Complex manifolds and Kähler Geometry

Lecture 7 of 16: Line bundles and divisors

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Plan of talk:

7.1. The Picard group

Let (X,J) be a complex manifold. The *Picard group* $\mathrm{Pic}(X)$ is defined to be the set of isomorphism classes $[L]$ of holomorphic line bundles $L\rightarrow X$, made into a group by defining multiplication $[L]\cdot[L']=[L\otimes L']$ using tensor product of line bundles, inverses $[L]^{-1}=[L^*]$ using duals of line bundles, and identity $0=[\mathcal{O}_X]$ the isomorphism class of the trivial line bundle $\mathcal{O}_X=\mathbb{C}\times X\rightarrow X$. It is an abelian group.

The first Chern class induces a map $c_1:\mathrm{Pic}(X)\rightarrow H^2(X;\mathbb{Z})$ by $c_1:[L]\mapsto c_1(L)$. As $c_1(L\otimes L')=c_1(L)+c_1(L')$, this is a *group homomorphism*. The inclusion $\mathbb{Z}\hookrightarrow\mathbb{C}$ induces a morphism $\Pi:H^2(X;\mathbb{Z})\rightarrow H^2_{\mathrm{dR}}(X;\mathbb{C})$, with kernel the *torsion* of $H^2(X;\mathbb{Z})$ (the elements of finite order), a finite group. Usually we don't distinguish between $H^2(X;\mathbb{Z})$ and $\Pi(H^2(X;\mathbb{Z}))\cong H^2(X;\mathbb{Z})/\text{torsion}$, but today we will.

The image of $c_1 : \mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$

For each $\alpha \in H^2(X;\mathbb{Z})$ there is a complex line bundle $L_\alpha \to X$, unique up to isomorphism, with $c_1(L_\alpha) = \alpha$. (N.B. complex line bundles are not holomorphic line bundles.) Choose any Hermitian metric h on the fibres of L_α, and connection ∇ on L_α preserving h. Then the curvature F_∇ of ∇ is $F_\nabla = i\eta$ for η a closed real 2-form on X with $[\eta] = 2\pi\,\Pi(\alpha)$ in $H^2_{\mathrm{dR}}(X;\mathbb{R})$.

Suppose (X, J, g) is a compact Kähler manifold. In §6.4 we showed that a necessary condition for L to be a *holomorphic* line bundle is that $\Pi(\alpha) \in H^{1,1}(X) \subset H^2_{\mathrm{dR}}(X;\mathbb{C})$. We now prove that this is sufficient. Suppose $\Pi(\alpha) \in H^{1,1}(X)$. Then there is a closed real (1,1)-form ζ with $[\zeta] = 2\pi\,\Pi(\alpha) = [\eta]$. So $\zeta - \eta$ is exact, and $\zeta - \eta = \mathrm{d}\beta$ for some real 1-form β.

Define a connection $\tilde\nabla$ on L by $\tilde\nabla s = \nabla s + i\, s \otimes \beta$ for $s \in C^\infty(L)$. Then $\tilde\nabla$ preserves h, and $F_{\tilde\nabla} = i\eta + i\mathrm{d}\beta = i\zeta$ is of type (1,1). So as in §6.2, $\tilde\nabla$ gives L the structure of a holomorphic line bundle. Therefore

$$\mathrm{Im}\bigl(c_1 : \mathrm{Pic}(X) \longrightarrow H^2(X;\mathbb{Z})\bigr) = \\ \bigl\{\alpha \in H^2(X;\mathbb{Z}) : \Pi(\alpha) \in H^{1,1}(X)\bigr\}.$$

Hence

$$\mathrm{Im}\bigl(\Pi \circ c_1 : \mathrm{Pic}(X) \longrightarrow H^2_{\mathrm{dR}}(X;\mathbb{C})\bigr) \\ = \Pi(H^2(X;\mathbb{Z})) \cap H^{1,1}(X).$$

The kernel of $\Pi \circ c_1 : \mathrm{Pic}(X) \to H^2_{\mathrm{dR}}(X;\mathbb{C})$

Let (X,J,g) be a compact Kähler manifold. Suppose L is a holomorphic line bundle on X, with $\bar\partial$-operator $\bar\partial_L$ and $\Pi \circ c_1(L) = 0$ in $H^2_{\mathrm{dR}}(X;\mathbb{C})$. Choose a Hermitian metric h on L. Then there is a unique connection ∇ on L preserving h with $\bar\partial$-operator $\bar\partial_L$. It has curvature $F_\nabla = i\eta$ for η a closed real (1,1)-form with $[\eta] = 2\pi\, \Pi \circ c_1(L) = 0$ in $H^2_{\mathrm{dR}}(X;\mathbb{C})$. Thus η is exact, and $\eta = \frac{1}{2}\mathrm{d}\mathrm{d}^c f$ for some smooth $f : X \to \mathbb{R}$ by the Global $\mathrm{d}\mathrm{d}^c$-Lemma in §4.2. Set $\hat h = e^f \cdot h$, and let $\hat\nabla$ be the unique connection on L preserving $\hat h$ with $\bar\partial$-operator $\bar\partial_L$. Then as in §6.4, $\hat\nabla$ has curvature $F_{\hat\nabla} = i\hat\eta$ with $\hat\eta = \eta - \frac{1}{2}\mathrm{d}\mathrm{d}^c f = 0$. So $F_{\hat\nabla} \equiv 0$, and $\hat\nabla$ is a flat connection, with group $\mathrm{U}(1)$. Such $(L,\hat\nabla)$ are classified up to isomorphism by their *holonomy*, which is a group morphism $\rho : \pi_1(X) \to \mathrm{U}(1)$ for $\pi_1(X)$ the fundamental group of X (supposing X connected).

Thus we see that

$$\begin{aligned}&\mathrm{Ker}\bigl(\Pi\circ c_1:\mathrm{Pic}(X)\longrightarrow H^2_{\mathrm{dR}}(X;\mathbb{C})\bigr)\\ &\cong \mathrm{Hom}\bigl(\pi_1(X),\mathrm{U}(1)\bigr)\\ &\cong \mathrm{Hom}\bigl(H_1(X;\mathbb{Z}),\mathrm{U}(1)\bigr),\end{aligned}$$

since $\mathrm{U}(1)$ is abelian and $H_1(X;\mathbb{Z})$ is the abelianization of $\pi_1(X)$. We have $H_1(X;\mathbb{Z})\cong\mathbb{Z}^{b_1(X)}\times G$, where $b_1(X)$ is the first Betti number of X, which is even by Cor. 5.1, and G is the torsion of $H_1(X;\mathbb{Z})$, a finite abelian group. Hence

$$\begin{aligned}&\mathrm{Ker}\bigl(\Pi\circ c_1:\mathrm{Pic}(X)\longrightarrow H^2_{\mathrm{dR}}(X;\mathbb{C})\bigr)\\ &\cong T^{b_1(X)}\times\mathrm{Hom}(G,U(1)),\end{aligned}$$

where $\mathrm{Hom}(G,U(1))$ is finite.

7.2. Describing $\mathrm{Pic}(X)$

Putting together the previous material shows that if (X,J,g) is a compact Kähler manifold we have an exact sequence of abelian groups

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}\bigl(H_1(X;\mathbb{Z}),\mathrm{U}(1)\bigr) \longrightarrow \mathrm{Pic}(X) \\ \longrightarrow \Pi(H^2(X;\mathbb{Z}))\cap H^{1,1}(X) \longrightarrow 0. \end{aligned}$$

Here $\Pi(H^2(X;\mathbb{Z}))\cap H^{1,1}(X)\cong\mathbb{Z}^k$ for some $k\leqslant b^2(X)$, and the sequence splits, so

$$\mathrm{Pic}(X)\cong T^{b_1(X)}\times\mathrm{Hom}(G,U(1))\times\mathbb{Z}^k.$$

In fact $\mathrm{Hom}(G, U(1)) \cong \mathrm{Ker}\bigl(\Pi : H^2(X;\mathbb{Z}) \to H^2_{\mathrm{dR}}(X;\mathbb{C})\bigr)$, so we have

$$\mathrm{Im}\bigl(c_1 : \mathrm{Pic}(X) \longrightarrow H^2(X;\mathbb{Z})\bigr) \cong \mathrm{Hom}(G, U(1)) \times \mathbb{Z}^k,$$
$$\mathrm{Ker}\bigl(c_1 : \mathrm{Pic}(X) \longrightarrow H^2(X;\mathbb{Z})\bigr) \cong T^{b_1(X)}.$$

Thus $\mathrm{Pic}(X)$ consists of a continuous part $T^{b_1(X)}$, parametrizing flat connections on the trivial line bundle $\mathbb{C} \times X \to X$, and a discrete part $\mathrm{Hom}(G, U(1)) \times \mathbb{Z}^k$, parametrizing the possible first Chern classes $c_1(L)$ of holomorphic line bundles L in $H^2(X;\mathbb{Z})$.

Since $\mathrm{Pic}(X)$ is the product of a manifold $T^{b_1(X)}$ with a discrete group $\mathrm{Hom}(G, U(1)) \times \mathbb{Z}^k$, it is a real manifold. In fact it has the structure of a complex manifold. We can naturally identify the torus $T^{b_1(X)}$ with $H^1(X;\mathbb{R})/\Pi(H^1(X;\mathbb{Z}))$. We have natural maps

$$\begin{aligned} &H^1(X;\mathbb{R}) \hookrightarrow H^1(X;\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) \\ &\longrightarrow H^1(X;\mathbb{C})/H^{1,0}(X) \cong H^{0,1}(X). \end{aligned}$$

This gives an isomorphism of real vector spaces $H^1(X;\mathbb{R}) \cong H^{0,1}(X)$, where $H^{0,1}(X)$ is a complex vector space. It is natural to write $T^{b_1(X)} \cong H^{0,1}(X)/\Pi(H^1(X;\mathbb{Z}))$, which makes $T^{b_1(X)}$ and $\mathrm{Pic}(X)$ into complex manifolds; this is the complex structure you get from regarding $\mathrm{Pic}(X)$ as a moduli space of holomorphic objects.

Line bundles on $\mathbb{CP}^n$

Let $n \geqslant 1$. Then $H^{2j}(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$, $H^{2j}_{\mathrm{dR}}(\mathbb{CP}^n;\mathbb{C}) = H^{j,j}(\mathbb{CP}^n) \cong \mathbb{C}$ for $j = 0,\ldots,n$, and $H^k(\mathbb{CP}^n;\mathbb{Z}) = H^k_{\mathrm{dR}}(\mathbb{CP}^n;\mathbb{C}) = 0$ otherwise. So §7.1–§7.2 show that $c_1 : \mathrm{Pic}(\mathbb{CP}^n) \to H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism, and $\mathrm{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$.

The *tautological line bundle* $L \to \mathbb{CP}^n$ is the holomorphic line bundle whose dual L^* has fibre the vector subspace $\langle (z_0,\ldots,z_n)\rangle_{\mathbb{C}}$ in $\mathbb{C}^{n+1}$ over $[z_0,\ldots,z_n]$. So L^* is a vector subbundle of the trivial bundle $\mathbb{C}^{n+1}\times\mathbb{CP}^n \to \mathbb{CP}^n$.

Then $c_1(L)$ generates $H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$, so $[L]$ generates $\mathrm{Pic}(X)$, and every line bundle on $\mathbb{CP}^n$ is isomorphic to L^k for some unique $k \in \mathbb{Z}$. Often one uses the notation $L = \mathcal{O}(1)$ and $L^k = \mathcal{O}(k)$.

One can show that the *canonical bundle* $K_{\mathbb{CP}^n}$ of $\mathbb{CP}^n$ is isomorphic to $L^{-n-1} = \mathcal{O}(-n-1)$.

7.3. Holomorphic and meromorphic sections

Let (X, J) be a complex manifold, and $L \to X$ a holomorphic line bundle. Then L has a $\bar\partial$-operator $\bar\partial : C^\infty(L) \to C^\infty(L \otimes \Lambda^{0,1}X)$. A *holomorphic section* of L is $s \in C^\infty(L)$ with $\bar\partial s = 0$. The holomorphic sections form a complex vector space $H^0(L)$, which is finite-dimensional if X is compact.

For example, for the line bundle $L^k \to \mathbb{CP}^n$, we have $H^0(L^k) = 0$ if $k < 0$, and $H^0(L^k)$ is isomorphic to the vector space of homogeneous polynomials on $\mathbb{C}^{n+1}$ of degree k if $k \geqslant 0$.

Since a holomorphic line bundle $L \to X$ locally looks like the trivial bundle $\mathbb{C} \times X \to X$, a holomorphic section s of L locally looks like a holomorphic function $f : X \to \mathbb{C}$. But globally they are different: for X compact all holomorphic functions are constant, but L can have many holomorphic sections, or none. One uses holomorphic sections of L as a substitute for holomorphic functions.

Meromorphic sections

If (X, J) is a complex manifold, a *meromorphic function* $f : X \dashrightarrow \mathbb{C}$ (or $\mathbb{C} \cup \{\infty\}$) is a function defined in a dense open subset of X, such that each $x \in X$ has an open neighbourhood U and holomorphic functions $g, h : U \to \mathbb{C}$, both not identically zero near x, with $f(u) = g(u)/h(u)$ for $u \in U$ in the domain of f. When $h(u) = 0$ and $g(u) \neq 0$ we can set $f(u) = \infty$, but $f(u)$ is undefined when $g(u) = h(u) = 0$, so f may not be defined on all of X.

In the same way, if $L \to X$ is a holomorphic line bundle, a *meromorphic section* s of X is a section of L over a dense open subset of X, such that each $x \in X$ has an open neighbourhood U, a holomorphic section g of L on U, and a holomorphic function $h : U \to \mathbb{C}$, both not identically zero near x, with $s(u) = g(u)/h(u)$ for $u \in U$ in the domain of s.

7.4. Divisors

Let (X, J) be a compact complex manifold. An *analytic hypersurface* V in X is a closed subset $V \subset X$ such that for each $v \in V$ there exists an open neighbourhood U of v in X and a holomorphic function $f : U \to \mathbb{C}$, not identically zero near v, such that $U \cap V = \{u \in U : f(u) = 0\}$. We call V *irreducible* if we cannot write $V = V_1 \cup V_2$ for analytic hypersurfaces $\emptyset \neq V_1 \neq V_2 \neq \emptyset$. Every analytic hypersurface is a finite union of irreducible analytic hypersurfaces.

If (X, J) is projective then Chow's Theorem shows that such V are actually algebraic, i.e. defined by the zeroes of polynomials.

A *divisor* on X is a finite formal sum $D = \sum_{j=1}^{k} a_j V_j$, where $a_1, \ldots, a_k \in \mathbb{Z}$ and $V_1, \ldots, V_k$ are irreducible analytic hypersurfaces. We call D *effective* if $a_j \geqslant 0$ for all j.

The *divisor group* $\mathrm{Div}(X)$ is the abelian group of divisors on X, with addition as group structure.

Suppose (X, J) is a compact complex manifold, and $f : X \dashrightarrow \mathbb{C}$ is a meromorphic function. Then one can associate a unique divisor $\mathrm{div}(f) = \sum_{j=1}^{k} a_j V_j$ to f, such that f has zeroes of order a_j on V_j when $a_j > 0$, and poles of order $-a_j$ on V_j when $a_j < 0$. That is, each $x \in X$ has an open neighbourhood U in X such that $f(u) = g(u) \prod_{j=1}^{l} f_j(u)^{a_j}$, where $f_j : U \to \mathbb{C}$ is a holomorphic function with $U \cap V_j = \{u \in U : f_j(u) = 0\}$, and f_j vanishes to order 1 on the smooth part of $U \cap V_j$, and $g : U \to \mathbb{C} \setminus \{0\}$ is holomorphic.

A divisor D is called *principal* if $D = \mathrm{div}(f)$ for some meromorphic function f. The subset of principal divisors in $\mathrm{Div}(X)$ is a subgroup, since $\mathrm{div}(f) + \mathrm{div}(g) = \mathrm{div}(fg)$, $-\mathrm{div}(f) = \mathrm{div}(f^{-1})$. Two divisors D_1, D_2 are called *linearly equivalent*, written $D_1 \sim D_2$, if $D_1 - D_2 = \mathrm{div}(f)$ for some meromorphic f. Write $[D]$ for the $\sim$-equivalence class of D, and $\mathrm{Div}(X)/\sim$ for the set of $[D]$. Then $\mathrm{Div}(X)/\sim$ is an abelian group, the quotient of $\mathrm{Div}(X)$ by the subgroup of principal divisors.

Now let L be a holomorphic line bundle, and s a meromorphic section of L. Then s has a divisor $\mathrm{div}(s)$, defined in the same way as $\mathrm{div}(f)$. If s is holomorphic then $\mathrm{div}(s)$ is effective (as s has no poles). If $f : X \dashrightarrow \mathbb{C}$ is a meromorphic function then fs is another meromorphic section of L, and $\mathrm{div}(fs) = \mathrm{div}(f) + \mathrm{div}(s)$, so that $\mathrm{div}(fs) \sim \mathrm{div}(s)$. Conversely, if t is another meromorphic section of L, then $f := t/s$ is a meromorphic function $X \dashrightarrow \mathbb{C}$, and $t = fs$, so $\mathrm{div}(t) = \mathrm{div}(f) + \mathrm{div}(s)$, and $\mathrm{div}(t) \sim \mathrm{div}(s)$.

This proves:

Lemma 7.1

Let (X, J) be a compact complex manifold and $L \to X$ a holomorphic line bundle which admits meromorphic sections s. Then the class $[\mathrm{div}(s)]$ in $\mathrm{Div}(X)/\sim$ is independent of the choice of meromorphic section s.

Conversely, given any divisor D on X, one can construct a holomorphic line bundle L and a meromorphic section s with $\mathrm{div}(s) = D$, and (L, s) are unique up to isomorphism. Thus $[L] \in \mathrm{Pic}(X)$ depends only on D. Also if $D' = D + \mathrm{div}(f)$ for f meromorphic then $\mathrm{div}(fs) = D'$. Thus the class $[L]$ depends only on the $\sim$-equivalence class $[D]$ of D.

Conclusions

If (X, J) is a compact complex manifold, there is a natural injective morphism $\mu : (\mathrm{Div}(X)/\sim) \hookrightarrow \mathrm{Pic}(X)$ mapping $\mu : [D] \mapsto [L]$, where L is a holomorphic line bundle with a meromorphic section s with $\mathrm{div}(s) = D$; if D is effective then s is holomorphic. The image of μ is the set of $[L]$ for which L admits a meromorphic section. We will show in §9 that if X is projective then every L admits meromorphic sections, so μ is an isomorphism.

Complex manifolds and Kähler Geometry

Lecture 8 of 16: Cohomology of holomorphic vector bundles

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Plan of talk:

8.1. Dolbeault-type cohomology for vector bundles

Let (X, J) be a complex manifold, and $E \to X$ a holomorphic vector bundle, with $\bar\partial$-operator

$$\bar\partial_E : C^\infty(E) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1}X).$$

As in §6.2, $\bar\partial_E$ extends to

$$\bar\partial_E^{p,q} : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q}X) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q+1}X)$$

with $\bar\partial_E^{p,q+1} \circ \bar\partial_E^{p,q} = 0$ for all p, q.

As for Dolbeault cohomology in §3.2, define the *cohomology of E* by

$$H^q(E) = \frac{\mathrm{Ker}\bigl(\bar\partial_E^{0,q} : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,q}X) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,q+1}X)\bigr)}{\mathrm{Im}\bigl(\bar\partial_E^{0,q-1} : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,q-1}X) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,q}X)\bigr)}.$$

This uses only $C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q}X)$ for $p = 0$. But we can interpret the $p \neq 0$ case in the same way:

$$E \otimes_{\mathbb{C}} \Lambda^{p,q}X \cong (E \otimes_{\mathbb{C}} \Lambda^{p,0}X) \otimes_{\mathbb{C}} \Lambda^{0,q}X$$

where $\Lambda^{p,0}X$ is the holomorphic vector bundle $\Lambda^p T^*X$, so $E \otimes_{\mathbb{C}} \Lambda^{p,0}X$ is a holomorphic vector bundle. So

$$H^q(E \otimes \Lambda^p T^*X) = \frac{\mathrm{Ker}\bigl(\bar\partial_E^{p,q} : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q}X) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q+1}X)\bigr)}{\mathrm{Im}\bigl(\bar\partial_E^{p,q-1} : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q-1}X) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q}X)\bigr)}.$$

Let X be compact. Choose Hermitian metrics g, h on X, E. Then we can define adjoint operators

$$(\bar\partial_E^{p,q-1})^* : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q}X) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q-1}X).$$

Define $\Delta_E^{p,q} : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q}X) \to C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q}X)$ by

$$\Delta_E^{p,q} = (\bar\partial_E^{p,q})^* \bar\partial_E^{p,q} + \bar\partial_E^{p,q-1} \circ (\bar\partial_E^{p,q-1})^*.$$

Then by Hodge theory we have

$$\mathcal{H}^{p,q}(E) := \operatorname{Ker} \Delta_E^{p,q} \cong H^q(E \otimes \Lambda^p T^*X),$$

so in particular $\mathcal{H}^{0,q}(E) \cong H^q(E)$.
As $\Delta_E^{p,q}$ is elliptic and X is compact, $\operatorname{Ker} \Delta_E^{p,q}$ is finite-dimensional. Hence $H^q(E)$ and $H^q(E \otimes \Lambda^p T^*X)$ are finite-dimensional complex vector spaces when X is compact.

Remarks

(a) $H^0(E)$ is the complex vector space of holomorphic sections of E.
(b) $\Lambda^p T^*X$ is a holomorphic vector bundle for $p = 0, \ldots, n$, and

$$H^q(\Lambda^p T^*X) = H^{p,q}_{\bar\partial}(X),$$

the Dolbeault cohomology of X.
(c) We only need g Hermitian, not Kähler. In §5 we wanted g Kähler so that $\Delta_{\bar\partial} = \frac{1}{2}\Delta_{\mathrm{d}}$, to relate de Rham and Dolbeault cohomology. Here we have no analogue of de Rham cohomology for E.

Remarks

(d) This is not the usual approach to defining cohomology of vector bundles. There is another way, which yields isomorphic groups, using *sheaf cohomology*. In this we define the *sheaf of holomorphic sections* of E (can do this analytically, or algebraically, if X, E are algebraic), and then define cohomology of the sheaf using Čech cohomology. The sheaf approach works over other fields, and for (quasi)coherent sheaves as well as for vector bundles.
(e) In algebraic geometry one defines *Ext groups* $\mathrm{Ext}^q(E, F)$ for E, F coherent sheaves, where $\mathrm{Ext}^0(E, F) = \mathrm{Hom}(E, F)$. When E, F are vector bundles we have $\mathrm{Ext}^q(E, F) \cong H^q(E^* \otimes F)$. But for general coherent sheaves E, F both duals E^* and tensor products $E^* \otimes F$ are problematic, so $\mathrm{Ext}^q(E, F) \cong H^q(E^* \otimes F)$ doesn't hold.

Euler characteristics

Definition

Let (X, J) be a compact complex manifold of complex dimension n, and $E \to X$ a holomorphic vector bundle. The *Euler–Poincaré characteristic of* E is

$$\chi(X, E) = \sum_{q=0}^{n} (-1)^q \dim_{\mathbb{C}} H^q(E).$$

For comparison, the *Euler characteristic* of X is

$$\chi(X) = \sum_{k=0}^{2n} (-1)^k \dim_{\mathbb{C}} H^k_{\mathrm{dR}}(X; \mathbb{C}).$$

8.2. The Hirzebruch–Riemann–Roch Theorem

Here is a very important result:

Theorem 8.1 (Hirzebruch–Riemann–Roch)

Let E be a holomorphic vector bundle on a compact complex manifold X. Then

$$\chi(X,E)=\int_X \mathrm{ch}(E)\,\mathrm{td}(X). \tag{8.1}$$

Here $\mathrm{ch}(E)\in H^{\mathrm{even}}(X;\mathbb{Q})$ is the *Chern character* of E, a polynomial in the Chern classes $c_i(E)$ and $\mathrm{rank}(E)$, and $\mathrm{td}(X)\in H^{\mathrm{even}}(X;\mathbb{Q})$ is the *Todd class* of X, a polynomial in the Chern classes of TX.

The r.h.s. of (8.1) means: multiply $\mathrm{ch}(E)$ and $\mathrm{td}(X)$ in $H^{\mathrm{even}}(X;\mathbb{Q})$, take the component in $H^{2n}(X;\mathbb{Q})$, and contract with the fundamental class $[X]\in H_{2n}(X;\mathbb{Q})$ to get a number.

Thus, the Hirzebruch–Riemann–Roch theorem says that $\chi(X,E)$ is a topological invariant, which we calculate using algebraic topology. The proof of the Hirzebruch–Riemann–Roch theorem is difficult. In our case it is a consequence of the *Atiyah–Singer Index Theorem*. Consider

$$\bar\partial_E^{0,*}+(\bar\partial_E^{0,*})^*:\bigoplus_{q\text{ even}}C^\infty(E\otimes_{\mathbb C}\Lambda^{0,q}X)\longrightarrow\bigoplus_{q\text{ odd}}C^\infty(E\otimes_{\mathbb C}\Lambda^{0,q}X).$$

It is a first-order complex elliptic operator on X with

$$\mathrm{Ker}\bigl(\bar\partial_E^{0,*}+(\bar\partial_E^{0,*})^*\bigr)=\bigoplus\nolimits_{q\text{ even}}\mathcal{H}^{0,q}(E),$$
$$\mathrm{Coker}\bigl(\bar\partial_E^{0,*}+(\bar\partial_E^{0,*})^*\bigr)=\bigoplus\nolimits_{q\text{ odd}}\mathcal{H}^{0,q}(E).$$

Hence

$$\mathrm{index}\bigl(\bar\partial_E^{0,*}+(\bar\partial_E^{0,*})^*\bigr)=$$
$$\sum\nolimits_{q\text{ even}}\dim_{\mathbb C}\mathcal{H}^{0,q}(E)-\sum\nolimits_{q\text{ odd}}\dim_{\mathbb C}\mathcal{H}^{0,q}(E)=\chi(X,E).$$

We use the Index Theorem to compute $\mathrm{index}\big(\bar\partial_E^{0,*}+(\bar\partial_E^{0,*})^*\big)$, and show it is $\int_X \mathrm{ch}(E)\mathrm{td}(X)$. The Hirzebruch–Riemann–Roch theorem also applies for coherent sheaves E, and over other fields, with the sheaf definition of $H^q(E)$, but this requires a different proof.

A good reference on characteristic classes and the Hirzebruch–Riemann–Roch Theorem is Hartshorne, *Algebraic Geometry*, Appendix A.

We have

$$\begin{aligned}\mathrm{ch}(E)&=\big[\mathrm{rank}(E)\big]+\big[c_1(E)\big]+\big[\tfrac12 c_1(E)^2-c_2(E)\big]+\cdots,\\ \mathrm{td}(X)&=\big[1\big]+\big[\tfrac12 c_1(TX)\big]+\big[\tfrac1{12}c_1(TX)^2+\tfrac1{12}c_2(TX)\big]+\cdots.\end{aligned}$$

So, for example, if L is a line bundle over a curve Σ_g of genus g then $\chi(\Sigma_g,L)=\deg L+1-g$.

If L is a line bundle on a surface X then

$$\chi(X,L)=\tfrac12\int_X c_1(L)\big(c_1(L)+c_1(TX)\big)+\chi(X,\mathcal{O}_X).$$

8.3. Serre duality

If X is a compact, oriented n-manifold then *Poincaré duality* says that $H^k_{\rm dR}(X;\mathbb{R})\cong H^{n-k}_{\rm dR}(X;\mathbb{R})^*$. A partial proof is to choose a metric g and use Hodge theory: if $\mathcal{H}^k$ is the harmonic k-forms then $H^k_{\rm dR}(X;\mathbb{R})\cong\mathcal{H}^k$. But the Hodge star gives an isomorphism $*:\mathcal{H}^k\to\mathcal{H}^{n-k}$, and the L^2-product an isomorphism $\mathcal{H}^{n-k}\cong(\mathcal{H}^{n-k})^*$. So we have isomorphisms

$$H^k_{\rm dR}(X;\mathbb{R})\cong\mathcal{H}^k\xrightarrow{*}\mathcal{H}^{n-k}\xrightarrow{L^2}(\mathcal{H}^{n-k})^*\cong H^{n-k}_{\rm dR}(X;\mathbb{R})^*.$$

The composition is independent of g.

In a similar way, if (X,J) is a compact complex manifold of complex dimension n, with canonical bundle K_X, and $E\to X$ is a holomorphic vector bundle, then *Serre duality* is a natural isomorphism

$$H^q(E)\cong H^{n-q}(E^*\otimes K_X)^*$$

for $q=0,\ldots,n$. We will give a partial proof.

Choose Hermitian metrics g, h on X, E. Then by §8.1

$$H^q(E) \cong \mathcal{H}^{0,q}(E) \subset C^\infty(E \otimes \Lambda^{0,q}X).$$

The Hodge star maps

$$* : \Lambda^{0,q}X \longrightarrow \Lambda^{n,n-q}X \cong \Lambda^{n,0}X \otimes_{\mathbb{C}} \Lambda^{0,n-q}X.$$

It is complex antilinear. The Hermitian metric h on E gives a complex antilinear isomorphism $h : E \to E^*$. Also $\Lambda^{n,0}X = K_X$. This gives

$$h \otimes * : E \otimes \Lambda^{0,q}X \to E^* \otimes K_X \otimes \Lambda^{0,n-q}X.$$

This $h \otimes *$ commutes with Δ_E and so induces a complex antilinear isomorphism

$$h \otimes * : \mathcal{H}^{0,q}(E) \longrightarrow \mathcal{H}^{0,n-q}(E^* \otimes K_X).$$

It is natural to identify the complex conjugate of $\mathcal{H}^{0,q}(E^*\otimes K_X)$ with $\mathcal{H}^{0,q}(E^*\otimes K_X)^*$ using the Hermitian L^2-inner product. So we have a complex isomorphism

$$\mathcal{H}^{0,q}(E)\longrightarrow \mathcal{H}^{0,n-q}(E^*\otimes K_X)^*.$$

Hence

$$H^q(E)\cong\mathcal{H}^{0,q}(E)\cong\mathcal{H}^{0,n-q}(E^*\otimes K_X)^*\cong H^{n-q}(E^*\otimes K_X)^*,$$

which is Serre duality.

Hirzebruch–Riemann–Roch for curves

Often what we are really interested in is the space $H^0(E)$, as this is the holomorphic sections of E. Also $H^n(E) \cong H^0(E^* \otimes K_X)^*$ by Serre duality, so $H^n(E)$ also has an interpretation in terms of holomorphic sections. When $n = 1$, this is all the cohomology groups $H^q(E)$. So the Hirzebruch–Riemann–Roch theorem for a holomorphic vector bundle E on a curve Σ_g of genus g becomes

$$\dim H^0(E) - \dim H^0(E^* \otimes K_X) = \deg E + (1-g)\,\mathrm{rank}(E). \quad (8.2)$$

8.4. Line bundles and vector bundles on $\mathbb{CP}^1$

From §7.2, all line bundles on $\mathbb{CP}^1$ are isomorphic to $L^n = \mathcal{O}(n)$ for $n \in \mathbb{Z}$, where $L \to \mathbb{CP}^1$ is the tautological line bundle with $c_1(L) = 1$, and the canonical bundle $K_{\mathbb{CP}^1}$ is $L^{-2} = \mathcal{O}(-2)$. We will compute $H^q(\mathcal{O}(n))$ for $q = 0, 1$ and all n in $\mathbb{Z}$. From §8.3 we have

$$\begin{aligned} &\dim H^0(\mathcal{O}(n)) - \dim H^0(\mathcal{O}(-2-n)) \\ &\quad = \deg \mathcal{O}(n) + (1-g)\,\mathrm{rank}(\mathcal{O}(n)) = n+1, \end{aligned} \tag{8.3}$$

as $\deg \mathcal{O}(n) = n$, $\mathrm{rank}\,\mathcal{O}(n) = 1$ and $g = 0$.

Now observe that

$$H^0(\mathcal{O}(-2)) \cong H^0(K_{\mathbb{CP}^1}) = H^0(\Lambda^{1,0}\mathbb{CP}^1) = H^{1,0}(\mathbb{CP}^1) = 0.$$

If $s \in H^0(\mathcal{O}(n))$ and $t \in H^0(\mathcal{O}(-2-n))$ are both nonzero, then $s \otimes t \in H^0(\mathcal{O}(-2))$ is also nonzero, a contradiction. Hence at least one of $H^0(\mathcal{O}(n))$ and $H^0(\mathcal{O}(-2-n))$ are zero. Therefore (8.3) implies that

$$\dim_{\mathbb{C}} H^0(\mathcal{O}(n)) = \begin{cases} n+1, & n \geqslant 0, \\ 0, & n < 0. \end{cases}$$

Serre duality gives

$$\begin{aligned} H^1(\mathcal{O}(n)) &\cong H^0(\mathcal{O}(n)^* \otimes K_{\mathbb{CP}^1})^* \\ &\cong H^0(\mathcal{O}(-2-n))^*. \end{aligned}$$

Therefore

$$\dim_{\mathbb{C}} H^1(\mathcal{O}(n)) = \begin{cases} -1-n, & n \leqslant -2, \\ 0, & n \geqslant -1. \end{cases}$$

The classification of vector bundles on $\mathbb{CP}^1$

Theorem 8.2 (Grothendieck Lemma)

Let E be a holomorphic vector bundle over $\mathbb{CP}^1$, of rank k. Then E is isomorphic to $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_k)$, for some unique integers $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$.

For curves of higher genus, and for complex manifolds of dimension > 1 including projective spaces, the classification of vector bundles is more complicated.

Proof of Theorem 8.2

We will prove the Grothendieck Lemma, by induction on k. When $k=1$ it follows from §7.2. Suppose it is true for all vector bundles of rank $<k$, for $k>1$, and let E have rank k and degree d. We claim that $H^0(E\otimes\mathcal{O}(n))$ is of large dimension for $n\gg 0$, and is zero for $n\ll 0$. To see this, note that $E\otimes\mathcal{O}(n)$ has rank k and degree $d+nk$, so by (8.2) we have

$$\dim H^0(E\otimes\mathcal{O}(n))-\dim H^0(E^*\otimes\mathcal{O}(-2-n))=d+(n+1)k.$$

So $\dim H^0(E\otimes\mathcal{O}(n))\gg 0$ for $n\gg 0$.

Proof of Theorem 8.2

As $\dim H^0(\mathcal{O}(1)) = 2$, considering the map

$$H^0(E \otimes \mathcal{O}(n)) \otimes H^0(\mathcal{O}(1)) \longrightarrow H^0(E \otimes \mathcal{O}(n+1))$$

shows that if $H^0(E \otimes \mathcal{O}(n)) \neq 0$ then

$$\dim H^0(E \otimes \mathcal{O}(n+1)) > \dim H^0(E \otimes \mathcal{O}(n)).$$

Thus $\dim H^0(E \otimes \mathcal{O}(n))$ is strictly increasing when it is nonzero, which forces $H^0(E \otimes \mathcal{O}(n)) = 0$ for $n \ll 0$.

Proof of Theorem 8.2

Let a_1 be greatest with $H^0(E\otimes\mathcal{O}(-a_1))\neq 0$, and choose $0\neq s\in H^0(E\otimes\mathcal{O}(-a_1))$. If $s=0$ at any $x\in\mathbb{CP}^1$ then $s=t\otimes u$, where $0\neq t\in H^0(E\otimes\mathcal{O}(-a_1-1))$, and $0\neq u\in H^0(\mathcal{O}(1))$ is zero at x. But then $H^0(E\otimes\mathcal{O}(-a_1-1))\neq 0$, contradicting definition of a_1. So $s\neq 0$ everywhere.

Regard s as a morphism $\mathcal{O}(a_1)\to E$. As $s\neq 0$ everywhere this embeds $\mathcal{O}(a_1)$ as a vector subbundle of E, and the quotient bundle $E'=E/s(\mathcal{O}(a_1))$ is a vector bundle of rank $k-1$. So by induction $E'=\mathcal{O}(a_2)\oplus\cdots\oplus\mathcal{O}(a_k)$ for unique $a_2\geqslant\cdots\geqslant a_k$.

Proof of Theorem 8.2

Taking morphisms from the exact sequence

$$0 \longrightarrow \mathcal{O}(a_1) \xrightarrow{s} E \longrightarrow E' \longrightarrow 0.$$

to $\mathcal{O}(a_1)$, and using $H^1(\mathcal{O}(0)) = 0$, gives an exact sequence

$$0 \longrightarrow H^0((E')^* \otimes \mathcal{O}(a_1)) \longrightarrow H^0(E^* \otimes \mathcal{O}(a_1)) \xrightarrow{\circ s} H^0(\mathcal{O}(0)) \longrightarrow 0.$$

As $H^0(\mathcal{O}(0)) = \mathbb{C}$, there exists $t \in H^0(E^* \otimes \mathcal{O}(a_1))$ with $t \circ s = \mathrm{id}_{\mathcal{O}(a_1)}$, regarding s, t as morphisms

$$\mathcal{O}(a_1) \xrightarrow{s} E \xrightarrow{t} \mathcal{O}(a_1).$$

So the sequence $0 \to \mathcal{O}(a_1) \to E \to E' \to 0$ splits, and $E \cong \mathcal{O}(a_1) \oplus E'$, where as a subbundle of E we have $E' = \operatorname{Ker} t$. Therefore $E \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_k)$. Also $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$, as $a_1 < a_2$ would contradict the definition of a_1. This completes the inductive step, and the proof.