow{\scriptstyle q_1^{-1}} & & \searrow{\scriptstyle \phi_1}\ {\scriptstyle q_2}\swarrow\nearrow{\scriptstyle q_2^{-1}} & & \searrow{\scriptstyle \phi_2} & & {\scriptstyle q_k}\swarrow\nearrow{\scriptstyle q_k^{-1}} & & \searrow{\scriptstyle \phi_k} \\ E^\bullet = A_0^\bullet & & A_1^\bullet & & A_2^\bullet & \cdots & A_{k-1}^\bullet & & A_k^\bullet = F^\bullet, \end{array}$$

where the q_i are quasi-isomorphisms and the inverses q_i^{-1} need not actually exist; but this is not very helpful. There are techniques which do give a good understanding of the morphisms in $D(\mathcal{A})$.

7.2. Derived functors, mapping cones, and triangles

There are many natural examples of abelian categories $\mathcal{A}, \mathcal{B}$ and functors $F : \mathcal{A} \to \mathcal{B}$ such that F does not take (short) exact sequences $0 \to U \to V \to W \to 0$ in $\mathcal{A}$ to (short) exact sequences $0 \to F(U) \to F(V) \to F(W) \to 0$ in $\mathcal{B}$. That is, F is not an *exact functor*. Often exactness fails only at $F(W)$ (i.e. $F(V) \to F(W)$ may not be surjective), when F is called *left exact*, or only at $F(U)$, when F is called *right exact*.

Example

(a) Let X be a $\mathbb{K}$-scheme. The *global sections functor* $\Gamma : \mathrm{coh}(X) \to \mathrm{Vect}_{\mathbb{K}}$ mapping $\mathcal{E} \mapsto \mathcal{E}(X)$ is left exact, but not exact.
(b) Let X be a $\mathbb{K}$-scheme and $H \in \mathrm{coh}(X)$. Then the functor $- \otimes H : \mathrm{coh}(X) \to \mathrm{coh}(X)$ mapping $E \mapsto E \otimes H$ is right exact, but not exact in general, though it is exact if H is a vector bundle. If $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles, then $0 \to E \otimes H \to F \otimes H \to G \otimes H \to 0$ is exact for general H.

Example 7.2

Let $f : X \to Y$ be a $\mathbb{K}$-scheme morphism. Then the pullback functor $f^* : \mathrm{coh}(Y) \to \mathrm{coh}(X)$ is right exact, but not exact in general. However, f^* does take exact sequences of vector bundles to exact sequences of vector bundles.

Consider the exact sequence in $\mathrm{coh}(\mathbb{CP}^1)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^1} \xrightarrow{\;y\;} \mathcal{O}_{\mathbb{CP}^1}(1) \longrightarrow \mathcal{O}_{[1,0]} \longrightarrow 0. \tag{7.1}$$

Let $f : * = \mathrm{Spec}\,\mathbb{C} \to \mathbb{CP}^1$ map $f(*) = [1,0]$. Then f^* of (7.1) is

$$0 \longrightarrow \mathcal{O}_* \xrightarrow{\;0\;} \mathcal{O}_* \xrightarrow{\;\mathrm{id}\;} \mathcal{O}_* \longrightarrow 0,$$

which is right exact, but not exact.

In the last two examples, although F is not exact, its restriction to the subcategory $\mathrm{Vect}(Y) \subset \mathrm{coh}(Y)$ maps exact sequences to exact sequences. We say the subcategory $\mathrm{Vect}(Y)$ is *adapted to* the functor F.

Left and right derived functors

We use Example 7.2 to illustrate the idea of *derived functor*. Take Y to be a projective $\mathbb{K}$-scheme. Then:

(i) $f^*:\mathrm{coh}(Y)\to\mathrm{coh}(X)$ is right exact.

(ii) We have an additive subcategory $\mathrm{Vect}(Y)\subset\mathrm{coh}(Y)$ such that $f^*|_{\mathrm{Vect}(Y)}$ preserves exact sequences.

(iii) As Y is projective, for every object $E\in\mathrm{coh}(Y)$ there exists a surjective morphism $\phi:E'\to E$ with $E'\in\mathrm{Vect}(Y)$.

Using these properties, we will explain how to define the *left derived functors* $L^kf^*:\mathrm{coh}(Y)\to\mathrm{coh}(X)$ for $k=1,2,\ldots$, which have the property that if $0\to E\to F\to G\to 0$ is an exact sequence in $\mathrm{coh}(Y)$ then the following is exact in $\mathrm{coh}(X)$:

$$\cdots\to L^2f^*(G)\to L^1f^*(E)\to L^1f^*(F)\to L^1f^*(G)\to f^*(E)\to f^*(F)\to f^*(G)\to 0.$$

Thus the L^if^* measure the failure of f^* to be exact.

Let $E \in \mathrm{coh}(Y)$. By (iii) we can choose $\mathcal{E}^0 \in \mathrm{Vect}(Y)$ and surjective $\mathrm{d}^0 = \phi^0 : \mathcal{E}^0 \to E$. Next we choose $\mathcal{E}^{-1} \in \mathrm{Vect}(Y)$ and surjective $\pi^{-1} : \mathcal{E}^{-1} \to \mathrm{Ker}\,\mathrm{d}^0$. Let $\mathrm{d}^{-1} : \mathcal{E}^{-1} \to \mathcal{E}^0$ be the composition $\mathcal{E}^{-1} \xrightarrow{\pi^{-1}} \mathrm{Ker}\,\mathrm{d}^0 \hookrightarrow \mathcal{E}^0$. Then $\mathrm{d}^0 \circ \mathrm{d}^{-1} = 0$, with $\mathrm{Im}\,\mathrm{d}^{-1} = \mathrm{Ker}\,\mathrm{d}^0$. By induction we choose $\mathcal{E}^k \in \mathrm{Vect}(Y)$ and $\mathrm{d}^k : \mathcal{E}^k \to \mathcal{E}^{k+1}$ for $k = -1, -2, \ldots$, such that $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$, with $\mathrm{Im}\,\mathrm{d}^k = \mathrm{Ker}\,\mathrm{d}^{k+1}$. This gives an exact sequence in $\mathrm{coh}(Y)$

$$\cdots \xrightarrow{\mathrm{d}^{-3}} \mathcal{E}^{-2} \xrightarrow{\mathrm{d}^{-2}} \mathcal{E}^{-1} \xrightarrow{\mathrm{d}^{-1}} \mathcal{E}^0 \xrightarrow{\phi^0} E \longrightarrow 0.$$

We rewrite this as the diagram in $\mathrm{Com}^-(\mathrm{coh}(Y))$:

$$\begin{array}{ccccccccccccccc}
\mathcal{E}^\bullet = \bigl(\cdots & \xrightarrow{\mathrm{d}^{-3}} & \mathcal{E}^{-2} & \xrightarrow{\mathrm{d}^{-2}} & \mathcal{E}^{-1} & \xrightarrow{\mathrm{d}^{-1}} & \mathcal{E}^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots\bigr) \\
\downarrow \phi & & \downarrow 0 & & \downarrow 0 & & \downarrow \phi^0 & & \downarrow 0 & & \downarrow 0 & & \\
E = \bigl(\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & E & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots\bigr)
\end{array}$$

Then $\mathcal{E}^\bullet$ is a complex of objects in $\mathrm{Vect}(Y)$, and $\phi : \mathcal{E}^\bullet \to E$ is a quasi-isomorphism (regarding E as an object in $\mathrm{Com}(\mathrm{coh}(Y))$), as $\mathcal{E}^\bullet, E$ both have cohomology E in degree 0 and 0 otherwise.

Now consider the complex in $\mathrm{coh}(X)$:

$$f^*(\mathcal{E}^\bullet) = \left(\cdots \xrightarrow{f^*(\mathrm{d}^{-3})} f^*(\mathcal{E}^{-2}) \xrightarrow{f^*(\mathrm{d}^{-2})} f^*(\mathcal{E}^{-1}) \xrightarrow{f^*(\mathrm{d}^{-1})} f^*(\mathcal{E}^{0}) \longrightarrow 0 \longrightarrow \cdots\right).$$

Define $L^k f^*(E) = H^{-k}(f^*(\mathcal{E}^\bullet))$. It turns out that this is independent of the choice of $\mathcal{E}^\bullet$, with $L^0 f^*(E) = f^*(E)$, and extends to $L^k f^* : \mathrm{coh}(Y) \to \mathrm{coh}(X)$ with the claimed properties.

Principle

It is often helpful to replace $E^\bullet \in \mathrm{Com}(\mathcal{A})$ by a quasi-isomorphic object $\mathcal{E}^\bullet \in \mathrm{Com}(\mathcal{A})$, such that the $\mathcal{E}^k$ all have a special property.

It turns out there is a *derived functor* $Lf^* : D^- \mathrm{coh}(Y) \to D^- \mathrm{coh}(X)$ such that for $E \in \mathrm{coh}(Y) \subset D^- \mathrm{coh}(Y)$ we have $Lf^*(E) = f^*(\mathcal{E}^\bullet)$, so that $L^k f^*(E) = H^k(Lf^*(E))$. On any $\mathcal{E}^\bullet \in D^- \mathrm{coh}(Y)$ with $\mathcal{E}^k \in \mathrm{Vect}(Y)$ for all k we may define $Lf^*(\mathcal{E}^\bullet) = f^*(\mathcal{E}^\bullet)$. Moreover, Lf^* is an *exact functor* of triangulated categories. Thus Lf^* fixes the failure of exactness of $f^* : \mathrm{coh}(Y) \to \mathrm{coh}(X)$.

Mapping cones and triangles in $\mathrm{Ho}(\mathrm{Com}(\mathcal{A}))$

If $E^\bullet$ is an object in $\mathrm{Com}(\mathcal{A})$ and $l \in \mathbb{Z}$, we define $E^\bullet[l]$ to be the same complex shifted l places to the left and with morphisms multiplied by $(-1)^l$, that is, $(E^\bullet[l])^k = E^{k+l}$ and $(\mathrm{d}[l])^k = (-1)^l \mathrm{d}^{k+l}$. This defines an equivalence of categories $[l] : \mathrm{Com}(\mathcal{A}) \to \mathrm{Com}(\mathcal{A})$. We call $[1]$ the *translation functor*.

Let $\phi : E^\bullet \to F^\bullet$ be a morphism in $\mathrm{Com}(\mathcal{A})$. The *mapping cone* $C(\phi)$ is the object in $\mathrm{Com}(\mathcal{A})$ with

$$C(\phi)^k = E^{k+1} \oplus F^k, \;\; \mathrm{d}^k_{C(\phi)} = \begin{pmatrix} -\mathrm{d}^{k+1}_{E^\bullet} & 0 \\ \phi^{k+1} & \mathrm{d}^k_{F^\bullet} \end{pmatrix}.$$

Define morphisms $i : F^\bullet \to C(\phi)$ and $\pi : C(\phi) \to E^\bullet[1]$ by $i^k = \begin{pmatrix} 0 & \mathrm{id}_{F^k} \end{pmatrix}$ and $\pi^k = \begin{pmatrix} \mathrm{id}_{E^{k+1}} \\ 0 \end{pmatrix}$. Then we have an exact sequence in the abelian category $\mathrm{Com}(\mathcal{A})$:

$$0 \longrightarrow F^\bullet \xrightarrow{\;i\;} C(\phi) \xrightarrow{\;\pi\;} E^\bullet[1] \longrightarrow 0.$$

Consider the long sequence in $\mathrm{Com}(\mathcal{A})$:

$$\cdots \longrightarrow E^\bullet \xrightarrow{\phi} F^\bullet \xrightarrow{i} C(\phi) \xrightarrow{\pi} E^\bullet[1] \xrightarrow{\phi[1]} F^\bullet[1] \xrightarrow{i[1]} C(\phi)[1] \longrightarrow \cdots$$

This is not a complex in $\mathrm{Com}(\mathcal{A})$: in general we have $\phi \circ i \neq 0$ and $\phi[1] \circ \pi \neq 0$. However, we do have $\phi \circ i \sim 0$ (take $\psi^k = \mathrm{id}_{E^k}$) and $\phi[1] \circ \pi \sim 0$. Thus, when we pass to the homotopy category $\mathrm{Ho}(\mathrm{Com}(\mathcal{A}))$ we have $[\phi] \circ [i] = 0$ and $[\phi][1] \circ [\pi] = 0$, so the following is a complex in $\mathrm{Ho}(\mathrm{Com}(\mathcal{A}))$:

$$\cdots \to E^\bullet \xrightarrow{[\phi]} F^\bullet \xrightarrow{[i]} C(\phi) \xrightarrow{[\pi]} E^\bullet[1] \xrightarrow{[\phi][1]} F^\bullet[1] \xrightarrow{[i][1]} C(\phi)[1] \to \cdots . \quad (7.2)$$

This is an example of a *distinguished triangle* in a triangulated category, which is the analogue of a short exact sequence $0 \to E \to F \to G \to 0$ in an abelian category. But triangulated categories have a cyclic symmetry: $E^\bullet \to F^\bullet \to C(\phi)$, and $F^\bullet \to C(\phi) \to E^\bullet[1]$, and $C(\phi) \to E^\bullet[1] \to F^\bullet[1]$, are on the same level.

7.3. Triangulated categories

Triangulated categories are a class of categories with extra structure, like abelian categories. Under good conditions, the derived categories $D(\mathcal{A}), D^b(\mathcal{A}), D^\pm(\mathcal{A})$ of an abelian category $\mathcal{A}$ are triangulated categories. The definition is not obvious.

Definition 7.3

A *triangulated category* is an additive category $\mathcal{T}$ equipped with the extra data:

(a) A strict isomorphism $\Sigma : \mathcal{T} \to \mathcal{T}$ called the *shift functor*. Then $\Sigma^n : \mathcal{T} \to \mathcal{T}$ is defined for $n \in \mathbb{Z}$, and we write $\Sigma^n = [n]$.

(b) A class of *distinguished triangles* (X, Y, Z, u, v, w), which are diagrams $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ in $\mathcal{T}$ with $v \circ u = 0$, $w \circ v = 0$, and $u[1] \circ w = 0$.

[Definition continues on next slide.]

Definition (Continued.)

These must satisfy the properties:

(i) For each $X \in \mathcal{T}$, $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{0} 0 \xrightarrow{0} X[1]$ is distinguished.

(ii) For each morphism $u : X \to Y$ in $\mathcal{T}$ there is a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$. We call Z the *cone* $C(u)$.

(iii) Distinguished triangles are closed under isomorphisms.

(iv) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished, then so are the rotated triangles $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ and $Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{u} Y \xrightarrow{v} Z$.

(v) Suppose we are given a diagram of morphisms '$\to$'

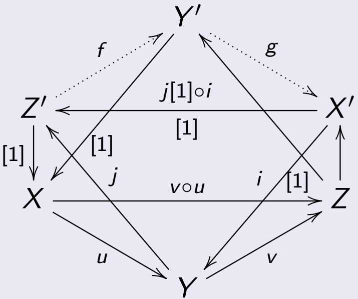
$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \vdots\, h & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

with distinguished rows, such that the left square commutes. Then there exists h making the whole diagram commute.

Aside: in (v) we do *not* require h to be unique.

Definition (Continued.)

(vi) (The *octahedral axiom*.) Given a diagram of morphisms '$\rightarrow$'



such that the faces XYZ', YZX' and XZY' are distinguished and the faces XYZ, $X'YZ'$ commute, there exist f, g as shown such that $X'Y'Z'$ is distinguished and $Z'Y'X$, $ZY'X'$ commute.

Remarks

- When $\mathcal{T} = D(\mathcal{A})$, the shift functor Σ shifts complexes left by one, and distinguished triangles come from mapping cones (7.2) as in §7.2.
- Distinguished triangles are a kind of mix of short and long exact sequences. For $\mathcal{A} \subset D(\mathcal{A})$, if $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is exact in $\mathcal{A}$, then $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished in $D(\mathcal{A})$, where $w \in \mathrm{Ext}^1(Z, X)$ classifies the short exact sequence. Also, if $E^\bullet \xrightarrow{u} F^\bullet \xrightarrow{v} G^\bullet \xrightarrow{w} E^\bullet[1]$ is distinguished in $D(\mathcal{A})$, taking cohomology of complexes gives a long exact sequence in $\mathcal{A}$:

$$\cdots \to H^k(E^\bullet) \to H^k(F^\bullet) \to H^k(G^\bullet) \to H^k(E^\bullet[1]) = H^{k+1}(E^\bullet) \to \cdots .$$

- In (v), h is not unique, as we can replace it by $h' = h + v' \circ x \circ w$ for any $x : X[1] \to Y'$. So in the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \longrightarrow & C(u) & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \downarrow C(u,u',f,g) & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \longrightarrow & C(u') & \longrightarrow & X'[1] \end{array}$$

there is no canonical morphism $C(u, u', f, g)$. This is known as *nonfunctoriality of the cone*. It is a sign we need ∞-categories.

Derived Algebraic Geometry

Lecture 8 of 14: Triangulated categories and derived categories II

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Summer Term 2022

Helpful references for this lecture:
S.I. Gelfand and Y.I. Manin, *Methods of Homological Algebra*, 2003.
D. Huybrechts, *'Fourier–Mukai transforms in Algebraic Geometry'*, 2006.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

8.1. Basic ideas on derived categories
Projective/injective objects and resolutions

Definition

Let $\mathcal{A}$ be an abelian category. An object P in $\mathcal{A}$ is called *projective* if given any surjective $\phi : X \twoheadrightarrow Y$ and any $\psi : P \to Y$ in $\mathcal{A}$, there exists $\bar{\psi} : P \to X$ with $\psi = \phi \circ \bar{\psi}$, in a diagram

$$\begin{array}{ccc} & & X \\ & \overset{\bar{\psi}}{\nearrow} & \big\downarrow \phi \\ P & \xrightarrow{\psi} & Y. \end{array}$$

We say that $\mathcal{A}$ has *enough projectives* if for all $X \in \mathcal{A}$ there exist a projective object P and a surjective morphism $\pi : P \twoheadrightarrow X$.

Similarly, $I \in \mathcal{A}$ is *injective* if given any injective $\phi : Y \hookrightarrow X$ and any $\psi : Y \to I$ in $\mathcal{A}$, there exists $\bar{\psi} : X \to I$ with $\psi = \bar{\psi} \circ \phi$. We say that $\mathcal{A}$ has *enough injectives* if for all $X \in \mathcal{A}$ there exist an injective object P and an injective morphism $\iota : X \hookrightarrow I$.

These are dual concepts (pass to opposite category, reverse arrows).

Definition

Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object $X \in \mathcal{A}$ has a *projective resolution*, an exact sequence

$$\cdots \longrightarrow P^{-2} \xrightarrow{\mathrm{d}^{-2}} P^{-1} \xrightarrow{\mathrm{d}^{-1}} P^0 \xrightarrow{\phi^0} X \longrightarrow 0, \qquad (8.1)$$

with all P^i projective. To choose such a resolution, choose surjective $\phi^0 : P^0 \twoheadrightarrow X$ with P^0 projective, then choose surjective $\mathrm{d}^{-1} : P^{-1} \twoheadrightarrow \mathrm{Ker}\,\phi$ with P^{-1} projective, and so on by induction. We rewrite (8.1) as a diagram in $\mathrm{Com}^-(\mathcal{A})$:

$$\begin{array}{ccccccccccccc} P^\bullet = (\cdots & \xrightarrow{\mathrm{d}^{-3}} & P^{-2} & \xrightarrow{\mathrm{d}^{-2}} & P^{-1} & \xrightarrow{\mathrm{d}^{-1}} & P^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots) \\ \downarrow\phi & & \downarrow 0 & & \downarrow 0 & & \downarrow \phi^0 & & \downarrow 0 & & \downarrow 0 & & \\ X = (\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots). \end{array}$$

Then $P^\bullet$ is a *projective complex* (a complex all of whose objects are projective), and ϕ is a quasi-isomorphism.
Similarly, if $\mathcal{A}$ has enough injectives then every $X \in \mathcal{A}$ has an *injective resolution* $I^\bullet = (I^0 \to I^1 \to \cdots)$ in $\mathrm{Com}^+(\mathcal{A})$ with a quasi-isomorphism $\iota : X \to I^\bullet$.

Proposition 8.1

Let $\mathcal{A}$ be an abelian category with enough projectives. Write $\mathrm{Com}^-(\mathcal{A})_{\mathrm{proj}}, D^-(\mathcal{A})_{\mathrm{proj}}$ for the full subcategories of $\mathrm{Com}^-(\mathcal{A}), D^-(\mathcal{A})$ whose objects are projective complexes. Then:

(a) *Every object of $D^-(\mathcal{A})$ is isomorphic to a projective complex. Hence $D^-(\mathcal{A})_{\mathrm{proj}} \hookrightarrow D^-(\mathcal{A})$ is an equivalence of categories.*

(b) *The functor $\mathrm{Ho}(\mathrm{Com}^-(\mathcal{A})_{\mathrm{proj}}) \to D^-(\mathcal{A})_{\mathrm{proj}}$ induces bijections on all Hom groups $\mathrm{Hom}(P^\bullet, Q^\bullet)$, and so is a strict isomorphism of categories. Thus $\mathrm{Ho}(\mathrm{Com}^-(\mathcal{A})_{\mathrm{proj}})$ is an equivalent category to $D^-(\mathcal{A})$.*

The dual statement is true for injective resolutions.

In §7.1 we noted that the definitions of $D(\mathcal{A}), D^\pm(\mathcal{A}), D^b(\mathcal{A})$ by localizing quasi-isomorphisms tells us almost nothing useful about what the morphism sets $\mathrm{Hom}_{D^*(\mathcal{A})}(E^\bullet, F^\bullet)$ actually are. But if $\mathcal{A}$ has enough projectives then we can replace $E^\bullet, F^\bullet$ by quasi-isomorphic projective resolutions $\tilde{E}^\bullet, \tilde{F}^\bullet$, and then $\mathrm{Hom}_{D^-(\mathcal{A})}(E^\bullet, F^\bullet) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Com}^-(\mathcal{A}))}(\tilde{E}^\bullet, \tilde{F}^\bullet)$.

Example 8.2

(a) The abelian category **Ab** of abelian groups has enough projectives. The projective objects are free abelian groups. Also **Ab** has enough injectives, and $G \in \mathbf{Ab}$ is injective iff multiplication by $0 \neq m \in \mathbb{Z}$ is surjective $m : G \twoheadrightarrow G$, e.g. $G = \mathbb{Q}/\mathbb{Z}$ is injective.

(b) The category R-mod of left modules over a ring or $\mathbb{K}$-algebra R has enough projectives and injectives.

(c) Let X be a noetherian scheme. In general $\mathrm{coh}(X), \mathrm{qcoh}(X)$ do not have enough projectives, and $\mathrm{coh}(X)$ not enough injectives, but $\mathrm{qcoh}(X)$ has enough injectives.

Because of this, a good way to study $D^b \mathrm{coh}(X), D^+ \mathrm{coh}(X)$ is to embed them in $D^+\mathrm{qcoh}(X)$, and use injectives in $\mathrm{qcoh}(X)$. For some purposes you can use vector bundles in $\mathrm{coh}(X)$ as like projective objects.

Ext groups and morphisms in $D^b \operatorname{coh}(X)$

Let X be a projective $\mathbb{K}$-scheme. Then for E, F in $\operatorname{coh}(X)$ and $i \geqslant 0$ one can define *Ext groups* $\operatorname{Ext}^i(E, F)$, finite-dimensional $\mathbb{K}$-vector spaces with $\operatorname{Hom}(E, F) = \operatorname{Ext}^0(E, F)$. They have the property that if $0 \to E \to F \to G \to 0$ is a short exact sequence in $\operatorname{coh}(X)$ and $H \in \operatorname{coh}(X)$, there are long exact sequences

$$0 \longrightarrow \operatorname{Ext}^0(H, E) \to \operatorname{Ext}^0(H, F) \to \operatorname{Ext}^0(H, G) \to \operatorname{Ext}^1(H, E) \to \cdots,$$

$$\cdots \to \operatorname{Ext}^1(E, H) \to \operatorname{Ext}^0(G, H) \to \operatorname{Ext}^0(F, H) \to \operatorname{Ext}^0(E, H) \longrightarrow 0.$$

These are examples of derived functors: $\operatorname{Hom}(H, -) : \operatorname{coh}(X) \to \operatorname{Vect}_{\mathbb{K}}$ is left exact, and $\operatorname{Ext}^i(H, -)$ for $i \geqslant 0$ are its right derived functors, and similarly $\operatorname{Hom}(-, H)$ is right exact, and $\operatorname{Ext}^i(-, H)$ are its left derived functors.

The Ext groups in $\mathrm{coh}(X)$ can be interpreted as Hom groups in the derived category $D^b\,\mathrm{coh}(X)$. If $E,F\in\mathrm{coh}(X)$ and $i\in\mathbb{Z}$ then

$$\mathrm{Ext}^i_{\mathrm{coh}(X)}(E,F)=\mathrm{Hom}_{D^b\,\mathrm{coh}(X)}(E,F[i]),$$

where $[i]$ shifts complexes i places to the left.
If $\mathcal{T}$ is a triangulated category, and $E\to F\to G\to E[1]$ a distinguished triangle in $\mathcal{T}$, and $H\in\mathcal{T}$, we have long exact sequences

$$\cdots\longrightarrow\mathrm{Hom}(H,E[k])\longrightarrow\mathrm{Hom}(H,F[k])\to\mathrm{Hom}(H,G[k])\to\mathrm{Hom}(H,E[k+1])\to\cdots,$$

$$\cdots\to\mathrm{Hom}(E[k+1],H)\to\mathrm{Hom}(G[k],H)\to\mathrm{Hom}(F[k],H)\longrightarrow\mathrm{Hom}(E[k],H)\longrightarrow\cdots.$$

T-structures

Definition

Let $\mathcal{T}$ be a triangulated category. A *t-structure* $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ on $\mathcal{T}$ is a pair of full subcategories $\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0} \subseteq \mathcal{T}$, closed under isomorphisms, satisfying

(i) If $X \in \mathcal{T}^{\leqslant 0}$ and $Y \in \mathcal{T}^{\geqslant 0}$ then $\mathrm{Hom}(X, Y[-1]) = 0$.

(ii) If $X \in \mathcal{T}^{\leqslant 0}$ then $X[1] \in \mathcal{T}^{\leqslant 0}$. If $Y \in \mathcal{T}^{\geqslant 0}$ then $Y[-1] \in \mathcal{T}^{\geqslant 0}$.

(iii) If $A \in \mathcal{T}$ there is a distinguished triangle $X \to A \to Y[-1] \to X[1]$ with $X \in \mathcal{T}^{\leqslant 0}$ and $Y \in \mathcal{T}^{\geqslant 0}$.

The *heart* is $\mathcal{H} = \mathcal{T}^{\leqslant 0} \cap \mathcal{T}^{\geqslant 0}$. It is an abelian category.

If $\mathcal{T}$ is a derived category $D(\mathcal{A}), D^{\pm}(\mathcal{A})$ or $D^b(\mathcal{A})$, we can define $\mathcal{T}^{\leqslant 0}$ to be the subcategory of complexes $E^\bullet$ with $H^i(E^\bullet) = 0$ for $i > 0$, and $\mathcal{T}^{\geqslant 0}$ to be the subcategory of $E^\bullet$ with $H^i(E^\bullet) = 0$ for $i < 0$. Then $\mathcal{H} = \mathcal{A} \subset D(\mathcal{A})$. So, a t-structure is the data we need to recover an abelian category from its derived category.

8.2. Exact functors and derived functors

Definition

Let $\mathcal{T}, \mathcal{T}'$ be triangulated categories. A functor $F : \mathcal{T} \to \mathcal{T}'$ is called *exact*, or *triangulated*, if F is additive, commutes with translation functors [1], and takes distinguished triangles to distinguished triangles.

This is like an exact functor of abelian categories. (Note that we don't define analogues of left exact or right exact.)

Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor of abelian categories. Then under good conditions we can define an exact *derived functor* $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ such that if $E \in \mathcal{A} \subset D^-(\mathcal{A})$ then $H^0(LF(E)) \cong F(E)$, and $H^{-k}(LF(E)) \cong L^k F(E)$ for $k \geqslant 0$ with $L^k F : \mathcal{A} \to \mathcal{B}$ the left derived functors of F, and $H^k(LF(E)) = 0$ for $k > 0$.

That is, right exact functors $\mathcal{A} \to \mathcal{B}$ of abelian categories transform to *fully exact* functors $D^-(\mathcal{A}) \to D^-(\mathcal{B})$ of derived categories.

Similarly, under good conditions a left exact functor $F : \mathcal{A} \to \mathcal{B}$ transforms to a fully exact functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$.
Being exact is much better than just being left or right exact.
One should think of the derived functors LF, RF as being "correct", and the abelian category versions as being truncations or approximations.
Note that LF, RF do not preserve the t-structures on $D^\pm(\mathcal{A}), D^\pm(\mathcal{B})$.

Serre duality

Let X be a smooth projective $\mathbb{K}$-scheme of dimension m. *Serre duality* gives functorial isomorphisms for all $E, F \in \mathrm{coh}(X)$

$$\mathrm{Ext}^k(E,F) \cong \mathrm{Ext}^{m-k}(F, E \otimes K_X)^*.$$

In the derived category $D^b \mathrm{coh}(X)$ we may write this as

$$\mathrm{Hom}_{D^b \mathrm{coh}(X)}(E, F[k]) \cong \mathrm{Hom}_{D^b \mathrm{coh}(X)}(F[k], E \otimes K_X[m])^*.$$

Define the *Serre functor* $S : D^b \mathrm{coh}(X) \to D^b \mathrm{coh}(X)$ to act by $S : E^\bullet \mapsto (E^\bullet \otimes K_X)[m]$. Then there are functorial isomorphisms

$$\mathrm{Hom}_{D^b \mathrm{coh}(X)}(E^\bullet, F^\bullet) \cong \mathrm{Hom}_{D^b \mathrm{coh}(X)}(F^\bullet, S(E^\bullet))^*$$

for all $E^\bullet, F^\bullet$ in $D^b \mathrm{coh}(X)$.

Verdier duality

Let X be a smooth projective $\mathbb{K}$-scheme. We have subcategories $\mathrm{Vect}(X) \subset \mathrm{coh}(X) \subset D^b \mathrm{coh}(X)$. There is a natural equivalence of categories $\mathbb{D}_X : \mathrm{Vect}(X) \to \mathrm{Vect}(X)^{\mathrm{op}}$ taking a vector bundle $E \to X$ to its dual vector bundle $E^* \to X$, where $E^* = \mathcal{H}om(E, \mathcal{O}_X)$. The square $\mathbb{D}_X \circ \mathbb{D}_X$ is naturally isomorphic to the identity.

Duality does not extend nicely to $\mathrm{coh}(X)$. However, there is an exact functor $\mathbb{D}_X : D^b \mathrm{coh}(X) \to D^b \mathrm{coh}(X)^{\mathrm{op}}$ called *Verdier duality*, which is an equivalence of categories, whose square is naturally isomorphic to the identity. The restriction of $\mathbb{D}_X$ to $\mathrm{Vect}(X) \subset D^b \mathrm{coh}(X)$ is $\mathbb{D}_X : \mathrm{Vect}(X) \to \mathrm{Vect}(X)^{\mathrm{op}}$.

If $\mathcal{E}^\bullet = (\cdots \to \mathcal{E}^k \to \mathcal{E}^{k+1} \to \cdots)$ lies in $D^b \mathrm{coh}(X)$ with each $\mathcal{E}^k \in \mathrm{Vect}(X)$ then $\mathbb{D}_X(\mathcal{E}^\bullet) = (\cdots \to (\mathcal{E}^{-k})^* \to (\mathcal{E}^{-1-k})^* \to \cdots)$ is the obvious dual complex.

Verdier duality does not take $\mathrm{coh}(X) \subset D^b \mathrm{coh}(X)$ to itself: the Verdier dual of a coherent sheaf is a complex in general.

Functors of derived categories $D^b\operatorname{coh}(X)$

Let X, Y be noetherian $\mathbb{K}$-schemes and $f : X \to Y$ a morphism. Then:

- If f is proper then $f_* : \operatorname{coh}(X) \to \operatorname{coh}(Y)$ has a right derived functor $Rf_* : D^b\operatorname{coh}(X) \to D^b\operatorname{coh}(Y)$.
- In general $f^* : \operatorname{coh}(Y) \to \operatorname{coh}(X)$ has a left derived functor $Lf^* : D^b\operatorname{coh}(Y) \to D^b\operatorname{coh}(X)$. It is left adjoint to Rf_* for f proper.
- If X, Y are smooth we define $f^! : D^b\operatorname{coh}(Y) \to D^b\operatorname{coh}(X)$ by $f^!(E^\bullet) = Lf^*(E^\bullet) \otimes K_X \otimes f^*(K_Y)^{-1}[\dim X - \dim Y]$. If f is proper then $f^!$ is right adjoint to Rf_*.
- We have $\mathbb{D}_Y \circ Rf_* \simeq Rf_* \circ \mathbb{D}_X$ and $f^! \simeq \mathbb{D}_X \circ Lf^* \circ \mathbb{D}_Y^{-1}$.
- There is a biexact *derived tensor product* $\otimes^L : D^b\operatorname{coh}(X) \times D^b\operatorname{coh}(X) \to D^b\operatorname{coh}(X)$.

If X, Y are smooth projective then all of $Rf_*, Lf^*, f^!$ are defined. This is an example of *Grothendieck's six functor formalism*.

Fourier–Mukai transforms

Definition

Let X, Y be smooth projective $\mathbb{K}$-schemes and $\mathcal{E}^\bullet \in D^b \operatorname{coh}(X \times Y)$. The *Fourier–Mukai transform* $F_{\mathcal{E}^\bullet} : D^b \operatorname{coh}(X) \to D^b \operatorname{coh}(Y)$ is the exact functor

$$F_{\mathcal{E}^\bullet} : \mathcal{G}^\bullet \longmapsto R(\pi_Y)_*\big(L(\pi_X)^*(\mathcal{G}^\bullet) \otimes^L \mathcal{E}^\bullet\big).$$

Mukai showed that $F_{\mathcal{E}^\bullet}$ has left and right adjoints, which are the Fourier–Mukai transforms by $\mathbb{D}_{X\times Y}(\mathcal{E}^\bullet) \otimes^L \pi_Y^*(K_Y)[\dim Y]$ and $\mathbb{D}_{X\times Y}(\mathcal{E}^\bullet) \otimes^L \pi_X^*(K_X)[\dim X]$.

Orlov showed that any exact functor $F : D^b \operatorname{coh}(X) \to D^b \operatorname{coh}(Y)$ with left and right adjoints is naturally isomorphic to $F_{\mathcal{E}^\bullet}$ for some $\mathcal{E}^\bullet$ in $D^b \operatorname{coh}(X \times Y)$, which is unique up to isomorphism.

For example, if $f : X \to Y$ is a morphism then Rf_* can be identified with $F_{\mathcal{E}^\bullet}$ for $\mathcal{E}^\bullet = \mathcal{O}_{\Gamma_f}$ with $\Gamma_f \subset X \times Y$ the graph of f.

Sometimes you can prove $F_{\mathcal{E}^\bullet}$ is an equivalence of categories.

8.3. Further topics
Spectra in Algebraic Topology

Write $\mathbf{Top}_*^{\mathbf{ho}}$ for the category of pointed topological spaces (X, x_0) of a topological space X (possibly weakly equivalent to a CW complex) with a base point x_0, with morphisms $[f] : (X, x_0) \to (Y, y_0)$ of homotopy classes of continuous $f : X \to Y$ with $f(x_0) = y_0$. There is a *suspension functor* $\Sigma : \mathbf{Top}_*^{\mathbf{ho}} \to \mathbf{Top}_*^{\mathbf{ho}}$ which maps $(X, x_0) \mapsto ((X \times [0,1])/\sim, \tilde{x}_0)$, where $\sim$ collapses $X \times \{0,1\}$ and $\{x_0\} \times [0,1]$ down to one point $\tilde{x}_0$. There is a triangulated category **Spectra** of *spectra*, called the *stable homotopy category*, with a functor $\Sigma^\infty : \mathbf{Top}_*^{\mathbf{ho}} \to \mathbf{Spectra}$ which takes Σ to the shift functor [1].

There are lots of cool things you can do with spectra. For example, generalized cohomology theories $H^* : (\mathbf{Top}_*^{\mathbf{ho}})^{\mathrm{op}} \to \mathbf{Ab}$ may be written as $(X, x_0) \mapsto \mathrm{Hom}(\Sigma^\infty(X, x_0), \boldsymbol{S})$ for some ring object $\boldsymbol{S}$ in **Spectra**.

Homological Mirror Symmetry

In Physics in the '80s, String Theorists made mysterious conjectures about 'Mirror Symmetry' relating pairs $X, \check{X}$ of Calabi–Yau m-folds (usually for $m = 3$). Kontsevich's 1994 *Homological Mirror Symmetry Conjecture* expressed Mirror Symmetry as equivalences of triangulated categories

$$D^b \operatorname{coh}(X) \simeq D^b \mathscr{F}(\check{X}), \qquad D^b \mathscr{F}(X) \simeq D^b \operatorname{coh}(\check{X}), \tag{8.2}$$

where $\mathscr{F}(X)$ is the *Fukaya category* of X as a symplectic manifold, whose objects are (roughly) Lagrangians in X.

This is one reason why derived categories and triangulated categories have become very important in Geometry. It is *necessary* to pass to the derived category before anything like (8.2) can be true, for example $\operatorname{coh}(X) \simeq \mathscr{F}(\check{X})$ is clearly nonsense.

A lot of the mathematical data about X which String Theory sees seems to be encoded in the triangulated categories $D^b \operatorname{coh}(X)$ ('B-model') and $D^b \mathscr{F}(X)$ ('A-model').

Interesting equivalences of derived categories

As in the HMS Conjecture, there are many interesting examples of equivalences between triangulated categories. For instance:

- There are equivalences $D^b\,\mathrm{coh}(\mathbb{CP}^n)\simeq D^b\text{mod-}\mathbb{C}Q/I$ for a certain 'quiver with relations' (Q,I). Here mod-$\mathbb{C}Q/I$ is much simpler than $\mathrm{coh}(\mathbb{CP}^n)$, so it helps us understand $D^b\,\mathrm{coh}(\mathbb{CP}^n)$.
- If X is a $K3$ surface, $D^b\,\mathrm{coh}(X)$ may have a large automorphism group not coming from automorphisms of X – 'hidden symmetries', which can be classified.
- Fourier–Mukai transforms can induce equivalences $D^b\,\mathrm{coh}(X)\simeq D^b\,\mathrm{coh}(Y)$.

Nonfunctoriality of the cone

I would argue that triangulated categories are not quite the 'right' theory. However, they are a *very good* approximation – you can work with them for years and not notice the problems.

As a signal that there should be something more, recall that if $\mathcal{T}$ is a triangulated category, and $u : X \to Y$ a morphism in $\mathcal{T}$, there is a 'cone' $C(u) \in \mathcal{T}$, in a distinguished triangle $X \to Y \to C(u) \to X[1]$ in $\mathcal{T}$. This is begging to be turned into a *cone functor*: we would like a category $\mathrm{Mor}(\mathcal{T})$ of morphisms in $\mathcal{T}$, and a functor $C : \mathrm{Mor}(\mathcal{T}) \to \mathcal{T}$ mapping $u \mapsto C(u)$ on objects. To try to define C on morphisms in $\mathrm{Mor}(\mathcal{T})$, consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \longrightarrow & C(u) & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \vdots\, C(f,g) & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \longrightarrow & C(u') & \longrightarrow & X'[1], \end{array} \qquad (8.3)$$

where u, u' are objects in $\mathrm{Mor}(\mathcal{T})$, and (f, g) a morphism. The axioms say some $C(f, g)$ exists, but it is not unique, so we cannot define C.

The explanation is that $\mathcal{T}$ should really be an ∞-*category* $\boldsymbol{\mathcal{T}}$. Then n-morphisms in $\mathrm{Mor}(\boldsymbol{\mathcal{T}})$ correspond to $(n+1)$-morphisms in $\boldsymbol{\mathcal{T}}$. So to define C on (1-)morphisms in $\mathrm{Mor}(\boldsymbol{\mathcal{T}})$, we should be using 2-morphisms in $\mathcal{T}$. We replace (8.3) by the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \longrightarrow & C(u) & \longrightarrow & X[1] \\ {\scriptstyle f}\downarrow & \overset{\eta}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle C(f,g,\eta)} & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow[u']{} & Y' & \longrightarrow & C(u') & \longrightarrow & X'[1], \end{array} \tag{8.4}$$

where $\eta : u' \circ f \Rightarrow g \circ u$ is a 2-morphism in $\boldsymbol{\mathcal{T}}$. Then $C(f, g, \eta)$ in (8.4) should exist and be unique up to 2-isomorphism. Note that $C(f, g, \eta)$ depends on the *particular choice* of η. When we pass to the homotopy category $\mathcal{T} = \mathrm{Ho}(\boldsymbol{\mathcal{T}})$, turning (8.4) into (8.3), this choice of η is forgotten, which is why we lose uniqueness of $C(f, g)$. Note that as n-morphisms in $\mathrm{Mor}(\boldsymbol{\mathcal{T}}) \leftrightarrow (n+1)$-morphisms in $\boldsymbol{\mathcal{T}}$, if we want $\boldsymbol{\mathcal{T}}$ and $\mathrm{Mor}(\boldsymbol{\mathcal{T}})$ to be objects of the same type we cannot truncate to N-categories for any finite N — we need $N = \infty$.

Stable ∞-categories

Assume for the moment that we have a good theory of ∞-categories. A *stable ∞-category* $\boldsymbol{\mathcal{T}}$ is an ∞-category such that:

(i) $\boldsymbol{\mathcal{T}}$ has a zero object.

(ii) Every morphism in $\boldsymbol{\mathcal{T}}$ has a kernel and a cokernel.

(iii) A triangle in $\boldsymbol{\mathcal{T}}$ is exact if and only if it is coexact.

These are very simple axioms – much simpler than those for triangulated categories. It is a remarkable theorem that if $\boldsymbol{\mathcal{T}}$ is a stable ∞-category then the homotopy category $\mathcal{T} = \mathrm{Ho}(\boldsymbol{\mathcal{T}})$ is a triangulated category.

You should assume that all the nice triangulated categories you meet at parties, like $D^b \operatorname{coh}(X)$, **Spectra**, and so on, are actually the homotopy categories of stable ∞-categories. And nice exact functors should be truncations of ∞-functors. Occasionally there are things you need to do upstairs in the ∞-categories, rather than downstairs in the homotopy categories.