Graduate lecture course   Professor Joyce   16 lectures, HT 2022

Complex manifolds and Kähler geometry

Wednesdays 2pm-4pm on MS Teams, starting on 19th January and finishing on 9th March.

Complex manifolds – manifolds defined over the complex numbers – are studied in Complex Algebraic Geometry. Kähler metrics are a natural class of Riemannian metrics on complex manifolds. A Kähler manifold is a complex manifold with a Kähler metric. The study of Kähler manifolds combines Differential Geometry, Algebraic Geometry, and Analysis, and includes much beautiful mathematics.

The point of view taken in the lectures will be primarily that of Differential Geometry — we will regard complex and Kähler manifolds as real manifolds equipped with extra geometric structures — but I will also bring in material from Algebraic Geometry and Analysis as we have need of it.

Synopses:

Lecture 1: Complex manifolds. Definition using complex charts and holomorphic transition functions. Holomorphic maps, complex submanifolds. Complex projective space $\mathbb{CP}^n$, projective complex manifolds, Chow's Theorem.

Lecture 2: Almost complex structures. Almost complex structures, the Nijenhuis tensor, the Newlander–Nirenberg Theorem. Alternative, differential-geometric definition of complex manifolds. Symplectic manifolds.

Lecture 3: Exterior forms on complex manifolds. Summary of exterior forms and de Rham cohomology for real manifolds. (p, q)-forms, the ∂ and $\bar\partial$ operators. Dolbeault cohomology. Holomorphic $(p, 0)$-forms. The canonical bundle.

Lecture 4: Kähler metrics. Hermitian metrics and Kähler metrics. The Kähler class and Kähler potentials. The Fubini–Study metric on $\mathbb{CP}^n$; projective complex manifolds are Kähler. Exterior forms on Kähler manifolds, the operators $\bar\partial^*, \partial^*, L, \Lambda$. The Kähler identities.

Lecture 5: Hodge theory for Kähler manifolds. Summary of Hodge theory for compact Riemannian manifolds. Hodge theory for Kähler manifolds. Corollary: odd Betti numbers of compact Kähler manifolds are even. An example of a complex manifold with no Kähler metrics. The Hard Lefschetz Theorem. The Hodge Conjecture.

Lecture 6: Holomorphic vector bundles. Vector bundles on real manifolds, connections and curvature. Holomorphic vector bundles, $\bar{\partial}$-operators and connections, (0,2)-curvature. Relation between holomorphic vector bundles, and complex vector bundles with connections with curvature of type (1,1). Chern classes. Holomorphic line bundles.

Lecture 7: Line bundles and divisors. The Picard group $\mathrm{Pic}(X)$. Characterization of image and kernel of $c_1 : \mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ on a compact Kähler manifold, explicit description of $\mathrm{Pic}(X)$ in terms of $H^1(X;\mathbb{Z})$, $H^2(X;\mathbb{Z})$, $H^{1,1}(X)$. Line bundles on $\mathbb{CP}^n$. Holomorphic and meromorphic sections of line bundles. Divisors, the morphism $\mu : \mathrm{Div}(X)/\sim \to \mathrm{Pic}(X)$.

Lecture 8: Cohomology of holomorphic vector bundles. Dolbeault-cohomology-style definition of cohomology groups $H^q(E)$ for $E \to X$ a holomorphic vector bundle. The Hirzebruch–Riemann–Roch Theorem. Serre duality. Hirzebruch–Riemann–Roch for curves. Cohomology groups of line bundles on $\mathbb{CP}^1$. Classification of vector bundles on $\mathbb{CP}^1$.

Lecture 9: Vanishing Theorems and the Kodaira Embedding Theorem. Positive line bundles. The Kodaira Vanishing Theorem. The Serre Vanishing Theorem. Application to line bundles and divisors. The base locus of a holomorphic line bundle, morphisms to projective spaces. Bertini's Theorem. The Kodaira Embedding Theorem, and consequences.

Lecture 10: Topics on line bundles and divisors. Finite covers of projective complex manifolds are projective. Example: complex tori T^{2n}, a family of compact complex manifolds, some of which are projective and some of which aren't. The Lefschetz Hyperplane Theorem. The adjunction formula. The blow-up of a complex manifold along a closed complex submanifold. Canonical bundles of blow-ups. (Positive) line bundles on blow-ups.

Lecture 11: Curvature of Kähler manifolds. Riemann and Ricci curvature, the Ricci form. Ricci-flat Kähler manifolds and Calabi–Yau manifolds. Kähler–Einstein manifolds.

Lecture 12: The Calabi Conjecture. Statement of the Calabi Conjecture, and sketch of proof. Existence of Calabi-Yau metrics. Topological properties of compact, Ricci-flat Kähler manifolds (restrictions on fundamental group $\pi_1(X)$ and $H^{p,0}(X)$), and of compact complex manifolds with K_X positive or negative.

Lecture 13: Riemannian holonomy groups. Parallel transport, the holonomy group of a connection on a vector bundle. Riemannian holonomy groups, Berger's classification, sketch of proof. G-structures on manifolds.

Lecture 14: The Kähler holonomy groups. Kähler geometry from the point of view of Riemannian holonomy. Calabi–Yau and hyperkähler manifolds, their topological properties. Calabi–Yau 2-folds, $K3$ surfaces.

Lecture 15: Introduction to moduli spaces. Generalities on moduli problems, in differential geometry and algebraic geometry. Deformations of complex structure on compact complex manifolds: infinitesimal deformations, second-order obstructions.

Lecture 16: Deformation theory for compact complex manifolds. Theorems of Kodaira–Spencer and Kuranishi on deformations of compact complex manifolds (X, J); local models for the moduli space $\mathcal{M}_X$ of complex structures on X. Special cases: curves and del Pezzo surfaces. Deformations of Calabi–Yau m-folds, the Tian–Todorov Theorem, and the period map.

Bibliography:

Recommended:

W. Ballmann, *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics, European Mathematical Society, 2006.

D. Huybrechts, *Complex Geometry: an introduction*, Universitext, Springer, 2005.

A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts 69, Cambridge University Press, 2007.

Also useful:

P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, 1978, Chapter 0.

D.D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics 12, Oxford University Press, 2007, Chapters 5–7.

A.L. Besse, *Einstein manifolds*, Springer, 1987, Chapter 2.

C. Voisin, *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge studies in advanced mathematics 76, Cambridge University Press, 2002, Chapters 1–3.

M.A. de Cataldo, *The Hodge theory of projective manifolds*, Imperial College Press, 2007.

K. Kodaira, *Complex manifolds and deformation of complex structures*, Springer, 1986. (Not a well written book.)