Derived Differential Geometry

Lecture 7 of 14: Orbifolds

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Plan of talk:

7. Orbifolds

7.1. Introduction

Orbifolds $\mathfrak{X}$ are generalizations of manifolds which are locally modelled on $\mathbb{R}^n/\Gamma$ for Γ a finite group acting linearly on $\mathbb{R}^n$.
If Γ acts *effectively* on $\mathbb{R}^n$ (i.e. the morphism $\Gamma \to \mathrm{GL}(n,\mathbb{R})$ is injective, so that Γ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$) then $\mathfrak{X}$ is called an *effective orbifold*. Some authors include this in the definition.
Orbifolds were introduced in 1956 by Satake, who called them 'V-manifolds'. Thurston gave them the name 'orbifolds' in 1980.
Lots of differential geometry for manifolds also works for orbifolds, often with only minor changes. Orbifolds are important in some kinds of 'moduli space' and 'invariant' theories, particularly for J-holomorphic curves in symplectic geometry and Gromov–Witten invariants, where one must "count" 'Deligne–Mumford stable curves' with finite symetry groups Γ, which makes the moduli spaces (derived) orbifolds rather than (derived) manifolds.

Orbifolds
Kuranishi spaces
Introduction
Orbifold charts and coordinate changes
Stacks on topological spaces
The weak 2-category of orbifolds **Orb**

There are some subtle issues around defining 'smooth maps' of orbifolds, and so making orbifolds into a category (or higher category), and there are several *non-equivalent* definitions in the literature, both 'good' and 'bad'. For the 'bad' definitions, some differential-geometric operations such as transverse fibre products, or pullbacks of vector bundles, are not always defined.
The best answer is that orbifolds form a 2-*category* **Orb**, in which all 2-morphisms are 2-isomorphisms (i.e. a (2,1)-category).
Orbifolds are a kind of differential-geometric stack, and stacks form (2,1)-categories. There are at least five definitions of (strict or weak) 2-categories of orbifolds, giving equivalent 2-categories.
'Good' definitions of ordinary categories of orbifolds yield a category equivalent to $\mathrm{Ho}(\mathbf{Orb})$. In $\mathrm{Ho}(\mathbf{Orb})$, morphisms $[\mathfrak{f}] : \mathfrak{X} \to \mathfrak{Y}$ are not local (do not form a sheaf) on $\mathfrak{X}$. If you try to define an ordinary category of orbifolds in which smooth maps $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ are local on $\mathfrak{X}$, you get a 'bad' definition.

To see what these issues are, suppose $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a smooth map of orbifolds, and $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$ with $\mathfrak{f}(x) = y$, and $\mathfrak{X}, \mathfrak{Y}$ are modelled near x, y on $[U/\Gamma]$, $[V/\Delta]$ for $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ open and Γ, Δ finite groups acting linearly on $\mathbb{R}^m, \mathbb{R}^n$ preserving U, V.

Naïvely, we would expect $\mathfrak{f}$ to be locally given near x by a smooth map of manifolds $f' : U \to V$ and a group morphism $\rho : \Gamma \to \Delta$ such that $f'(\gamma \cdot u) = \rho(\gamma) \cdot f'(u)$ for all $u \in U$ and $\gamma \in \Gamma$, so that f' induces a map of sets $U/\Gamma \to V/\Delta$.

Note that the map of sets $\mathfrak{f} : U/\Gamma \to V/\Delta$ does not determine f' and ρ uniquely. For any $\delta \in \Delta$, we can always replace f', ρ by $\tilde{f}', \tilde{\rho}$ where $\tilde{f}'(u) = \delta \cdot f'(u)$ and $\tilde{\rho}(\gamma) = \delta\rho(\gamma)\delta^{-1}$. For some $\mathfrak{f}$, there is more choice of f', ρ than this.

The definition of smooth map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ needs to remember some information about allowed choices of (f', ρ). To see this is *2-categorical*, think of $(f', \rho), (\tilde{f}', \tilde{\rho})$ as 1-morphisms $(U, \Gamma) \to (V, \Delta)$, and $\delta : (f', \rho) \Rightarrow (\tilde{f}', \tilde{\rho})$ as a 2-morphism.

We will discuss some examples before formally defining orbifolds.

Example 7.1

Let X be a manifold, and G a finite group, so that $[*/G]$ is a (noneffective) orbifold. What are 'smooth maps' $\mathfrak{f} : X \to [*/G]$? The answer should be: in the 2-category **Orb**,

- 1-morphisms $\mathfrak{f} : X \to [*/G]$ should correspond to principal G-bundles $P \to X$.
- For 1-morphisms $\mathfrak{f}, \tilde{\mathfrak{f}} : X \to [*/G]$ corresponding to principal G-bundles $P, \tilde{P} \to X$, 2-morphisms $\eta : \mathfrak{f} \Rightarrow \tilde{\mathfrak{f}}$ should correspond to isomorphisms of principal bundles $P \cong \tilde{P}$.

Therefore in the homotopy category $\mathrm{Ho}(\mathbf{Orb})$, morphisms $[\mathfrak{f}] : X \to [*/G]$ correspond to isomorphism classes of principal G-bundles $P \to X$.

Non-locality of morphisms in $\mathrm{Ho}(\mathbf{Orb})$

Example 7.2

Let $X = \mathcal{S}^1 \subset \mathbb{R}^2$ and $G = \mathbb{Z}_2$. Then $X = U \cup V$ for $U = \mathcal{S}^1 \setminus \{(1,0)\}$, $V = \mathcal{S}^1 \setminus \{(-1,0)\}$. There are two principal $\mathbb{Z}_2$-bundles on $\mathcal{S}^1$ up to isomorphism, with monodromy 1 and -1 around $\mathcal{S}^1$. But on $U \cong \mathbb{R} \cong V$ there are only one principal $\mathbb{Z}_2$-bundle (the trivial bundle) up to isomorphism.
Therefore morphisms $[\mathfrak{f}] : \mathcal{S}^1 \to [*/\mathbb{Z}_2]$ in $\mathrm{Ho}(\mathbf{Orb})$ are not determined by their restrictions $[\mathfrak{f}]|_U, [\mathfrak{f}]|_V$ for the open cover $\{U, V\}$ of $\mathcal{S}^1$, so such $[\mathfrak{f}]$ do not form a sheaf on $\mathcal{S}^1$.

Regarding X as a quotient $[\mathcal{S}^1/\{1\}]$, this example also shows that morphisms $\mathfrak{f} : [U/\Gamma] \to [V/\Delta]$ in **Orb** or $\mathrm{Ho}(\mathbf{Orb})$ are *not* globally determined by a smooth map $f' : U \to V$ and morphism $\rho : \Gamma \to \Delta$, as $f' : \mathcal{S}^1 \to *$, $\rho : \{1\} \to \mathbb{Z}_2$ are unique, but $\mathfrak{f}$ is not.

Example 7.3 (Hilsum–Skandalis morphisms)

Suppose U, V are manifolds and Γ, Δ are finite groups acting smoothly on U, V, so that $\mathfrak{X} = [U/\Gamma]$, $\mathfrak{Y} = [V/\Delta]$ are orbifolds ('global quotient orbifolds'). The correct notion of 1-morphism $\mathfrak{X} \to \mathfrak{Y}$ in **Orb** is induced by a triple (P, π, f), where

- P is a manifold with a smooth action of $\Gamma \times \Delta$
- $\pi : P \to U$ is a Γ-equivariant, Δ-invariant smooth map making P into a principal Δ-bundle over U.
- $f : P \to V$ is a smooth Δ-equivariant and Γ-invariant map.

This is called a *Hilsum–Skandalis morphism*.
2-morphisms $\eta : (P, \pi, f) \Rightarrow (\tilde{P}, \tilde{\pi}, \tilde{f})$ are $\Gamma \times \Delta$-equivariant diffeomorphisms $\eta : P \to \tilde{P}$ with $\tilde{\pi} \circ \eta = \pi$, $\tilde{f} \circ \eta = f$.
If $(Q, \pi, g) : [V/\Delta] \to [W/K]$ is another morphism then composition of 1-morphisms is

$$(Q, \pi, g) \circ (P, \pi, f) = \big((P \times_{f,V,\pi} Q)/\Delta, \pi \circ \pi_P, g \circ \pi_Q\big),$$

where $P \times_{f,V,\pi} Q$ is a transverse fibre product of manifolds.

Suppose $(P,\pi,f):[U/\Gamma]\to[V/\Delta]$ is a Hilsum–Skandalis morphism with U connected, and $P=U\times\Delta$ is a trivial Δ-bundle, with Δ-action $\delta:(u,\delta')\mapsto(u,\delta\delta')$. The Γ-action on P commutes with the Δ-action and $\pi_U:U\times\Delta\to U$ is Γ-equivariant, so it must be of the form $\gamma:(u,\delta)\mapsto(\gamma\cdot u,\delta\rho(\gamma)^{-1})$ for $\rho:\Gamma\to\Delta$ a group morphism. Define $f':U\to V$ by $f'(u)=f(u,1)$. Then f Δ-equivariant implies that $f(u,\delta)=\delta\cdot f'(u)$, and f Γ-invariant implies that $f'(\gamma\cdot u)=\rho(\gamma)\cdot f'(u)$.

Thus, if P is a trivial Δ-bundle then (P,π,f) corresponds to the 'naïve' notion of morphisms $[U/\Gamma]\to[V/\Delta]$ discussed before.

Since every principal Δ-bundle is locally trivial, every Hilsum–Skandalis morphism is locally of the expected 'naïve' form.

Example 7.4 (Weighted projective spaces)

Let n and $a_0,\ldots,a_n$ be positive integers, with $\mathrm{hcf}(a_0,\ldots,a_n)=1$. Define the *weighted projective space* $\mathbb{CP}^n_{a_0,\ldots,a_n}=(\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}\setminus\{0\}$ by

$$\lambda:(z_0,z_1,\ldots,z_n)\longmapsto(\lambda^{a_0}z_0,\ldots,\lambda^{a_n}z_n).$$

Then $\mathbb{CP}^n_{a_0,\ldots,a_n}$ is a compact complex orbifold. Near $[z_0,\ldots,z_n]$ it is modelled on $\mathbb{C}^n/\mathbb{Z}_k$, where k is the highest common factor of those a_i for $i=0,\ldots,n$ with $z_i\neq 0$.

Example 7.5

$\mathbb{CP}^1_{2,1}$ is topologically a 2-sphere $\mathcal{S}^2$. It has one orbifold point $[1,0]$ where it is locally modelled on $\mathbb{C}/\{\pm 1\}$, and $\mathbb{CP}^1_{2,1}\setminus[1,0]\cong\mathbb{C}$. Suppose for a contradiction that $\mathbb{CP}^1_{2,1}\cong[U/\Gamma]$ for U a manifold and Γ a finite group. Let $[1,0]\cong u\Gamma$ and $[0,1]\cong v\Gamma$, and let $\Delta\subseteq\Gamma$ be the subgroup fixing u, so that $\Delta\cong\mathbb{Z}_2$. Write $U'=U\setminus u\Gamma$, $U''=U\setminus v\Gamma$ and $U'''=U'\cap U''$. Then $U'\to U'/\Gamma=\mathbb{CP}^1_{2,1}\setminus[1,0]=\mathbb{C}$ is a $|\Gamma|:1$ covering map, so that U' is $|\Gamma|$ copies of $\mathbb{C}$ as $\mathbb{C}$ is simply-connected, and U''' is $|\Gamma|$ copies of $\mathbb{C}\setminus\{0\}$. Also $U''/\Delta\to U''/\Gamma=\mathbb{CP}^1_{2,1}\setminus[0,1]=\mathbb{C}/\mathbb{Z}_2$ is a $|\Gamma|/2\!:\!1$ covering map, so U''/Δ is $|\Gamma|/2$ copies of $\mathbb{C}/\mathbb{Z}_2$, and U'' is $|\Gamma|/2$ copies of $\mathbb{C}$, and U''' is $|\Gamma|$ copies of $\mathbb{C}\setminus\{0\}$, a contradiction. Hence $\mathbb{CP}^1_{2,1}$ cannot be a global quotient $[U/\Gamma]$ for Γ finite.

This example shows we need to define orbifolds $\mathfrak{X}$ by covering $\mathfrak{X}$ by many open charts $\mathfrak{U}_i\subset\mathfrak{X}$ with $\mathfrak{U}_i\cong[U_i/\Gamma_i]$.

7.2. Orbifold charts and coordinate changes

We now give one definition of a weak 2-category of orbifolds **Orb**, taken from my arXiv:1409.6908, §4.5. It is a dry run for the definition of Kuranishi spaces **Kur**.

Definition 7.6

Let X be a topological space. An *orbifold chart* (V_i,Γ_i,ψ_i) on X is a manifold V_i, a finite group Γ_i acting smoothly on V_i, and a map $\psi_i:V_i/\Gamma_i\to X$ which is a homeomorphism with an open set $\mathrm{Im}\,\psi_i\subseteq X$. We write $\bar\psi_i:V_i\to X$ for the composition $V_i\to V_i/\Gamma_i\xrightarrow{\psi_i}X$. If $X'\subseteq X$ is open, the *restriction* of (V_i,Γ_i,ψ_i) to X' is $(V_i,\Gamma_i,\psi_i)|_{X'}:=(V_i',\Gamma_i,\psi_i')$, where $V_i'=\bar\psi_i^{-1}(X')$ and $\psi_i'=\psi_i|_{V_i'/\Gamma_i}$.

Definition 7.7

Let $(V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)$ be orbifold charts on X with $\mathrm{Im}\,\psi_i=\mathrm{Im}\,\psi_j$. A *coordinate change* $(P_{ij},\pi_{ij},\phi_{ij}):(V_i,\Gamma_i,\psi_i)\to(V_j,\Gamma_j,\psi_j)$ is $P_{ij},\pi_{ij},\phi_{ij}$, where

- P_{ij} is a manifold with a smooth action of $\Gamma_i\times\Gamma_j$.
- $\pi_{ij}:P\to V_i$ is a Γ_i-equivariant, Γ_j-invariant smooth map making P_{ij} into a principal Γ_j-bundle over V_i.
- $\phi_{ij}:P_{ij}\to V_j$ is a Γ_j-equivariant, Γ_i-invariant smooth map making P_{ij} into a principal Γ_i-bundle over V_j.

If $(P_{ij},\pi_{ij},\phi_{ij}),(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij}):(V_i,\Gamma_i,\psi_i)\to(V_j,\Gamma_j,\psi_j)$ are coordinate changes, a *2-morphism* $\eta:(P_{ij},\pi_{ij},\phi_{ij})\Rightarrow(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij})$ is a $\Gamma_i\times\Gamma_j$-equivariant diffeomorphism $\eta:P_{ij}\to\tilde P_{ij}$ with $\tilde\pi_{ij}\circ\eta=\pi_{ij}$, $\tilde\phi_{ij}\circ\eta=\phi_{ij}$. If $X'\subseteq X$ is open, we can *restrict* coordinate changes and 2-morphisms to X' by $(P_{ij},\pi_{ij},\phi_{ij})|_{X'}:=((\bar\psi_i\circ\pi_{ij})^{-1}(X'),\pi_{ij}|_{\dots},\phi_{ij}|_{\dots})$ and $\eta|_{X'}:=\eta|_{(\bar\psi_i\circ\pi_{ij})^{-1}(X')}$.

Definition 7.8

If $(P_{jk},\pi_{jk},\phi_{jk}):(V_j,\Gamma_j,\psi_j)\to(V_k,\Gamma_k,\psi_k)$ is another coordinate change then *composition of coordinate changes* is
$(P_{jk},\pi_{jk},\phi_{jk})\circ(P_{ij},\pi_{ij},\phi_{ij})=((P_{ij}\times_{\phi_{ij},V_j,\pi_{jk}}P_{jk})/\Gamma_j,\pi_{ij}\circ\pi_{P_{ij}},\phi_{jk}\circ\pi_{P_{jk}})$,
where $P_{ij}\times_{\phi_{ij},V_j,\pi_{jk}}P_{jk}$ is a transverse fibre product of manifolds.
If $\eta:(P_{ij},\pi_{ij},\phi_{ij})\Rightarrow(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij})$ and $\zeta:(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij})\to(\hat P_{ij},\hat\pi_{ij},\hat\phi_{ij})$ are 2-morphisms, the *vertical composition* is $\zeta\odot\eta=\zeta\circ\eta:P_{ij}\to\hat P_{ij}$.
If $\eta:(P_{ij},\pi_{ij},\phi_{ij})\Rightarrow(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij})$ and $\zeta:(P_{jk},\pi_{jk},\phi_{jk})\Rightarrow(\tilde P_{jk},\tilde\pi_{jk},\tilde\phi_{jk})$ are 2-morphisms, the *horizontal composition* is

$$\zeta*\eta=(\eta\times_{V_j}\zeta)/\Gamma_j:(P_{ij}\times_{V_j}P_{jk})/\Gamma_j\to(\tilde P_{ij}\times_{V_j}\tilde P_{jk})/\Gamma_j.$$

The *identity coordinate change* for (V_i,Γ_i,ψ_i) is $\mathrm{id}_{(V_i,\Gamma_i,\psi_i)}:=(P_{ii},\pi_{ii},\phi_{ii})$ where $P_{ii}=V_i\times\Gamma_i$ with $\Gamma_i\times\Gamma_i$-action $(\gamma_1,\gamma_2):(v,\delta)\mapsto(\gamma_1\cdot v,\gamma_2\delta\gamma_1^{-1})$, and $\pi_{ii}:(v,\gamma)\mapsto v$, $\phi_{ii}:(v,\gamma)\mapsto\gamma\cdot v$.
The *identity 2-morphism* for $(P_{ij},\pi_{ij},\phi_{ij})$ is $\mathrm{id}_{(P_{ij},\pi_{ij},\phi_{ij})}=\mathrm{id}_{P_{ij}}$.

Theorem 7.9

Let X be a topological space, and $S \subseteq X$ be open. Then we have defined a strict 2-category $\mathbf{Coord}_S(X)$ *with objects orbifold charts* (V_i, Γ_i, ψ_i) *on X with* $\mathrm{Im}\,\psi_i = S$*, and 1-morphisms coordinate changes* $(P_{ij}, \pi_{ij}, \phi_{ij}) : (V_i, \Gamma_i, \psi_i) \to (V_j, \Gamma_j, \psi_j)$*, and 2-morphisms* $\eta : (P_{ij}, \pi_{ij}, \phi_{ij}) \Rightarrow (\tilde P_{ij}, \tilde\pi_{ij}, \tilde\phi_{ij})$ *as above. All 1-morphisms in* $\mathbf{Coord}_S(X)$ *are 1-isomorphisms, and all 2-morphisms are 2-isomorphisms. If* $T \subseteq S \subseteq X$ *are open, then restriction* $|_T : \mathbf{Coord}_S(X) \to \mathbf{Coord}_T(X)$ *is a strict 2-functor.*

7.3. Stacks on topological spaces

In §3.2 we defined *sheaves of sets* $\mathcal{E}$ on a topological space X. There is a parallel notion of ‘sheaves of groupoids’ on X, which is called a *stack* (or *2-sheaf*) on X. As sets form a category **Sets**, but groupoids form a 2-category **Groupoids** (in fact, a (2,1)-category), stacks on X are a (2,1)-category generalization of sheaves.

The connection with stacks in algebraic geometry is that both are examples of ‘stacks on a site’, where here we mean the site of open sets in X, and in algebraic geometry we use the site of $\mathbb{K}$-algebras $\mathbf{Alg}_{\mathbb{K}}$, regarded as a kind of generalized topological space.

As for sheaves, we define *prestacks* and *stacks*. Sheaves are presheaves which satisfy a gluing property on open covers $\{V_i : i \in I\}$, involving data on V_i and conditions on double overlaps $V_i \cap V_j$. For the 2-category generalization we need data on V_i, $V_i \cap V_j$ and conditions on triple overlaps $V_i \cap V_j \cap V_k$.

Definition 7.10

Let X be a topological space. A *prestack* (or *prestack in groupoids*, or 2-*presheaf*) $\mathcal{E}$ on X, consists of the data of a groupoid $\mathcal{E}(S)$ for every open set $S \subseteq X$, and a functor $\rho_{ST} : \mathcal{E}(S) \to \mathcal{E}(T)$ called the *restriction map* for every inclusion $T \subseteq S \subseteq X$ of open sets, and a natural isomorphism of functors $\eta_{STU} : \rho_{TU} \circ \rho_{ST} \Rightarrow \rho_{SU}$ for all inclusions $U \subseteq T \subseteq S \subseteq X$ of open sets, satisfying the conditions that:

(i) $\rho_{SS} = \mathrm{id}_{\mathcal{E}(S)} : \mathcal{E}(S) \to \mathcal{E}(S)$ for all open $S \subseteq X$, and $\eta_{SST} = \eta_{STT} = \mathrm{id}_{\rho_{ST}}$ for all open $T \subseteq S \subseteq X$; and

(ii) $\eta_{SUV} \odot (\mathrm{id}_{\rho_{UV}} * \eta_{STU}) = \eta_{STV} \odot (\eta_{TUV} * \mathrm{id}_{\rho_{ST}}) : \rho_{UV} \circ \rho_{TU} \circ \rho_{ST} \Longrightarrow \rho_{SV}$ for all open $V \subseteq U \subseteq T \subseteq S \subseteq X$.

Definition (Continued)

A prestack $\mathcal{E}$ on X is called a *stack* (or *stack in groupoids*, or 2-*sheaf*) on X if whenever $S \subseteq X$ is open and $\{T_i : i \in I\}$ is an open cover of S, then (iii)–(v) hold, where:

(iii) If $\alpha, \beta : A \to B$ are morphisms in $\mathcal{E}(S)$ and $\rho_{ST_i}(\alpha) = \rho_{ST_i}(\beta) : \rho_{ST_i}(A) \to \rho_{ST_i}(B)$ in $\mathcal{E}(T_i)$ for all $i \in I$, then $\alpha = \beta$.

(iv) If A, B are objects of $\mathcal{E}(S)$ and $\alpha_i : \rho_{ST_i}(A) \to \rho_{ST_i}(B)$ are morphisms in $\mathcal{E}(T_i)$ for all $i \in I$ with

$$\eta_{ST_i(T_i\cap T_j)}(B) \circ \rho_{T_i(T_i\cap T_j)}(\alpha_i) \circ \eta_{ST_i(T_i\cap T_j)}(A)^{-1} = \eta_{ST_j(T_i\cap T_j)}(B) \circ \rho_{T_j(T_i\cap T_j)}(\alpha_j) \circ \eta_{ST_j(T_i\cap T_j)}(A)^{-1}$$

in $\mathcal{E}(T_i \cap T_j)$ for all $i, j \in I$, then there exists $\alpha : A \to B$ in $\mathcal{E}(S)$ (unique by (iii)) with $\rho_{ST_i}(\alpha) = \alpha_i$ for all $i \in I$.

Definition (Continued)

(v) If $A_i \in \mathcal{E}(T_i)$ for $i \in I$ and $\alpha_{ij} : \rho_{T_i(T_i\cap T_j)}(A_i) \to \rho_{T_j(T_i\cap T_j)}(A_j)$ are morphisms in $\mathcal{E}(T_i \cap T_j)$ for $i,j \in I$ with

$$\eta_{T_k(T_j\cap T_k)(T_i\cap T_j\cap T_k)}(A_k)\circ\rho_{(T_j\cap T_k)(T_i\cap T_j\cap T_k)}(\alpha_{jk})\circ\eta_{T_j(T_j\cap T_k)(T_i\cap T_j\cap T_k)}(A_j)^{-1}$$
$$\circ\,\eta_{T_j(T_i\cap T_j)(T_i\cap T_j\cap T_k)}(A_j)\circ\rho_{(T_i\cap T_j)(T_i\cap T_j\cap T_k)}(\alpha_{ij})\circ\eta_{T_i(T_i\cap T_j)(T_i\cap T_j\cap T_k)}(A_i)^{-1}$$
$$=\eta_{T_k(T_i\cap T_k)(T_i\cap T_j\cap T_k)}(A_k)\circ\rho_{(T_i\cap T_k)(T_i\cap T_j\cap T_k)}(\alpha_{ik})\circ\eta_{T_i(T_i\cap T_k)(T_i\cap T_j\cap T_k)}(A_i)^{-1}$$

for all $i,j,k \in I$, then there exist an object A in $\mathcal{E}(S)$ and morphisms $\beta_i : A_i \to \rho_{ST_i}(A)$ for $i \in I$ such that for all $i,j \in I$ we have

$$\eta_{ST_i(T_i\cap T_j)}(A)\circ\rho_{T_i(T_i\cap T_j)}(\beta_i) = \eta_{ST_j(T_i\cap T_j)}(A)\circ\rho_{T_j(T_i\cap T_j)}(\beta_j)\circ\alpha_{ij}.$$

In the examples we are interested in we have $\rho_{TU}\circ\rho_{ST} = \rho_{SU}$ and $\eta_{STU} = \mathrm{id}_{\rho_{SU}}$ for all open $U \subseteq T \subseteq S \subseteq X$, so all the $\eta_{\cdots}(\cdots)$ terms above can be omitted.

In Theorem 6.6 we showed that coordinate changes of M-Kuranishi neighbourhoods have a sheaf property. There is an analogous stack property for coordinate changes of orbifold charts.

Theorem 7.11

Let $(V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)$ be orbifold charts on X. For each open $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, write $\mathcal{C}oord\big((V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)\big)(S)$ for the groupoid of coordinate changes $(P_{ij},\pi_{ij},\phi_{ij}) : (V_i,\Gamma_i,\psi_i) \to (V_j,\Gamma_j,\psi_j)$ over S, and for all open $T \subseteq S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$ define

$$\rho_{ST} : \mathcal{C}oord\big((V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)\big)(S) \longrightarrow \mathcal{C}oord\big((V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)\big)(T) \text{ by } \rho_{ST} = |_T,$$

and for all open $U \subseteq T \subseteq S \subseteq X$ define
$\eta_{STU} = \mathrm{id}_{\rho_{SU}} : \rho_{TU}\circ\rho_{ST} = \rho_{SU} \Rightarrow \rho_{SU}$.
Then $\mathcal{C}oord\big((V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)\big)$ is a stack on $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$.

The nontrivial part of this is a gluing result for principal Γ_j-bundles on a cover of V_i, with given isomorphisms on double overlaps and an associativity condition on triple overlaps.

7.4. The weak 2-category of orbifolds **Orb**

Definition 7.12

Let X be a Hausdorff, second countable topological space. An *orbifold structure* $\mathcal{O}$ *on* X *of dimension* $n\in\mathbb{N}$ is data $\mathcal{O}=\bigl(I,(V_i,\Gamma_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \lambda_{ijk,\ i,j,k\in I}\bigr)$, where:

(a) I is an indexing set.

(b) (V_i,Γ_i,ψ_i) is an orbifold chart on X for each $i\in I$, with $\dim V_i=n$. Write $S_i=\mathrm{Im}\,\psi_i$, $S_{ij}=\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, etc.

(c) $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij}):(V_i,\Gamma_i,\psi_i)|_{S_{ij}}\to(V_j,\Gamma_j,\psi_j)|_{S_{ij}}$ is a coordinate change for all $i,j\in I$.

(d) $\lambda_{ijk}:\Phi_{jk}\circ\Phi_{ij}|_{S_{ijk}}\Rightarrow\Phi_{ik}|_{S_{ijk}}$ is a 2-morphism for all $i,j,k\in I$.

(e) $\bigcup_{i\in I}\mathrm{Im}\,\psi_i=X$. (f) $\Phi_{ii}=\mathrm{id}_{(V_i,\Gamma_i,\psi_i)}$ for all $i\in I$.

(g) $\lambda_{iij}=\lambda_{ijj}=\mathrm{id}_{\Phi_{ij}}$ for all $i,j\in I$.

(h) $\lambda_{ikl}\odot(\mathrm{id}_{\Phi_{kl}}*\lambda_{ijk})|_{S_{ijkl}}=\lambda_{ijl}\odot(\lambda_{jkl}*\mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}}:$ $\Phi_{kl}\circ\Phi_{jk}\circ\Phi_{ij}|_{S_{ijkl}}\Longrightarrow\Phi_{il}|_{S_{ijkl}}$ for all $i,j,k,l\in I$.

We call $\mathfrak{X}=(X,\mathcal{O})$ an *orbifold*, of *dimension* $\dim\mathfrak{X}=n$.

Recall that to define M-Kuranishi spaces in §6, which form a category, we specified data on S_i,S_{ij}, and imposed conditions on S_{ijk}. Here for orbifolds, which form a 2-category, we specify data on S_i,S_{ij},S_{ijk}, and impose conditions on quadruple overlaps S_{ijkl}.

We call $\mathfrak{X}$ an *effective orbifold* if the orbifold charts (V_i,Γ_i,ψ_i) are effective, that is, if Γ_i acts (locally) effectively on V_i for all $i\in I$.

We can also define 1-morphisms and 2-morphisms of orbifolds. To do this, given a continuous map $f:X\to Y$ and orbifold charts (V_i,Γ_i,ψ_i), (W_j,Δ_j,χ_j) on X,Y, we have to define 1-morphisms $(P_{ij},\pi_{ij},f_{ij}):(V_i,\Gamma_i,\psi_i)\to(W_j,\Delta_j,\chi_j)$ of orbifold charts over f, and 2-morphisms $\eta_{ij}:(P_{ij},\pi_{ij},f_{ij})\Rightarrow(\tilde P_{ij},\tilde\pi_{ij},\tilde f_{ij})$, compositions $\circ,\odot,*$, and identities. These generalize Definitions 7.7 and 7.8 for $f=\mathrm{id}_X$, so we leave them as an exercise.

Definition 7.13

Let $\mathfrak{X}=(X,\mathcal{O})$ and $\mathfrak{Y}=(Y,\mathcal{P})$ be orbifolds, with $\mathcal{O}=\bigl(I,(V_i,\Gamma_i,\psi_i)_{i\in I},\,\Phi_{ii',\,i,i'\in I},\,\lambda_{ii'i'',\,i,i',i''\in I}\bigr)$ and $\mathcal{P}=\bigl(J,(W_j,\Delta_j,\chi_j)_{j\in J},\,\Psi_{jj',\,j,j'\in J},\,\mu_{jj'j'',\,j,j',j''\in J}\bigr)$. A *1-morphism* $\mathfrak{f}:\mathfrak{X}\to\mathfrak{Y}$ is data $\mathfrak{f}=\bigl(f,\mathfrak{f}_{ij,\,i\in I,\,j\in J},\,F^{j,\,j\in J}_{ii',\,i,i'\in I},\,F^{jj',\,j,j'\in J}_{i,\,i\in I}\bigr)$, with:

(a) $f:X\to Y$ is a continuous map.

(b) $\mathfrak{f}_{ij}=(P_{ij},\pi_{ij},f_{ij}):(V_i,\Gamma_i,\psi_i)\to(W_j,\Delta_j,\chi_j)$ is a 1-morphism of orbifold charts over f for all $i\in I$, $j\in J$.

(c) $F^j_{ii'}:\mathfrak{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\mathfrak{f}_{ij}$ is a 2-morphism over f for $i,i'\in I$, $j\in J$.

(d) $F^{jj'}_i:\Psi_{jj'}\circ\mathfrak{f}_{ij}\Rightarrow\mathfrak{f}_{ij'}$ is a 2-morphism over f for $i\in I$, $j,j'\in J$.

(e) $F^j_{ii}=F^{jj}_i=\mathrm{id}_{\mathfrak{f}_{ij}}$.

(f) $F^j_{ii''}\odot(\mathrm{id}_{\mathfrak{f}_{i''j}}*\lambda_{ii'i''})=F^j_{ii'}\odot(F^j_{i'i''}*\mathrm{id}_{\Phi_{ii'}}):\mathfrak{f}_{i''j}\circ\Phi_{i'i''}\circ\Phi_{ii'}\Rightarrow\mathfrak{f}_{i''j}$.

(g) $F^{jj'}_i\odot(\mathrm{id}_{\Psi_{jj'}}*F^j_{ii'})=F^{j'}_{ii'}\odot(F^{jj'}_{i'}*\mathrm{id}_{\Phi_{ii'}}):\Psi_{jj'}\circ\mathfrak{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\mathfrak{f}_{ij'}$.

(h) $F^{j'j''}_i\odot(\mathrm{id}_{\Psi_{j'j''}}*F^{jj'}_i)=F^{jj''}_i\odot(\mu_{jj'j''}*\mathrm{id}_{\mathfrak{f}_{ij}}):\Psi_{j'j''}\circ\Psi_{jj'}\circ\mathfrak{f}_{ij}\Rightarrow\mathfrak{f}_{ij''}$.

Here (c)–(h) hold for all $i,j,\ldots,$ restricted to appropriate domains.

Definition 7.14

Let $\mathfrak{f},\mathfrak{g}:\mathfrak{X}\to\mathfrak{Y}$ be 1-morphisms of orbifolds, with
$\mathfrak{f}=\bigl(f,\mathfrak{f}_{ij,\,i\in I,\,j\in J},\,F^{j,\,j\in J}_{ii',\,i,i'\in I},\,F^{jj',\,j,j'\in J}_{i,\,i\in I}\bigr)$,
$\mathfrak{g}=\bigl(g,\mathfrak{g}_{ij,\,i\in I,\,j\in J},\,G^{j,\,j\in J}_{ii',\,i,i'\in I},\,G^{jj',\,j,j'\in J}_{i,\,i\in I}\bigr)$. Suppose the continuous maps $f,g:X\to Y$ satisfy $f=g$. A *2-morphism* $\eta:\mathfrak{f}\Rightarrow\mathfrak{g}$ is data $\eta=(\eta_{ij,\,i\in I,\,j\in J})$, where $\eta_{ij}:\mathfrak{f}_{ij}\Rightarrow\mathfrak{g}_{ij}$ is a 2-morphism of orbifold charts over $f=g$, satisfying:

(a) $G^j_{ii'}\odot(\eta_{i'j}*\mathrm{id}_{\Phi_{ii'}})=\eta_{ij}\odot F^j_{ii'}:\mathfrak{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\mathfrak{g}_{ij}$ for $i,i'\in I$, $j\in J$.

(b) $G^{jj'}_i\odot(\mathrm{id}_{\Psi_{jj'}}*\eta_{ij})=\eta_{ij'}\odot F^{jj'}_i:\Psi_{jj'}\circ\mathfrak{f}_{ij}\Rightarrow\mathfrak{g}_{ij'}$ for $i\in I$, $j,j'\in J$.

We can then define composition of 1- and 2-morphisms, identity 1- and 2-morphisms, and so on, making orbifolds into a weak 2-category **Orb**.

Composition of 1-morphisms $\mathfrak{g}\circ\mathfrak{f}$ is complicated: we have to use the analogue of the stack property Theorem 7.11 for 1-morphisms of orbifold charts to define $(\mathfrak{g}\circ\mathfrak{f})_{ik}$ in $\mathfrak{g}\circ\mathfrak{f}$. This only determines $(\mathfrak{g}\circ\mathfrak{f})_{ik}$ up to 2-isomorphism, so we have to make an arbitrary choice to define $\mathfrak{g}\circ\mathfrak{f}$. Because of this, we need not have $\mathfrak{h}\circ(\mathfrak{g}\circ\mathfrak{f})=(\mathfrak{h}\circ\mathfrak{g})\circ\mathfrak{f}$, instead we prove the existence of a natural 2-isomorphism $\alpha_{\mathfrak{h},\mathfrak{g},\mathfrak{f}}:\mathfrak{h}\circ(\mathfrak{g}\circ\mathfrak{f})\Rightarrow(\mathfrak{h}\circ\mathfrak{g})\circ\mathfrak{f}$. This is why **Orb** is a weak 2-category rather than a strict 2-category.

Note that the arguments used here are of two kinds. First in §7.1-§7.2 we use a lot of differential geometry to construct a 2-category of orbifold charts, and prove the stack property. But for the second part in §7.3, there is no differential geometry, we use 2-categories and stack theory to define the weak 2-category **Orb**.

To generalize to Kuranishi spaces, we first need to construct a 2-category of Kuranishi charts, and prove the stack property. The second part, construction of the weak 2-category **Kur** using 2-categories and stack theory, is standard, identical to **Orb**.

Kuranishi spaces

Lecture 8 of 14: Kuranishi spaces

Dominic Joyce, Oxford University
Summer 2015

These slides available at
http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

8. Kuranishi spaces

We now define a weak 2-category **Kur** of *Kuranishi spaces* **X**, a kind of derived orbifold, following my arXiv:1409.6908, §4. One can also define derived orbifolds using C^∞-algebraic geometry by generalizing the definition of d-manifolds, replacing C^∞-schemes $\underline{X}$ by Deligne–Mumford C^∞-stacks $\mathcal{X}$. This yields a strict 2-category **dOrb** of *d-orbifolds*, as in my arXiv:1208.4948. There is an equivalence of weak 2-categories **Kur** $\simeq$ **dOrb**, so Kuranishi spaces and d-orbifolds are interchangeable. Kuranishi spaces are simpler. The definition of Kuranishi spaces combines those of M-Kuranishi spaces in §6, and orbifolds in §7. We define 2-categories $\mathrm{Kur}_S(X)$ of 'Kuranishi neighbourhoods' on X supported on open $S \subseteq X$, with restriction functors $|_T : \mathrm{Kur}_S(X) \to \mathrm{Kur}_T(X)$ for open $T \subseteq S \subseteq X$, and show they satisfy the stack property. Then the same method as for orbifolds defines Kuranishi spaces as topological spaces with an atlas of Kuranishi neighbourhoods.

In fact 'Kuranishi spaces' (with a different, non-equivalent definition, which we will call 'FOOO Kuranishi spaces') have been used for many years in the work of Fukaya et al. in symplectic geometry (Fukaya and Ono 1999, Fukaya–Oh–Ohta–Ono 2009), as the geometric structure on moduli spaces of J-holomorphic curves. There are problems with their theory (e.g. there is no notion of morphism of FOOO Kuranishi space), and I claim my definition is the 'correct' definition of Kuranishi space, which should replace the FOOO definition. Any FOOO Kuranishi space $\mathbf{X}$ can be made into a Kuranishi space $\mathbf{X}'$ in my sense, uniquely up to equivalence in **Kur**, so this replacement can be done fairly painlessly.

8.1. Kuranishi neighbourhoods and coordinate changes

Definition 8.1

Let X be a topological space. A *Kuranishi neighbourhood* on X is a quintuple (V, E, Γ, s, ψ) such that:

(a) V is a smooth manifold.
(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.
(c) Γ is a finite group with compatible smooth actions on V and E preserving the vector bundle structure.
(d) $s : V \to E$ is a Γ-equivariant smooth section of E, the *Kuranishi section*.
(e) $\psi : s^{-1}(0)/\Gamma \to X$ is a homeomorphism with an open $\mathrm{Im}\,\psi \subseteq X$. We write $\bar{\psi}$ for the composition $s^{-1}(0) \to s^{-1}(0)/\Gamma \xrightarrow{\psi} X$.

If $S \subseteq X$ is open, we call (V, E, Γ, s, ψ) a *Kuranishi neighbourhood over S* if $S \subseteq \mathrm{Im}\,\psi \subseteq X$.

This is the same as Fukaya-Oh-Ohta-Ono Kuranishi neighbourhoods.

Definition 8.2

Let X be a topological space, $(V_i,E_i,\Gamma_i,s_i,\psi_i)$, $(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be Kuranishi neighbourhoods on X, and $S\subseteq \mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j\subseteq X$ be open. A 1-*morphism* $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\to(V_j,E_j,\Gamma_j,s_j,\psi_j)$ *over* S is a quadruple $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$ satisfying:

(a) P_{ij} is a manifold with a smooth action of $\Gamma_i\times\Gamma_j$, with the Γ_j-action free.

(b) $\pi_{ij}:P_{ij}\to V_i$ is Γ_i-equivariant, Γ_j-invariant, and étale. The image $V_{ij}:=\pi_{ij}(P_{ij})$ is a Γ_i-invariant open neighbourhood of $\bar\psi_i^{-1}(S)$ in V_i, and $\pi_{ij}:P_{ij}\to V_{ij}$ is a principal Γ_j-bundle.

(c) $\phi_{ij}:P_{ij}\to V_j$ is a Γ_i-invariant and Γ_j-equivariant smooth map.

(d) $\hat\phi_{ij}:\pi_{ij}^*(E_i)\to\phi_{ij}^*(E_j)$ is a $\Gamma_i\times\Gamma_j$-equivariant morphism of vector bundles on P_{ij}, using the given Γ_i-action and the trivial Γ_j-action on E_i, and vice versa for E_j.

(e) $\hat\phi_{ij}(\pi_{ij}^*(s_i))=\phi_{ij}^*(s_j)+O(\pi_{ij}^*(s_i)^2)$.

(f) $\bar\psi_i\circ\pi_{ij}=\bar\psi_j\circ\phi_{ij}$ on $\pi_{ij}^{-1}(s_i^{-1}(0))\subseteq P_{ij}$.

Here $[V_{ij}/\Gamma_i]\subseteq[V_i/\Gamma_i]$ is an open sub-orbifold, and $\boldsymbol{\phi}_{ij}=(P_{ij},\pi_{ij},\phi_{ij}):[V_{ij}/\Gamma_i]\to[V_j/\Gamma_j]$ is a Hilsum–Skandalis morphism of orbifolds, as in §7.1. We can interpret E_i,E_j as *orbifold vector bundles* over $[V_i/\Gamma_i],[V_j/\Gamma_j]$ with sections s_i,s_j, and $\hat\phi_{ij}$ as a morphism $E_i\to\boldsymbol{\phi}_{ij}^*(E_j)$ of vector bundles on $[V_{ij}/\Gamma_i]$ with $\hat\phi_{ij}(s_i)=\boldsymbol{\phi}_{ij}^*(s_j)+O(s_i^2)$.

Thus, $(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$ is the orbifold analogue of the morphisms $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ of M-Kuranishi neighbourhoods in §6.1, with $(P_{ij},\pi_{ij},\phi_{ij})$ in place of ϕ_{ij}.

For M-Kuranishi spaces, we took equivalence classes $[V_{ij},\phi_{ij},\hat\phi_{ij}]$ of triples $(V_{ij},\phi_{ij},\hat\phi_{ij})$. Here we do not take equivalence classes for 1-morphisms, but we will for 2-morphisms.

Definition 8.3

Let $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be 1-morphisms of Kuranishi neighbourhoods over $S \subseteq \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \subseteq X$, where $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij})$ and $\Phi'_{ij} = (P'_{ij}, \pi'_{ij}, \phi'_{ij}, \hat\phi'_{ij})$.

Consider triples $(\dot P_{ij}, \lambda_{ij}, \hat\lambda_{ij})$ satisfying:

(a) $\dot P_{ij}$ is a $\Gamma_i \times \Gamma_j$-invariant open neighbhd of $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$ in P_{ij}.

(b) $\lambda_{ij} : \dot P_{ij} \to P'_{ij}$ is a $\Gamma_i \times \Gamma_j$-equivariant smooth map with $\pi'_{ij} \circ \lambda_{ij} = \pi_{ij}|_{\dot P_{ij}}$. This implies that λ_{ij} is a diffeomorphism with a $\Gamma_i \times \Gamma_j$-invariant open set $\lambda_{ij}(\dot P_{ij})$ in P'_{ij}.

(c) $\hat\lambda_{ij} : \pi_{ij}^*(E_i)|_{\dot P_{ij}} \to \phi_{ij}^*(TV_j)|_{\dot P_{ij}}$ is a Γ_i- and Γ_j-invariant smooth morphism of vector bundles on $\dot P_{ij}$, satisfying

$$\begin{aligned} \phi'_{ij} \circ \lambda_{ij} &= \phi_{ij}|_{\dot P_{ij}} + \hat\lambda_{ij} \cdot \pi_{ij}^*(s_i) + O\bigl(\pi_{ij}^*(s_i)^2\bigr) \quad \text{and} \\ \lambda_{ij}^*(\hat\phi'_{ij}) &= \hat\phi_{ij}|_{\dot P_{ij}} + \hat\lambda_{ij} \cdot \phi_{ij}^*(\mathrm{d}s_j) + O\bigl(\pi_{ij}^*(s_i)\bigr) \quad \text{on } \dot P_{ij}. \end{aligned} \tag{8.1}$$

Definition (Continued)

Define an equivalence relation $\approx$ (or $\approx_S$) on such triples by $(\dot P_{ij}, \lambda_{ij}, \hat\lambda_{ij}) \approx (\dot P'_{ij}, \lambda'_{ij}, \hat\lambda'_{ij})$ if there exists an open neighbourhood $\ddot P_{ij}$ of $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$ in $\dot P_{ij} \cap \dot P'_{ij}$ with

$$\lambda_{ij}|_{\ddot P_{ij}} = \lambda'_{ij}|_{\ddot P_{ij}} \quad \text{and} \quad \hat\lambda_{ij}|_{\ddot P_{ij}} = \hat\lambda'_{ij}|_{\ddot P_{ij}} + O\bigl(\pi_{ij}^*(s_i)\bigr) \quad \text{on } \ddot P_{ij}. \tag{8.2}$$

Write $[\dot P_{ij}, \lambda_{ij}, \hat\lambda_{ij}]$ for the $\approx$-equivalence class of $(\dot P_{ij}, \lambda_{ij}, \hat\lambda_{ij})$. We say that $[\dot P_{ij}, \lambda_{ij}, \hat\lambda_{ij}] : \Phi_{ij} \Rightarrow \Phi'_{ij}$ is a *2-morphism of 1-morphisms of Kuranishi neighbourhoods on X over S*, or just a *2-morphism over S*. We often write $\Lambda_{ij} = [\dot P_{ij}, \lambda_{ij}, \hat\lambda_{ij}]$ or $\boldsymbol{\lambda}_{ij} = [\dot P_{ij}, \lambda_{ij}, \hat\lambda_{ij}]$.

Here (8.1) is the orbifold version of standard model 2-morphisms of d-manifolds in §5.3, and (8.2) the orbifold version of when two standard model 2-morphisms are equal from Theorem 5.3(b).

A 2-morphism $[\dot P_{ij},\lambda_{ij},\hat\lambda_{ij}]:\Phi_{ij}\Rightarrow\Phi'_{ij}$ really consists of three pieces of data:

(i) An open neighbourhood $[\dot V_{ij}/\Gamma_i]$ of $\psi_i^{-1}(S)$ in $[V_{ij}/\Gamma_i]\cap[V'_{ij}/\Gamma_i]\subseteq[V_i/\Gamma_i]$, where $\dot P_{ij}=\pi_{ij}^{-1}(\dot V_{ij})$.

(ii) A 2-morphism of orbifolds $\lambda_{ij}:(P_{ij},\phi_{ij},\phi_{ij})|_{[\dot V_{ij}/\Gamma_i]}\Rightarrow(P'_{ij},\phi'_{ij},\phi'_{ij})|_{[\dot V_{ij}/\Gamma_i]}$, in the sense of §7.

(iii) A 'standard model' 2-morphism of derived manifolds $\hat\lambda_{ij}$, lifted to derived orbifolds.

There is little interaction between (ii) and (iii); the 'orbifold' and 'derived manifold' generalizations of manifolds are more-or-less independent.

We can define composition of 1- and 2-morphisms of Kuranishi neighbourhoods, and identity 1- and 2-morphisms, by combining the orbifold story in §7 with the derived manifold story in §5–§6. In this way we obtain a strict 2-category $\mathrm{Kur}_S(X)$ of Kuranishi neighbourhoods over $S\subseteq X$.

If $T\subseteq S\subseteq X$ are open there is a restriction 2-functor $|_T:\mathrm{Kur}_S(X)\to\mathrm{Kur}_T(X)$. On objects $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ and 1-morphisms $(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$, this just acts as the identity. But for 2-morphisms $[\dot P_{ij},\lambda_{ij},\hat\lambda_{ij}]$ in $\mathrm{Kur}_S(X)$, the equivalence relation $\approx_S$ on triples $(\dot P_{ij},\lambda_{ij},\hat\lambda_{ij})$ depends on S, as (8.2) must hold in a neighbourhood of $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$. So $|_T$ maps the $\approx_S$-equivalence class $[\dot P_{ij},\lambda_{ij},\hat\lambda_{ij}]_S$ to the $\approx_T$-equivalence class $[\dot P_{ij},\lambda_{ij},\hat\lambda_{ij}]_T$.

Definition 8.4

A 1-morphism $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\to(V_j,E_j,\Gamma_j,s_j,\psi_j)$ of Kuranishi neighbourhoods over $S\subseteq X$ is a *coordinate change* over S if it is an equivalence in the 2-category $\mathrm{Kur}_S(X)$.

Theorems 5.7 and 6.5 gave criteria for a standard model 1-morphism to be an equivalence, and a morphism of M-Kuranishi neighbourhoods to be a coordinate change. Here is the orbifold analogue:

Theorem 8.5

Let $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ *be a 1-morphism of Kuranishi neighbourhoods over* $S\subseteq X$*. Let* $p\in\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))\subseteq P_{ij}$*, and set* $v_i=\pi_{ij}(p)\in V_i$ *and* $v_j=\phi_{ij}(p)\in V_j$*. Consider the complex of real vector spaces:*

$$0\rightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}s_i|_{v_i}\oplus(\mathrm{d}\phi_{ij}|_p\circ\mathrm{d}\pi_{ij}|_p^{-1})}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{-\hat\phi_{ij}|_p\oplus\mathrm{d}s_j|_{v_j}}E_j|_{v_j}\rightarrow 0. \quad (8.3)$$

Also consider the morphism of finite groups

$$\rho_p:\bigl\{(\gamma_i,\gamma_j)\in\Gamma_i\times\Gamma_j:(\gamma_i,\gamma_j)\cdot p=p\bigr\}\longrightarrow\bigl\{\gamma_j\in\Gamma_j:\gamma_j\cdot v_j=v_j\bigr\},$$
$$\rho_p:(\gamma_i,\gamma_j)\longmapsto\gamma_j. \quad (8.4)$$

Then Φ_{ij} *is a coordinate change over* S *iff* (8.3) *is exact and* (8.4) *is an isomorphism for all* $p\in\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$.

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Example 8.6

In Fukaya–Oh–Ohta–Ono Kuranishi spaces, a 'coordinate change' $(V_{ij},\rho_{ij},\varphi_{ij},\hat\varphi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ consists of a Γ_i-invariant open $V_{ij}\subseteq V_i$, a group morphism $\rho_{ij}:\Gamma_i\rightarrow\Gamma_j$, a ρ_{ij}-equivariant embedding of submanifolds $\varphi_{ij}:V_{ij}\hookrightarrow V_j$, and a ρ_{ij}-equivariant embedding of vector bundles $\hat\varphi_{ij}:E_i|_{V_{ij}}\hookrightarrow\varphi_{ij}^*(E_j)$ with $\hat\varphi_{ij}\circ s_i=\varphi_{ij}^*(s_j)$, such that the induced morphism $(\mathrm{d}s_i)_*:\varphi_{ij}^*(TV_j)/TV_{ij}\rightarrow\varphi_{ij}^*(E_j)/E_i$ is an isomorphism near $s_i^{-1}(0)$, and ρ restricts to an isomorphism $\mathrm{Stab}_{\Gamma_i}(v)\rightarrow\mathrm{Stab}_{\Gamma_j}(\varphi_{ij}(v))$ for all $v\in\bar\psi_i^{-1}(S)$.
By Theorem 8.5 we can show that this induces a coordinate change $(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$ in our sense, with $P_{ij}=V_i\times\Gamma_j$ the trivial principal Γ_j-bundle over V_i. But FOOO coordinate changes are very special examples of ours; they only exist if $\dim V_i\leqslant\dim V_j$. Our coordinate changes are more flexible, and are invertible up to 2-isomorphisms.

As for Theorems 6.6 and 7.11 we have:

Theorem 8.7

Let $(V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be Kuranishi neighbourhoods on X. For each open $S\subseteq \mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, write $\mathcal{C}oord\bigl((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)\bigr)(S)$ for the groupoid of coordinate changes $(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\to(V_j,E_j,\Gamma_j,s_j,\psi_j)$ over S, and for all open $T\subseteq S\subseteq \mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$ define

$$\rho_{ST}:\mathcal{C}oord\bigl((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)\bigr)(S)\longrightarrow \mathcal{C}oord\bigl((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)\bigr)(T)\ \text{ by }\ \rho_{ST}=|_T,$$

and for all open $U\subseteq T\subseteq S\subseteq X$ define $\eta_{STU}=\mathrm{id}_{\rho_{SU}}:\rho_{TU}\circ\rho_{ST}=\rho_{SU}\Rightarrow\rho_{SU}$. Then $\mathcal{C}oord\bigl((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)\bigr)$ is a **stack** *on the topological space $\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$.*

The important, nontrivial part is a gluing result for coordinate changes on an open cover, with given 2-isomorphisms on double overlaps and an associativity condition on triple overlaps.

8.2. The 2-category of Kuranishi spaces **Kur**

Definition 8.8

Let X be a Hausdorff, second countable topological space. A *Kuranishi structure* $\mathcal{K}$ on X of *virtual dimension* $n\in\mathbb{Z}$ is data $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \Lambda_{ijk,\ i,j,k\in I}\bigr)$, where:

(a) I is an indexing set.

(b) $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ is a Kuranishi neighbourhood on X for $i\in I$, with $\dim V_i-\mathrm{rank}\,E_i=n$. Write $S_{ij}=\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, etc.

(c) $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\to(V_j,E_j,\Gamma_j,s_j,\psi_j)$ is a coordinate change over S_{ij} for $i,j\in I$.

(d) $\Lambda_{ijk}:\Phi_{jk}\circ\Phi_{ij}\Rightarrow\Phi_{ik}$ is a 2-morphism over S_{ijk} for $i,j,k\in I$.

(e) $\bigcup_{i\in I}\mathrm{Im}\,\psi_i=X$. (f) $\Phi_{ii}=\mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}$ for $i\in I$.

(g) $\Lambda_{iij}=\Lambda_{ijj}=\mathrm{id}_{\Phi_{ij}}$ for $i,j\in I$.

(h) $\Lambda_{ikl}\odot(\mathrm{id}_{\Phi_{kl}}*\Lambda_{ijk})|_{S_{ijkl}}=\Lambda_{ijl}\odot(\Lambda_{jkl}*\mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}}:$ $\Phi_{kl}\circ\Phi_{jk}\circ\Phi_{ij}|_{S_{ijkl}}\Longrightarrow\Phi_{il}|_{S_{ijkl}}$ for $i,j,k,l\in I$.

We call $\mathbf{X}=(X,\mathcal{K})$ a *Kuranishi space*, with $\mathrm{vdim}\,\mathbf{X}=n$.

Definition 8.8 is a direct analogue of orbifolds in Definition 7.12, replacing orbifold charts by Kuranishi neighbourhoods. It covers X by an 'atlas of charts' $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ over $\mathrm{Im}\,\psi_i \subseteq X$, with coordinate changes Φ_{ij} on double overlaps $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, and 2-isomorphisms $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij}|_{S_{ijk}} \Rightarrow \Phi_{ik}|_{S_{ijk}}$ on triple overlaps $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_k$, with associativity $\Lambda_{ikl} \odot (\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk})|_{S_{ijkl}} = \Lambda_{ijl} \odot (\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}}$ on quadruple overlaps $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_k \cap \mathrm{Im}\,\psi_l$.

Once you have grasped the idea that Kuranishi neighbourhoods over $S \subseteq X$ form a 2-category, with restriction $|_T$ to open subsets $T \subseteq S \subseteq X$, Definition 8.8, although complicated, is obvious, and necessary: it is the only sensible way to make a global space by gluing local charts in the world of 2-categories.

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Kuranishi neighbourhoods and coordinate changes
The 2-category of Kuranishi spaces **Kur**

We can also define 1-morphisms and 2-morphisms of Kuranishi spaces. To do this, given a continuous map $f : X \to Y$ and Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(W_j, F_j, \Delta_j, t_j, \chi_j)$ on X, Y, and open $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$, we first have to define 1-morphisms $\mathbf{f}_{ij} = (P_{ij}, \pi_{ij}, f_{ij}, \hat{f}_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (W_j, F_j, \Delta_j, t_j, \chi_j)$ of Kuranishi neighbourhoods over S and f, and 2-morphisms $\Lambda_{ij} : (P_{ij}, \pi_{ij}, f_{ij}, \hat{f}_{ij}) \Rightarrow (P'_{ij}, \pi'_{ij}, f'_{ij}, \hat{f}'_{ij})$, compositions $\circ, \odot, *$, and identities. These generalize Definitions 8.2 and 8.3 for $f = \mathrm{id}_X$, so we leave them as an exercise.

Definition 8.9

Let $\mathbf{X}=(X,\mathcal{K})$ and $\mathbf{Y}=(Y,\mathcal{L})$ be Kuranishi spaces, with $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\ \Phi_{ii',\,i,i'\in I},\ \Lambda_{ii'i'',\,i,i',i''\in I}\bigr)$ and $\mathcal{L}=\bigl(J,(W_j,F_j,\Delta_j,t_j,\chi_j)_{j\in J},\ \Psi_{jj',\,j,j'\in J},\ \mathrm{M}_{jj'j'',\,j,j',j''\in J}\bigr)$. A 1-*morphism* $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is $\mathbf{f}=\bigl(f,\mathbf{f}_{ij,\,i\in I,\,j\in J},\ F_{ii',\,i,i'\in I}^{j,\,j\in J},\ F_{i,\,i\in I}^{jj',\,j,j'\in J}\bigr)$, with: (a) $f:X\to Y$ is a continuous map.

(b) $\mathbf{f}_{ij}=(P_{ij},\pi_{ij},f_{ij},\hat f_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\to(W_j,F_j,\Delta_j,t_j,\chi_j)$ is a 1-morphism of Kuranishi neighbourhoods over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for $i\in I$, $j\in J$.

(c) $F_{ii'}^j:\mathbf{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\mathbf{f}_{ij}$ is a 2-morphism over f for $i,i'\in I$, $j\in J$.

(d) $F_i^{jj'}:\Psi_{jj'}\circ\mathbf{f}_{ij}\Rightarrow\mathbf{f}_{ij'}$ is a 2-morphism over f for $i\in I$, $j,j'\in J$.

(e) $F_{ii}^j=F_i^{jj}=\mathrm{id}_{\mathbf{f}_{ij}}$.

(f) $F_{ii''}^j\odot(\mathrm{id}_{\mathbf{f}_{i''j}}*\Lambda_{ii'i''})=F_{ii'}^j\odot(F_{i'i''}^j*\mathrm{id}_{\Phi_{ii'}}):\mathbf{f}_{i''j}\circ\Phi_{i'i''}\circ\Phi_{ii'}\Rightarrow\mathbf{f}_{i''j}$.

(g) $F_i^{jj'}\odot(\mathrm{id}_{\Psi_{jj'}}*F_{ii'}^j)=F_{ii'}^{j'}\odot(F_{i'}^{jj'}*\mathrm{id}_{\Phi_{ii'}}):\Psi_{jj'}\circ\mathbf{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\mathbf{f}_{ij'}$.

(h) $F_i^{j'j''}\odot(\mathrm{id}_{\Psi_{j'j''}}*F_i^{jj'})=F_i^{jj''}\odot(\mathrm{M}_{jj'j''}*\mathrm{id}_{\mathbf{f}_{ij}}):\Psi_{j'j''}\circ\Psi_{jj'}\circ\mathbf{f}_{ij}\Rightarrow\mathbf{f}_{ij''}$.

Here (c)–(h) hold for all $i,j,\ldots$, restricted to appropriate domains.

Definition 8.10

Let $\mathbf{f},\mathbf{g}:\mathbf{X}\to\mathbf{Y}$ be 1-morphisms of Kuranishi spaces, with $\mathbf{f}=\bigl(f,\mathbf{f}_{ij,\,i\in I,\,j\in J},\ F_{ii',\,i,i'\in I}^{j,\,j\in J},\ F_{i,\,i\in I}^{jj',\,j,j'\in J}\bigr)$, $\mathbf{g}=\bigl(g,\mathbf{g}_{ij,\,i\in I,\,j\in J},\ G_{ii',\,i,i'\in I}^{j,\,j\in J},\ G_{i,\,i\in I}^{jj',\,j,j'\in J}\bigr)$. Suppose the continuous maps $f,g:X\to Y$ satisfy $f=g$. A 2-*morphism* $\boldsymbol{\Lambda}:\mathbf{f}\Rightarrow\mathbf{g}$ is data $\boldsymbol{\Lambda}=(\Lambda_{ij,\,i\in I,\,j\in J})$, where $\Lambda_{ij}:\mathbf{f}_{ij}\Rightarrow\mathbf{g}_{ij}$ is a 2-morphism of Kuranishi neighbourhoods over $f=g$, satisfying:

(a) $G_{ii'}^j\odot(\Lambda_{i'j}*\mathrm{id}_{\Phi_{ii'}})=\Lambda_{ij}\odot F_{ii'}^j:\mathbf{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\mathbf{g}_{ij}$ for $i,i'\in I$, $j\in J$.

(b) $G_i^{jj'}\odot(\mathrm{id}_{\Psi_{jj'}}*\Lambda_{ij})=\Lambda_{ij'}\odot F_i^{jj'}:\Psi_{jj'}\circ\mathbf{f}_{ij}\Rightarrow\mathbf{g}_{ij'}$ for $i\in I$, $j,j'\in J$.

Definitions 8.9, 8.10 are direct analogues of the orbifold versions. We can then define composition of 1- and 2-morphisms, identity 1- and 2-morphisms, and so on, making Kuranishi spaces into a weak 2-category **Kur**. Composition of 1-morphisms needs the stack property of 1-morphisms of Kuranishi neighbhds, as in Theorem 8.7.

Write $\mathbf{Kur_{tr\Gamma}}$ for the full 2-subcategory of **Kur** with objects $\mathbf{X}=(X,\mathcal{K})$ in which the Kuranishi neighbourhoods $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ in $\mathcal{K}$ have $\Gamma_i=\{1\}$ for all $i\in I$. Write $\mathbf{Kur_{trG}}$ for the full 2-subcategory of $\mathbf{X}$ in which Γ_i acts freely on $s_i^{-1}(0)\subseteq V_i$ for all $i\in I$. Then $\mathbf{Kur_{tr\Gamma}}\subset\mathbf{Kur_{trG}}\subset\mathbf{Kur}$. Both $\mathbf{Kur_{tr\Gamma}},\mathbf{Kur_{trG}}$ are 2-categories of derived manifolds. In the notation of next time, $\mathbf{Kur_{trG}}$ is the full 2-subcategory of Kuranishi spaces $\mathbf{X}$ with trivial orbifold groups $G_x\mathbf{X}=\{1\}$ for all $x\in\mathbf{X}$.

Theorem 8.11

There are equivalences of weak 2-categories

$$\mathbf{Kur_{tr\Gamma}}\simeq\mathbf{Kur_{trG}}\simeq\mathbf{dMan},\qquad \mathbf{Kur}\simeq\mathbf{dOrb},$$

and equivalences of (homotopy) categories

$$\mathbf{MKur}\simeq\mathrm{Ho}(\mathbf{Kur_{tr\Gamma}})\simeq\mathrm{Ho}(\mathbf{Kur_{trG}})\simeq\mathrm{Ho}(\mathbf{dMan}).$$

So for most purposes Kuranishi spaces (with trivial orbifold groups) and d-orbifolds (or d-manifolds) are interchangeable.