Lie brackets and vertex algebra structures on the homology of moduli spaces, and wall-crossing formulae

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Plan of talk:

1. Introduction to Ringel–Hall algebras

Let $\mathbb{K}$ be a field, and $\mathcal{A}$ a $\mathbb{K}$-linear abelian category satisfying some conditions, e.g. $\mathcal{A}$ could be the category $\text{mod-}\mathbb{K}Q$ of representations of a quiver Q, or the category $\text{coh}(X)$ of coherent sheaves on a smooth projective $\mathbb{K}$-scheme X. Write $\mathfrak{M}$ for the moduli stack of objects in $\mathcal{A}$, which should be an Artin $\mathbb{K}$-stack, locally of finite type, and $\mathfrak{M}(\mathbb{K})$ for the set of $\mathbb{K}$-points.

There are several versions of the *Ringel–Hall algebra* $\mathcal{H}$ associated to $\mathcal{A}$. In one version, $\mathcal{H}$ is a $\mathbb{Q}$-vector space of some class of functions $f : \mathfrak{M}(\mathbb{K}) \to \mathbb{Q}$ (e.g. functions with finite support, or constructible functions $\mathcal{H} = \text{CF}(\mathfrak{M})$) equipped with an associative multiplication $*$ making $\mathcal{H}$ into a $\mathbb{Q}$-algebra, with unit $\delta_0 : \mathfrak{M}(\mathbb{K}) \to \mathbb{Q}$ the function which is 1 on $0 \in \mathcal{A}$ and 0 otherwise.

Introduction to Ringel–Hall algebras
Ringel–Hall vertex algebras and Lie algebras on homology
Examples from quivers and coherent sheaves
Enumerative invariants and wall-crossing

Write $\mathfrak{Exact}$ for the moduli stack of exact sequences $E_\bullet = 0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}$, with projections $\Pi_i : \mathfrak{Exact} \to \mathfrak{M}$ mapping $E_\bullet \to E_i$ for $i = 1, 2, 3$. Then $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is

$$f * g = (\Pi_2)_* \circ (\Pi_1, \Pi_3)^*(f \boxtimes g).$$

Here $\Pi_2 : \mathfrak{Exact} \to \mathfrak{M}$ is a representable morphism, and $(\Pi_1, \Pi_3) : \mathfrak{Exact} \to \mathfrak{M} \times \mathfrak{M}$ is a finite type morphism, and pushforwards (pullbacks) of $\mathcal{H}$-type functions should be defined for representable (finite type) morphisms of Artin $\mathbb{K}$-stacks.
Ringel–Hall algebras are studied in Geometric Representation Theory, for instance to construct Quantum Groups from $\mathcal{A} = \text{mod-}\mathbb{K}Q$ for Q an ADE quiver.
Note that $\mathcal{H}$ is also a *Lie algebra*, with Lie bracket $[f, g] = f * g - g * f$, the Jacobi identity follows from $*$ associative.

Ringel–Hall algebras of constructible functions were used in my work on 'Configurations in abelian categories' to study wall-crossing under change of stability condition. Let $K(\mathcal{A})$ be a quotient group of the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ (e.g. $K(\mathcal{A}) = K^{\mathrm{num}}(\mathcal{A})$ could be the numerical Grothendieck group) such that $\mathfrak{M} = \coprod_{\alpha\in K(\mathcal{A})} \mathfrak{M}_\alpha$ with $\mathfrak{M}_\alpha$ the moduli stack of objects $E \in \mathcal{A}$ in class α in $K(\mathcal{A})$ an open and closed substack in $\mathfrak{M}$, and $\mathfrak{M}_0 = \{0\}$. Write $C(\mathcal{A}) = \{0 \neq \alpha \in K(\mathcal{A}) : \mathfrak{M}_\alpha \neq \emptyset\}$, the 'positive cone' in $K(\mathcal{A})$. A *stability condition* $(\tau, T, \leqslant)$ on $\mathcal{A}$ is a total order $(T, \leqslant)$ and a map $\tau : C(\mathcal{A}) \to T$ such that if $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then $\tau(\alpha) \leqslant \tau(\beta) \leqslant \tau(\gamma)$ or $\tau(\alpha) \geqslant \tau(\beta) \geqslant \tau(\gamma)$. Then we can define when an object $E \in \mathcal{A}$ is *τ-semistable*. Write $\mathfrak{M}^{\mathrm{ss}}_\alpha(\tau) \subseteq \mathfrak{M}_\alpha \subset \mathfrak{M}$ for the moduli stack of τ-semistable objects in class α in $\mathcal{A}$.

We call τ *permissible* if $\mathfrak{M}^{\mathrm{ss}}_\alpha(\tau)$ is a finite type Artin $\mathbb{K}$-stack for all $\alpha \in C(\mathcal{A})$. Then the characteristic function $\delta_{\mathfrak{M}^{\mathrm{ss}}_\alpha(\tau)}$ is an element of the constructible Ringel–Hall algebra $\mathcal{H} = \mathrm{CF}(\mathfrak{M})$.
If $\tau, \tilde\tau$ are two permissible stability conditions, I gave a *universal wall-crossing formula* which wrote $\delta_{\mathfrak{M}^{\mathrm{ss}}_\alpha(\tilde\tau)}$ as a sum of products of $\delta_{\mathfrak{M}^{\mathrm{ss}}_\beta(\tau)}$ in $\mathcal{H}$, with combinatorial coefficients depending on $\tau, \tilde\tau$.
I also defined elements $\epsilon_\alpha(\tau)$ in $\mathcal{H}$ for $\alpha \in C(\mathcal{A})$ by

$$\epsilon_\alpha(\tau) = \sum_{\substack{n\geqslant 1,\ \alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\\ \tau(\alpha_i)=\tau(\alpha),\ \alpha_1+\cdots+\alpha_n=\alpha}} \frac{(-1)^n}{n} \cdot \delta_{\mathfrak{M}^{\mathrm{ss}}_{\alpha_1}(\tau)} * \cdots * \delta_{\mathfrak{M}^{\mathrm{ss}}_{\alpha_n}(\tau)}.$$

I showed the $\epsilon_\alpha(\tau)$ also satisfy a universal wall-crossing formula in $\mathcal{H}$ under change of stability condition, which can be written using only the Lie bracket on $\mathcal{H}$. That is, we can write $\epsilon_\alpha(\tilde\tau)$ as a $\mathbb{Q}$-linear combination of multiple Lie brackets $[\epsilon_{\beta_1}(\tau), [\epsilon_{\beta_2}(\tau), [\cdots]]]$.
This has applications to wall-crossing of DT invariants for C–Y 3-folds.

2. Ringel–Hall vertex algebras and Lie algebras on homology

Let $\mathcal{A}$ be a $\mathbb{K}$-linear abelian category as before, and $\mathfrak{M}$ the moduli stack of objects in $\mathcal{A}$, an Artin $\mathbb{K}$-stack, locally of finite type.
Suppose we have a homology theory $H_*(-)$ of Artin $\mathbb{K}$-stacks over a commutative ring R (e.g. $R=\mathbb{Q}$), satisfying some axioms.
Given some extra data on $\mathfrak{M}$, we will define a *vertex algebra* structure on the homology $H_*(\mathfrak{M})$. We also define a *graded Lie bracket* $[\,,\,]$ on $H_*(\mathfrak{M})$ (or rather, a modification of this), making $H_*(\mathfrak{M})$ into a *graded Lie (super)algebra* (with a nonstandard grading). This is analogous to the Ringel–Hall Lie algebra $(\mathrm{CF}(\mathfrak{M}),[\,,\,])$, but with $\mathrm{CF}(\mathfrak{M})$ replaced by $H_*(\mathfrak{M})$.
There are lots of interesting applications:

- Lie algebras in Geometric Representation Theory from quivers, etc.
- Explain Grojnowski–Nakajima on (co)homology of Hilbert schemes.
- Wall-crossing for virtual cycles in enumerative invariant problems.
- A differential-geometric version for use in gauge theory.

The extra data we need

We have $f*g=(\Pi_2)_*\circ(\Pi_1,\Pi_3)^*(f\boxtimes g)$ for Ringel–Hall algebras of constructible functions $\mathrm{CF}(\mathfrak{M})$. If we replace $\mathrm{CF}(\mathfrak{M})$ by $H_*(\mathfrak{M})$ then the pushforward $(\Pi_2)_*$ is natural, but the pullback $(\Pi_1,\Pi_3)^*$ is not. To define our substitute for $(\Pi_1,\Pi_3)^*$ we need some extra data, a perfect complex $\Theta^\bullet$ on $\mathfrak{M}\times\mathfrak{M}$ satisfying some assumptions; the formula for $[\,,\,]$ involves $\operatorname{rank}\Theta^\bullet$ and $c_i(\Theta^\bullet)$.
We also need signs $\epsilon_{\alpha,\beta}$ related to 'orientation data' for $\mathcal{A}$.
For graded antisymmetry of $[\,,\,]$ we need $\sigma^*(\Theta^\bullet)\cong(\Theta^\bullet)^\vee[2n]$ for some $n\in\mathbb{Z}$, where $\sigma:\mathfrak{M}\times\mathfrak{M}\to\mathfrak{M}\times\mathfrak{M}$ exchanges the factors, as then $c_i(\sigma^*(\Theta^\bullet))=(-1)^i c_i(\Theta^\bullet)$.
In our examples there is a natural perfect complex $\mathcal{E}xt^\bullet$ on $\mathfrak{M}\times\mathfrak{M}$ with $H^i(\mathcal{E}xt^\bullet)|_{([E],[F])}\cong\mathrm{Ext}^i_{\mathcal{A}}(E,F)$ for $E,F\in\mathcal{A}$ and $i\in\mathbb{Z}$. If $\mathcal{A}$ is a $2n$-Calabi–Yau category then $\sigma^*((\mathcal{E}xt^\bullet)^\vee)\cong\mathcal{E}xt^\bullet[2n]$, and we put $\Theta^\bullet=(\mathcal{E}xt^\bullet)^\vee$. Otherwise we put $\Theta^\bullet=(\mathcal{E}xt^\bullet)^\vee+\sigma^*(\mathcal{E}xt^\bullet)[2n]$.
Thus examples split into 'even Calabi–Yau' and 'general' Lie algebras.

More detail on the basic set-up

Let $K(\mathcal{A})$ be a quotient group of the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ such that $\mathfrak{M}=\coprod_{\alpha\in K(\mathcal{A})}\mathfrak{M}_\alpha$, with $\mathfrak{M}_\alpha$ the moduli stack of objects $E\in\mathcal{A}$ in class α in $K(\mathcal{A})$ an open and closed substack in $\mathfrak{M}$. We suppose we are given a biadditive map $\chi:K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$ called the *Euler form*, with $\chi(\alpha,\beta)=\chi(\beta,\alpha)$. The restriction $\Theta^\bullet_{\alpha,\beta}=\Theta^\bullet|_{\mathfrak{M}_\alpha\times\mathfrak{M}_\beta}$ should have $\operatorname{rank}\Theta^\bullet_{\alpha,\beta}=\chi(\alpha,\beta)$.

There should be an Artin stack morphism $\Phi:\mathfrak{M}\times\mathfrak{M}\to\mathfrak{M}$ mapping $\Phi(\mathbb{K}):([E],[F])\mapsto[E\oplus F]$ on $\mathbb{K}$-points, from direct sum in $\mathcal{A}$. It is associative and commutative. In perfect complexes on $\mathfrak{M}_\alpha\times\mathfrak{M}_\beta\times\mathfrak{M}_\gamma$ for $\alpha,\beta,\gamma\in K(\mathcal{A})$ we should have

$$(\Phi_{\alpha,\beta}\times\mathrm{id}_{\mathfrak{M}_\gamma})^*(\Theta^\bullet_{\alpha+\beta,\gamma})\cong\Pi^*_{\mathfrak{M}_\alpha\times\mathfrak{M}_\gamma}(\Theta^\bullet_{\alpha,\gamma})\oplus\Pi^*_{\mathfrak{M}_\beta\times\mathfrak{M}_\gamma}(\Theta^\bullet_{\beta,\gamma}),$$

needed for the graded Jacobi identity for $[\,,\,]$, and corresponding to

$$\mathrm{Ext}^i_{\mathcal{A}}(E\oplus F,G)^*\cong\mathrm{Ext}^i_{\mathcal{A}}(E,G)^*\oplus\mathrm{Ext}^i_{\mathcal{A}}(F,G)^*.$$

The stack $[*/\mathbb{G}_m]$ and morphism Ψ

Write $\mathbb{G}_m=\mathbb{K}\setminus\{0\}$ as an algebraic $\mathbb{K}$-group under multiplication, and $[*/\mathbb{G}_m]$ for the quotient stack, where $*=\operatorname{Spec}\mathbb{K}$ is the point. If S is an Artin $\mathbb{K}$-stack and $s\in S(\mathbb{K})$ a $\mathbb{K}$-point there is an *isotropy group* $\mathrm{Iso}_S(s)$, an algebraic $\mathbb{K}$-group. We have $\mathrm{Iso}_{\mathfrak{M}}([E])\cong\mathrm{Aut}(E)$ for $E\in\mathcal{A}$. There is a natural morphism $\mathbb{G}_m\to\mathrm{Aut}(E)$ mapping $\lambda\mapsto\lambda\cdot\mathrm{id}_E\in\mathrm{Aut}(E)\subset\mathrm{Hom}_{\mathcal{A}}(E,E)$.

There should be an Artin stack morphism $\Psi:[*/\mathbb{G}_m]\times\mathfrak{M}\to\mathfrak{M}$ mapping $(*,[E])\mapsto[E]$ on $\mathbb{K}$-points, and acting on isotropy groups by

$$\Psi_*:\mathrm{Iso}_{[*/\mathbb{G}_m]\times\mathfrak{M}}(*,[E])\cong\mathbb{G}_m\times\mathrm{Aut}(E)\longrightarrow\mathrm{Iso}_{\mathfrak{M}}([E])\cong\mathrm{Aut}(E),$$

$$\Psi_*:(\lambda,\mu)\longmapsto(\lambda\cdot\mathrm{id}_E)\circ\mu.$$

Here $[*/\mathbb{G}_m]$ is a *group stack*, and Ψ is an *action of* $[*/\mathbb{G}_m]$ *on* $\mathfrak{M}$, which is free except over $[0]\in\mathfrak{M}$. This Ψ encodes the natural morphisms $\mathbb{G}_m\to\mathrm{Iso}_{\mathfrak{M}}([E])$ for all $[E]\in\mathfrak{M}(\mathbb{K})$.

We require a compatibility between Ψ and $\Theta^\bullet$, roughly that

$$(\Psi \times \mathrm{id}_{\mathfrak{M}})^*(\Theta^\bullet) \cong \Pi^*_{[*/\mathbb{G}_m]}(L) \otimes \Pi^*_{\mathfrak{M}\times\mathfrak{M}}(\Theta^\bullet)$$

where L is the line bundle on $[*/\mathbb{G}_m]$ associated to the obvious representation of $\mathbb{G}_m$ on $\mathbb{K}$. This corresponds to $\lambda\,\mathrm{id}_E \in \mathrm{Aut}(E)$ acting by multiplication by $\lambda \in \mathbb{G}_m$ on $\mathrm{Ext}^i(E,F)^*$.

We should be given $\epsilon_{\alpha,\beta} = \pm 1$ for $\alpha, \beta \in K(\mathcal{A})$ satisfying

$$\epsilon_{\alpha,\beta} \cdot \epsilon_{\beta,\alpha} = (-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)},$$
$$\epsilon_{\alpha,\beta} \cdot \epsilon_{\alpha+\beta,\gamma} = \epsilon_{\alpha,\beta+\gamma} \cdot \epsilon_{\beta,\gamma}.$$

They are needed to correct signs in defining $[\,,\,]$. Such $\epsilon_{\alpha,\beta}$ always exist. They are related to 'orientation data' as follows: if we have chosen 'orientations' for $\mathfrak{M}_\alpha, \mathfrak{M}_\beta, \mathfrak{M}_{\alpha+\beta}$, then $\epsilon_{\alpha,\beta}$ should be the natural sign comparing the orientations at $[E] \in \mathfrak{M}_\alpha(\mathbb{K})$, $[F] \in \mathfrak{M}_\beta(\mathbb{K})$ and $[E \oplus F] = \Phi([E],[F]) \in \mathfrak{M}_{\alpha+\beta}(\mathbb{K})$.

The homology of $[*/\mathbb{G}_m]$, and its action on $H_*(\mathfrak{M})$

Let $H_*(-)$ be a homology theory of Artin $\mathbb{K}$-stacks over a commutative ring R, satisfying some natural axioms. Then

$$H_i([*/\mathbb{G}_m]) \cong \begin{cases} R, & i = 0, 2, 4, 6, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

(This holds as the 'classifying space' of $[*/\mathbb{G}_m]$ is $\mathbb{KP}^\infty$.) So we may write $H_*([*/\mathbb{G}_m]) \cong R[t]$, for t a formal variable of degree 2, such that t^n is a basis element for $H_{2n}([*/\mathbb{G}_m])$.
Let $\Omega : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ be the stack morphism induced by the group morphism $\omega : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ mapping $\omega : (\lambda, \mu) \mapsto \lambda\mu$. Define $\star : H_*([*/\mathbb{G}_m]) \times H_*([*/\mathbb{G}_m]) \to H_*([*/\mathbb{G}_m])$ by $\zeta \star \eta = H_*(\Omega)(\zeta \boxtimes \eta)$. Then $\star$ makes $H_*([*/\mathbb{G}_m]) \cong R[t]$ into a commutative R-algebra, with $t^m \star t^n = \binom{m+n}{m} t^{m+n}$.
Define $\diamond : H_*([*/\mathbb{G}_m]) \times H_*(\mathfrak{M}) \to H_*(\mathfrak{M})$ by $\zeta \diamond \theta = H_*(\Psi)(\zeta \boxtimes \theta)$. Then $\diamond$ makes $H_*(\mathfrak{M})$ into a module over $H_*([*/\mathbb{G}_m]) \cong R[t]$.

Bilinear operations $[\,,\,]_n$ on $H_*(\mathfrak{M})$ and vertex algebras

Let $\alpha,\beta\in K(\mathcal{A})$ and $a,b,n\geqslant 0$. Define an R-bilinear operation

$$[\,,\,]_n : H_a(\mathfrak{M}_\alpha)\times H_b(\mathfrak{M}_\beta)\longrightarrow H_{a+b-2n-2\chi(\alpha,\beta)-2}(\mathfrak{M}_{\alpha+\beta})$$

by, for all $\zeta\in H_a(\mathfrak{M}_\alpha)$ and $\eta\in H_b(\mathfrak{M}_\beta)$,

$$[\zeta,\eta]_n=\sum_{\substack{i\geqslant 0:\ 2i\leqslant a+b,\\ i\geqslant n+\chi(\alpha,\beta)+1}}\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}\cdot H_{a+b-2n-2\chi(\alpha,\beta)-2}(\Phi_{\alpha,\beta}\circ(\Psi_\alpha\times\mathrm{id}_{\mathfrak{M}_\beta}))\bigl(t^{i-n-\chi(\alpha,\beta)-1}\boxtimes\bigl[(\zeta\boxtimes\eta)\cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr),$$

where $t^k\in H_{2k}([*/\mathbb{G}_m])$ as above. These are *not* Lie brackets, nor are they $R[t]$-bilinear. However, the operations $t^k\diamond -$ and $[\,,\,]_n$ and 'vacuum vector' $1\in H_0(\mathfrak{M}_0)$ make $H_*(\mathfrak{M})$ into a *graded vertex algebra*, a complicated algebraic structure from Conformal Field Theory and String Theory.

Question

What is the interpretation of these vertex algebras in Physics?

The '$t=0$' Lie algebra

There are many different versions of our Lie algebra construction. Here is one of the simplest, which is well known in the theory of vertex algebras. Write $I_t=\langle t,t^2,t^3,\ldots\rangle_R$ for the ideal in $H_*([*/\mathbb{G}_m])=R[t]$ spanned over R by all positive powers of t. For each $\alpha\in K(\mathcal{A})$, define

$$H_*(\mathfrak{M}_\alpha)^{t=0}=H_*(\mathfrak{M}_\alpha)/(I_t\diamond H_*(\mathfrak{M}_\alpha)),$$

using the representation $\diamond$ of $(R[t],\star)$ on $H_*(\mathfrak{M}_\alpha)$. Now define

$$[\,,\,]^{t=0}: H_a(\mathfrak{M}_\alpha)^{t=0}\times H_b(\mathfrak{M}_\beta)^{t=0}\longrightarrow H_{a+b-2\chi(\alpha,\beta)-2}(\mathfrak{M}_{\alpha+\beta})^{t=0}$$

by $\bigl[\zeta+(I_t\diamond H_*(\mathfrak{M}_\alpha)),\eta+(I_t\diamond H_*(\mathfrak{M}_\beta))\bigr]^{t=0}=[\zeta,\eta]_0+(I_t\diamond H_*(\mathfrak{M}_{\alpha+\beta}))$.

Define an alternative grading on $H_*(\mathfrak{M}_\alpha)^{t=0}$ by

$$\tilde H_i(\mathfrak{M}_\alpha)^{t=0}=H_{i+2-\chi(\alpha,\alpha)}(\mathfrak{M}_\alpha)^{t=0}.$$

Then using $\chi(\alpha,\beta)=\chi(\beta,\alpha)$ we find that $[\,,\,]^{t=0}$ maps

$$[\,,\,]^{t=0}:\tilde H_{\tilde a}(\mathfrak{M}_\alpha)^{t=0}\times\tilde H_{\tilde b}(\mathfrak{M}_\beta)^{t=0}\longrightarrow\tilde H_{\tilde a+\tilde b}(\mathfrak{M}_{\alpha+\beta})^{t=0}.$$

Using identities on the $[\,,\,]_n$, we find that if $\zeta\in\tilde H_{\tilde a}(\mathfrak{M}_\alpha)^{t=0}$, $\eta\in\tilde H_{\tilde b}(\mathfrak{M}_\beta)^{t=0}$ and $\theta\in\tilde H_{\tilde c}(\mathfrak{M}_\gamma)^{t=0}$ then

$$[\eta,\zeta]^{t=0}=(-1)^{\tilde a\tilde b+1}[\zeta,\eta]^{t=0},$$

$$(-1)^{\tilde c\tilde a}[[\zeta,\eta]^{t=0},\theta]^{t=0}+(-1)^{\tilde a\tilde b}[[\eta,\theta]^{t=0},\zeta]^{t=0}$$
$$+(-1)^{\tilde b\tilde c}[[\theta,\zeta]^{t=0},\eta]^{t=0}=0.$$

That is, $[\,,\,]^{t=0}$ is a *graded Lie bracket* on $\tilde H_*(\mathfrak{M})^{t=0}=\bigoplus_{\alpha\in K(\mathcal{A})}\tilde H_*(\mathfrak{M}_\alpha)^{t=0}$, as we want.

In general there is no associative multiplication $*$ on $\tilde H_*(\mathfrak{M})^{t=0}$ with $[\zeta,\eta]^{t=0}=\zeta*\eta-\eta*\zeta$, in contrast to constructible functions case.

The ‘projective linear’ Lie algebra

A disadvantage of the ‘$t=0$’ version is that $H_*(\mathfrak{M})^{t=0}$ is not presented as the homology of a nice space. The ‘projective linear’ version corrects this. Recall that $[*/\mathbb{G}_m]$ is a group stack, and $\Psi:[*/\mathbb{G}_m]\times\mathfrak{M}\to\mathfrak{M}$ is an action of $[*/\mathbb{G}_m]$ on $\mathfrak{M}$, which is free on $\mathfrak{M}'=\mathfrak{M}\setminus\{[0]\}$. We can form a quotient $\mathfrak{M}^{\rm pl}=\mathfrak{M}/[*/\mathbb{G}_m]$ called the ‘projective linear moduli stack’, with a morphism $\Pi^{\rm pl}:\mathfrak{M}'\to\mathfrak{M}^{\rm pl}$ which is a principal $[*/\mathbb{G}_m]$-bundle.

Then $\mathbb{K}$-points of $\mathfrak{M}^{\rm pl}$ are isomorphism classes $[E]$ of nonzero $E\in\mathcal{A}$, and isotropy groups are

$$\mathrm{Iso}_{\mathfrak{M}^{\rm pl}}([E])\cong\mathrm{Aut}(E)/(\mathbb{G}_m\cdot\mathrm{id}_E).$$

That is, we make $\mathfrak{M}^{\rm pl}$ from $\mathfrak{M}'$ by quotienting out $\mathbb{G}_m$ from each isotropy group, a process called ‘rigidification’. For moduli of stable coherent sheaves, the stable moduli scheme is the rigidification of the stable moduli stack.

Under some assumptions (including R a $\mathbb{Q}$-algebra) we can show that $H_*(\Pi^{\rm pl}) : H_*(\mathfrak{M}') \to H_*(\mathfrak{M}^{\rm pl})$ induces an isomorphism $H_*(\mathfrak{M}')^{t=0} \cong H_*(\mathfrak{M}^{\rm pl})$. Thus, the Lie bracket $[\, ,\,]^{t=0}$ on $H_*(\mathfrak{M}')^{t=0}$ induces a Lie bracket $[\, ,\,]^{\rm pl}$ on $H_*(\mathfrak{M}^{\rm pl})$. Actually, even without an isomorphism $H_*(\mathfrak{M}')^{t=0} \cong H_*(\mathfrak{M}^{\rm pl})$ we can define a graded Lie bracket $[\, ,\,]^{\rm pl}$ on $H_*(\mathfrak{M}^{\rm pl})$ in a different way.
Here $[\, ,\,]^{\rm pl}$ is graded for the alternative grading

$$\tilde{H}_i(\mathfrak{M}^{\rm pl}_\alpha) = H_{i+2-\chi(\alpha,\alpha)}(\mathfrak{M}^{\rm pl}_\alpha).$$

We should interpret $2 - \chi(\alpha, \alpha)$ as the (homological) *virtual dimension* of $\mathfrak{M}^{\rm pl}_\alpha$, where the 2 is the (real) dimension of $\mathbb{G}_m$, which we quotiented from the isotropy groups to make $\mathfrak{M}^{\rm pl}$.
There is also a triangulated category version of the construction, using higher stacks, which we can apply to moduli of objects in categories such as $D^b \operatorname{coh}(X)$ for X a smooth projective $\mathbb{K}$-scheme.

2. Examples from quivers and coherent sheaves

Let $Q = (Q_0, Q_1, h, t)$ be a quiver and $\mathbb{K} = \mathbb{C}$, and apply our constructions to the abelian category $\mathcal{A} = \text{mod-}\mathbb{C}Q$ and the triangulated category $\mathcal{T} = D^b \text{mod-}\mathbb{C}Q$. Write $K(\mathcal{A}) = K(\mathcal{T}) = \mathbb{Z}^{Q_0}$ for the lattice of dimension vectors. Define $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ by $\chi(\boldsymbol{d}, \boldsymbol{e}) = \sum_{v,w\in Q_0} a_{vw}\boldsymbol{d}(v)\boldsymbol{e}(w)$, where $a_{vw} = 2\delta_{vw} - n_{vw} - n_{wv}$, for n_{vw} the number of edges $\overset{v}{\bullet} \to \overset{w}{\bullet}$ in Q, so that $A = (a_{vw})_{v,w\in Q_0}$ is the generalized Cartan matrix of Q. Write $\mathfrak{M}$ and $\bar{\mathfrak{M}}$ for the (higher) moduli stacks of objects in $\mathcal{A}$ and $\mathcal{T}$. Then we can work everything out very explicitly. We find:

- The vertex algebra $H_*(\bar{\mathfrak{M}})$ is the lattice vertex algebra of $(\mathbb{Z}^{Q_0}, \chi)$.
- The full Lie algebra $\tilde{H}_*(\bar{\mathfrak{M}}^{\rm pl})$ is rather large, but (for Q with no vertex loops) $\tilde{H}_0(\bar{\mathfrak{M}}^{\rm pl})$ contains the derived Kac–Moody algebra $\mathfrak{g}'(A)$ with Cartan matrix A, with $\tilde{H}_0(\bar{\mathfrak{M}}^{\rm pl}) = \mathfrak{g}'(A)$ if A is positive definite. Similarly, $\tilde{H}_0(\mathfrak{M}^{\rm pl})$ contains/equals the positive part $\mathfrak{n}_+$ of $\mathfrak{g}'(A)$.
- If $Q = \bullet$ has one vertex and no edges then $\tilde{H}_0(\bar{\mathfrak{M}}^{\rm pl}) \cong \mathfrak{sl}(2, \mathbb{C})$.

Let X be a smooth projective $\mathbb{C}$-scheme, and apply our theory to the abelian category $\mathcal{A} = \mathrm{coh}(X)$, with moduli stack $\mathfrak{M}$, and the triangulated category $\mathcal{T} = D^b \mathrm{coh}(X)$, with moduli stack $\bar{\mathfrak{M}}$. We either take X to be $2n$-Calabi–Yau and set $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, or we set $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee + \sigma^*(\mathcal{E}xt^\bullet)$ for any X. Note that if X is $(2n+1)$-Calabi–Yau this gives $c_i(\Theta^\bullet) = 0$, so our vertex algebras and Lie algebras are abelian, and boring.

I haven't worked out the details yet, but here are some highlights:

- For some nice classes of X (e.g. curves, some surfaces) we can compute $H_*(\bar{\mathfrak{M}})$ fairly explicitly as a vertex algebra. It is the tensor product of a lattice-type vertex algebra defined using $K^0(X)$ or $H^{\mathrm{even}}(X)$, and a fermion vertex algebra defined using $K^1(X)$ or $H^{\mathrm{odd}}(X)$. For general X we can produce vertex algebra morphisms from $H_*(\mathfrak{M}), H_*(\bar{\mathfrak{M}})$ to an explicit vertex algebra of this type.
- The Heisenberg algebra acting on homology of Hilbert schemes in Grojnowski–Nakajima should appear as a Lie subalgebra of $\tilde{H}_*(\bar{\mathfrak{M}}^{\mathrm{pl}}_{\mathrm{dim}\,0})$ for dimension 0 sheaves and complexes on X.

3. Enumerative invariants and wall-crossing

The rest of the lecture is conjectural, but we hope to prove.
There are several theories of enumerative invariants 'counting' τ-(semi)stable moduli spaces for a stability condition τ on an abelian category $\mathcal{A} = \mathrm{coh}(X)$:

- Mochizuki's invariants counting coherent sheaves on a surface (an algebraic version of Donaldson invariants).
- Donaldson–Thomas invariants of a Fano 3-fold.
- Donaldson–Thomas type invariants of a Calabi–Yau 4-fold. (Borisov–Joyce, Cao–Leung.)

(We exclude Donaldson–Thomas invariants of Calabi–Yau 3-folds.)
I hope to treat the definitions of the invariants in the strictly semistable case, and wall-crossing for the invariants under change of stability condition, in a uniform way for all these theories, using the 'projective linear' Ringel–Hall Lie algebra.

These theories have the following structure:

- For each $\alpha \in K(\mathcal{A})$ we form τ-(semi)stable moduli schemes $\mathcal{M}_\alpha^{\rm st}(\tau) \subseteq \mathcal{M}_\alpha^{\rm ss}(\tau)$. Here $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is proper, and $\mathcal{M}_\alpha^{\rm st}(\tau)$ has a perfect obstruction theory $\mathcal{E}^\bullet$ (excluding the CY4 case).
- If $\mathcal{M}_\alpha^{\rm st}(\tau) = \mathcal{M}_\alpha^{\rm ss}(\tau)$ (the easy case) then $\mathcal{M}_\alpha^{\rm st}(\tau)$ is proper with a perfect obstruction theory, so by Behrend–Fantechi it has a virtual class $[\mathcal{M}_\alpha^{\rm st}(\tau)]_{\rm virt}$ in (Chow) homology $H_*(\mathcal{M}_\alpha^{\rm st}(\tau))$.
- We can make numerical invariants by integrating natural cohomology classes on $\mathcal{M}_\alpha^{\rm st}(\tau)$ over $[\mathcal{M}_\alpha^{\rm st}(\tau)]_{\rm virt}$.
- If $\mathcal{M}_\alpha^{\rm st}(\tau) \neq \mathcal{M}_\alpha^{\rm ss}(\tau)$ (the difficult case), we need to work harder (e.g. by considering moduli schemes of pairs) to define invariants 'counting' $\mathcal{M}_\alpha^{\rm ss}(\tau)$. It may not be clear what we are really counting.

Stable moduli schemes and the 'projective linear' stack $\mathfrak{M}^{\rm pl}$

If $E \in \mathcal{A}$ is τ-stable then $\mathrm{Aut}(E) \cong \mathbb{G}_m$. Thus, the τ-stable moduli stacks $\mathfrak{M}_\alpha^{\rm st}(\tau)$ have isotropy groups $\mathrm{Iso}_{\mathfrak{M}_\alpha^{\rm st}(\tau)}([E]) \cong \mathbb{G}_m$. So the 'projective linear' τ-stable moduli stacks $\mathfrak{M}_\alpha^{\rm st,pl}(\tau)$ have isotropy groups $\mathrm{Iso}_{\mathfrak{M}_\alpha^{\rm st,pl}(\tau)}([E]) \cong \mathbb{G}_m/\mathbb{G}_m = \{1\}$. That is, the stack $\mathfrak{M}_\alpha^{\rm st,pl}(\tau)$ is actually the scheme $\mathcal{M}_\alpha^{\rm st}(\tau)$, and $\mathcal{M}_\alpha^{\rm st}(\tau)$ is open in the projective linear $\mathfrak{M}_\alpha^{\rm pl}$. Hence, if $\mathcal{M}_\alpha^{\rm st}(\tau) = \mathcal{M}_\alpha^{\rm ss}(\tau)$ then $[\mathcal{M}_\alpha^{\rm st}(\tau)]_{\rm virt}$ defines a class in $H_*(\mathfrak{M}_\alpha^{\rm pl})$. In fact $[\mathcal{M}_\alpha^{\rm st}(\tau)]_{\rm virt}$ has degree $2-\chi(\alpha,\alpha)$, so $[\mathcal{M}_\alpha^{\rm st}(\tau)]_{\rm virt}$ lies in $\tilde H_0(\mathfrak{M}_\alpha^{\rm pl})$, the degree 0 part of our 'projective linear' Lie algebra.

Conjecture 1 (joint work with Yuuji Tanaka)

There is a natural way to define a virtual class $[\mathfrak{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in homology $\tilde H_0(\mathfrak{M}_\alpha^{\rm pl})$ over $R = \mathbb{Q}$ for all $\alpha \in K(\mathcal{A})$. It involves blowing up $\mathfrak{M}_\alpha^{\rm ss,pl}(\tau)$ to get a proper Deligne–Mumford stack with perfect obstruction theory (except the CY4 case).

Conjecture 2

Under change of stability condition, the virtual classes $[\mathfrak{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ *in Conjecture 1 transform according to the universal Lie algebra wall-crossing formula for the* $\epsilon_\alpha(\tau)$ *discussed in* $\S 1$*, but in the Lie algebra* $\bigl(\tilde H_0(\mathfrak{M}^{\rm pl}_\alpha),[\,,\,]^{\rm pl}\bigr)$.

Now consider:

- Mochizuki's invariants counting coherent sheaves on a surface.
- Donaldson–Thomas invariants of a Fano 3-fold.
- Donaldson–Thomas type invariants of a Calabi–Yau 4-fold.

In each case, Conjectures 1 and 2 give both an extension of of the invariants to the strictly τ-semistable case, and an explicit prediction for how the extended invariants (both virtual classes, and the numerical invariants made by integrating cohomology classes on them) transform under change of stability condition.