Introduction to Differential Geometry

Lecture 5 of 10: Orientations and integration

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These slides available at
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Plan of talk:

5. Orientations and integration

5.1. Orientations on real vector spaces

Let V be a real vector space of dimension n, and $(v_1,\ldots,v_n)$, $(v'_1,\ldots,v'_n)$ be two bases for V. Then $v'_i=\sum_{j=1}^n A_{ij}v_j$ for $A_{ij}\in\mathbb{R}$, and $(A_{ij})_{i,j=1}^n$ is an invertible real matrix, so it has a determinant $\det(A_{ij})\in\mathbb{R}\setminus 0$. Define an equivalence relation on such bases by $(v_1,\ldots,v_n)\sim(v'_1,\ldots,v'_n)$ if $\det(A_{ij})>0$. Write $[(v_1,\ldots,v_n)]$ for the $\sim$-equivalence class of $(v_1,\ldots,v_n)$. An *orientation* O on V is a choice of $\sim$-equivalence class $[(v_1,\ldots,v_n)]$. There are two possible orientations, $[(v_1,\ldots,v_n)]$ and $[(-v_1,v_2,\ldots,v_n)]$. Given an orientation O, we call a basis $(v_1,\ldots,v_n)$ for V *oriented* if $(v_1,\ldots,v_n)\in O$, and *anti-oriented* otherwise. Given an orientation O on V, the *opposite orientation* $-O$ is the other one.
A basis $(v_1,\ldots,v_n)$ for V corresponds to a dual basis $(v^1,\ldots,v^n)$ for V^*, and orientations on V correspond naturally to orientations on V^*, such that $(v_1,\ldots,v_n)$ is oriented iff $(v^1,\ldots,v^n)$ is oriented.

An orientation on $\mathbb{R}^2$ corresponds to notions of 'clockwise' and 'anticlockwise'. An orientation on $\mathbb{R}^3$ corresponds to notions of 'left-handed' and 'right-handed'.
Reflection in a mirror changes orientation.
We can write orientations in terms of the top exterior power $\Lambda^n V$. It has dimension $\binom{\dim V}{n}=1$, so $\Lambda^n V\cong\mathbb{R}$. If $(v_1,\ldots,v_n)$ is a basis for V then $v_1\wedge\cdots\wedge v_n\in\Lambda^n V\setminus\{0\}$. If $(v'_1,\ldots,v'_n)$ is another basis with $v'_i=\sum_{j=1}^n A_{ij}v_j$ then

$$v'_1\wedge\cdots\wedge v'_n=\det(A_{ij})\cdot v_1\wedge\cdots\wedge v_n.$$

Thus, an orientation on V corresponds to a choice of one of the two connected components of $\Lambda^n V\setminus\{0\}$, where $\Lambda^n V\setminus\{0\}\cong\mathbb{R}\setminus\{0\}=(-\infty,0)\amalg(0,\infty)$.
Given an orientation on V, we call $\alpha\in\Lambda^n V\setminus\{0\}$ *positive* if $\alpha=C\,v_1\wedge\cdots\wedge v_n$ for $C>0$ whenever $(v_1,\ldots,v_n)$ is an oriented basis, and *negative* otherwise.

5.2. Orientations on manifolds and top degree forms

Definition

Let X be a manifold, of dimension n. An *orientation* on X is an orientation on T_xX for each $x \in X$ (or equivalently, on T_x^*X for each $x \in X$) which depends continuously on x.
Orientations may not exist. If X admits an orientation, it is called *orientable*. If X has a choice of orientation, it is called *oriented*.

Thus, if X is oriented, we divide bases $(v^1, \ldots, v^n)$ for T_xX, $x \in X$, into *oriented* bases and *anti-oriented* bases, and under continuous deformations of $(x, v^1, \ldots, v^n)$ the oriented / anti-oriented remains constant. Define a nonvanishing top degree form $\alpha \in C^\infty(\Lambda^n T^*X)$ to be *positive* (or *negative*) if $\alpha|_x \cdot (v^1 \wedge \cdots \wedge v^n) > 0$ (or $\alpha|_x \cdot (v^1 \wedge \cdots \wedge v^n) < 0$) whenever $x \in X$ and $(v^1, \ldots, v^n)$ is an oriented basis for T_xX.

A nonvanishing top degree form $\alpha \in C^\infty(\Lambda^n T^*X)$ determines a unique orientation on X such that α is positive.
A connected orientable manifold has exactly two orientations.

Example

The Möbius strip (§1.5) is a non-orientable 2-manifold. So is the projective plane $\mathbb{RP}^2$, and the 'Klein bottle'.

5.3. Integration on manifolds

We are all familiar with integrals in one or more variables such as $\int_0^1 f(t)\mathrm{d}t$ or $\int_{\mathbb{R}^2} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y$. These happen in subsets $U \subseteq \mathbb{R}^n$, and involve a *particular choice of coordinates* t, (x, y), ... on U. But we also know formulae for how integrals behave under change of coordinates, for instance

$$\int_a^b \left[f(y(x)) \frac{\mathrm{d}y}{\mathrm{d}x}(x) \right] \mathrm{d}x = \int_c^d f(y)\, \mathrm{d}y \qquad (5.1)$$

if $y : [a, b] \to [c, d]$ is differentiable and increasing with $y(a) = c$, $y(b) = d$, changes coordinates from x to $y = y(x)$. You may have been taught that '$\mathrm{d}t$', '$\mathrm{d}x\mathrm{d}y$', ... are simply notation, and don't mean anything.

In Differential Geometry, choosing coordinates is considered bad style, especially in theory rather than examples. So we can ask: How should one interpret integration in Differential Geometry, without choosing coordinates?
Also, do we integrate functions, or something else?

Principle

In Differential Geometry, one should write integrals as $\int_X \alpha \in \mathbb{R}$, where X is an oriented n-dimensional manifold (possibly with boundary or corners), and α is an n-form on X, so $\alpha \in C^\infty(\Lambda^n T^*X)$. We can allow α to be non-smooth, e.g. $\alpha \in L^1(\Lambda^n T^*X)$.

Example

Consider the integral $\int_c^d f(y)\,\mathrm{d}y$, as in (5.1). This is of the form $\int_X \alpha$, where $X = [c,d]$, as a 1-manifold with boundary, oriented such that $\frac{\partial}{\partial y}$ is an oriented basis of T_xX for all $x \in [c,d]$, and $\alpha = f(y)\mathrm{d}y$ is a 1-form on X.
Note that $\mathrm{d}y$ is not just notation: $\mathrm{d}y \in C^\infty(\Lambda^1 T^*X)$ is now a 1-form on X, as is $f(y)\mathrm{d}y$.
Suppose $x : X \to [a,b]$ is another global coordinate on X, so that $y = y(x)$. Then we have

$$f(y(x))\tfrac{\mathrm{d}y}{\mathrm{d}x}(x)\,\mathrm{d}x = f(y)\,\mathrm{d}y \quad \text{in } C^\infty(\Lambda^1 T^*X).$$

So in (5.1) both sides are $\int_X \alpha$, we are just rewriting the integral in terms of two different bases of sections $\mathrm{d}x$ and $\mathrm{d}y$ for $\Lambda^1 T^*X$.

Example

Let $X = [a,b]$, and $f : X \to \mathbb{R}$ be smooth. As $\Lambda^0 T^*X = \mathbb{R}$, we regard $f \in C^\infty(\Lambda^0 T^*X)$ as a 0-form, so $\mathrm{d}f$ is a 1-form, as in §4.1. We write $\mathrm{d}f = \frac{\mathrm{d}f}{\mathrm{d}x}(x)\,\mathrm{d}x$, using the coordinate x on $X = [a,b]$. But as a 1-form, $\mathrm{d}f$ is independent of coordinates. As usual we have

$$\int_X \mathrm{d}f = \int_{[a,b]} \frac{\mathrm{d}f}{\mathrm{d}x}(x)\,\mathrm{d}x = f(b) - f(a).$$

However, if we had chosen $y = -x$ as coordinate on X, identifying X with $[-b,-a]$, and defined $g : [-b,-a] \to \mathbb{R}$ by $g(y) = f(-y)$, then in 1-forms we have $\mathrm{d}(g(y)) = \mathrm{d}(f(x))$, so we expect

$$\int_X \mathrm{d}f = \int_X \mathrm{d}g = \int_{[-b,-a]} \frac{\mathrm{d}g}{\mathrm{d}y}(y)\,\mathrm{d}y = g(-a) - g(-b) = f(a) - f(b).$$

What went wrong? We changed orientations, from $\frac{\partial}{\partial x}$ oriented basis of T_pX to $\frac{\partial}{\partial y}$ oriented, changing the sign of the integral.

Stokes' Theorem

Theorem (Stokes' Theorem)

Let X be a compact, oriented n-dimensional manifold with boundary (or with corners). Then the boundary ∂X is a compact $(n-1)$-dimensional manifold (with corners if X has corners), with a natural orientation induced from the orientation of X.
Let $\alpha \in C^\infty(\Lambda^{n-1}T^*X)$ be an $(n-1)$-form on X, so that $\mathrm{d}\alpha \in C^\infty(\Lambda^n T^*X)$. We may restrict α to the boundary (that is, pull back $i^*(\alpha)$ by the inclusion $i : \partial X \hookrightarrow X$) to form $\alpha|_{\partial X} \in C^\infty(\Lambda^{n-1}T^*\partial X)$. Then

$$\int_X \mathrm{d}\alpha = \int_{\partial X} \alpha|_{\partial X}. \tag{5.2}$$

Example

If $f : [a, b] \to \mathbb{R}$ is differentiable then

$$\int_{[a,b]} \frac{\mathrm{d}f}{\mathrm{d}x}(x)\, dx = f(b) - f(a).$$

This is an example of Stokes' Theorem with $X = [a, b]$ and $\alpha = f$, as a 0-form (function). So $\frac{\mathrm{d}f}{\mathrm{d}x}(x)\, dx = \mathrm{d}f$.
We have $\partial X = \{a\} \amalg \{b\}$, as a 0-manifold, where $\{b\}$ has positive orientation and $\{a\}$ negative orientation. So $\int_{\partial X} f|_{\partial X} = f(b) - f(a)$.

Example (Green's Theorem)

Let C be a simple, smooth, closed curve in $\mathbb{R}^2$, oriented anticlockwise. Then $C = \partial D$ for $D \subseteq \mathbb{R}^2$ a (topological) closed disc. Suppose $L, M : D \to \mathbb{R}$ are smooth. Then

$$\oint_C L\mathrm{d}x + M\mathrm{d}y = \int_D \Big(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\Big)\,\mathrm{d}x\,\mathrm{d}y.$$

This is an example of Stokes' Theorem, with $X = D$, and α the 1-form $L(x,y)\mathrm{d}x + M(x,y)\mathrm{d}y$, so that $\mathrm{d}\alpha = \big(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\big)\,\mathrm{d}x \wedge \mathrm{d}y$.

5.4. Applications to de Rham cohomology
Integrating over submanifolds, and homology

Suppose X is a manifold, and $Y \hookrightarrow X$ is a compact, oriented, k-dimensional submanifold. Define a linear map

$$[Y]\cdot\ : H^k_{\mathrm{dR}}(X,\mathbb{R}) \longrightarrow \mathbb{R}$$

by $[Y]\cdot(\alpha + \mathrm{Im}\,\mathrm{d}) = \int_Y (\alpha|_Y)$ for each $\alpha \in C^\infty(\Lambda^k T^*X)$ with $\mathrm{d}\alpha = 0$. This is well defined since if $\alpha + \mathrm{Im}\,\mathrm{d} = \alpha' + \mathrm{Im}\,\mathrm{d}$ then $\alpha = \alpha' + \mathrm{d}\beta$, and $\int_Y \mathrm{d}\beta = \int_{\partial Y} \beta = 0$ by Stokes' Theorem, as $\partial Y = \emptyset$. Thus each compact, oriented k-submanifold Y in X defines a class $[Y]$ in the dual vector space $H^k_{\mathrm{dR}}(X,\mathbb{R})^*$, which is the homology group $H_k(X,\mathbb{R})$. In fact we can define $[Y] \in H_k(X,\mathbb{Z})$ in homology over $\mathbb{Z}$.

Poincaré duality

Let X be a compact, oriented n-manifold. Then as above we have a linear map $[X]\cdot\ : H^n_{\rm dR}(X,\mathbb{R})\to\mathbb{R}$. For each $k=0,\ldots,n$, define a bilinear pairing

$$(\ ,\ ): H^k_{\rm dR}(X,\mathbb{R})\times H^{n-k}_{\rm dR}(X,\mathbb{R})\longrightarrow\mathbb{R}$$

by $(\alpha,\beta)=[X]\cdot(\alpha\cup\beta)$. *Poincaré duality* says that this is a perfect pairing, that is, it induces an isomorphism of dual vector spaces $H^k_{\rm dR}(X,\mathbb{R})\cong H^{n-k}_{\rm dR}(X,\mathbb{R})^*$. Hence the Betti numbers satisfy $b^k(X)=b^{n-k}(X)$. This can be false if X is not compact, or not orientable. For example, the Betti numbers of $\mathbb{R}^n$ (oriented but noncompact) are $b^0=1$ and $b^k=0$ for $k>0$, and the Betti numbers of $\mathbb{RP}^2$ (compact but not orientable) are $b^0=1$ and $b^1=b^2=0$, so Poincaré duality fails in both cases.

5.5. The classification of compact 2-manifolds

It is an important problem to classify manifolds of a given dimension up to diffeomorphism. Usually one restricts to compact manifolds (since noncompact manifolds may be infinitely complex). As any compact manifold is the disjoint union of finitely many compact, connected manifolds, we can restrict to connected manifolds. The difficulty of the classification problem increases with dimension (well, modulo problems in dimensions 3 and 4). In 1 dimension, the only compact, connected 1-manifold is $\mathcal{S}^1$. We will explain the classification of compact, connected 2-manifolds. We begin with the notion of *connect sum*.

Connect sums

Suppose X and Y are connected, oriented manifolds of dimension n. We will explain how to define a connected, oriented n-manifold $X\#Y$ called the *connect sum* of X and Y.

Pick points $x_0 \in X$ and $y_0 \in Y$. Cut out small open balls $B_\epsilon(x_0)$ about x_0 in X and $B_\epsilon(y_0)$ about y_0 in Y, to give manifolds with boundary $X \setminus B_\epsilon(x_0)$ and $Y \setminus B_\epsilon(y_0)$. These have boundaries $S_\epsilon(x_0)$, $S_\epsilon(y_0)$ which are small oriented $(n-1)$-spheres $\mathcal{S}^{n-1}$. Glue $X \setminus B_\epsilon(x_0)$ and $Y \setminus B_\epsilon(y_0)$ by an orientation-reversing diffeomorphism along their common boundary $\mathcal{S}^{n-1}$ to get a connected, oriented n-manifold $X\#Y$, which we think of as X and Y joined by a small neck. Up to oriented diffeomorphism, it depends only on X, Y as oriented manifolds.

If X, Y are compact then $X\#Y$ is compact.

Connect sum is a kind of addition operation on (compact) oriented n-manifolds. It is commutative and associative, with $X\#\mathcal{S}^n \cong X$.

We can also define $X\#Y$ if X, Y are not oriented, but then we have to choose an orientation for the gluing map $S_\epsilon(x_0) \cong S_\epsilon(y_0)$.

Additivity properties of Euler characteristics imply that $\chi(X\#Y) = \chi(X) + \chi(Y) - \chi(\mathcal{S}^n)$, where $\chi(\mathcal{S}^n)$ is 2 for n even and 0 for n odd.

Theorem (Classification of compact 2-manifolds)

Let X be a compact, connected 2-manifold. Then either:
(a) *X is orientable. Then X is diffeomorphic to the connect sum $T^2 \# T^2 \cdots \# T^2$ of g tori T^2 for $g = 0, 1, \ldots,$ with $X \cong \mathcal{S}^2$ when $g = 0$. We call g the* **genus** *of X, and we call X a* **genus** *g* **surface**. *We have $b^0(X) = 1$, $b^1(X) = 2g$, $b^2(X) = 1$, and $\chi(X) = 2 - 2g$.*
(b) *X is not orientable. Then X is diffeomorphic to the connect sum $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ of h projective planes $\mathbb{RP}^2$ for $h = 1, 2, \ldots$. We have $b^0(X) = 1$, $b^1(X) = h$, $b^2(X) = 0$, and $\chi(X) = 1 - h$.*

Compact 2-manifolds are generated under connect sum $\#$ by T^2 and $\mathbb{RP}^2$, with the relation $T^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.
The *Klein bottle* is $K = \mathbb{RP}^2 \# \mathbb{RP}^2$.
A compact 2-manifold X can be embedded in $\mathbb{R}^3$ iff it is orientable.

Introduction to Differential Geometry

Lecture 6 of 10: Connections and curvature

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Plan of talk:

6. Connections and curvature
6.1. Differentiation in Differential Geometry

Let X be a manifold. If $f : X \to \mathbb{R}$ is smooth, the 'derivative' of f is the 1-form $\mathrm{d}f \in C^\infty(T^*X)$ (regarding f as a 0-form).
Now let $E \to X$ be a vector bundle (e.g. $E = \bigotimes^k TX \otimes \bigotimes^l T^*X$), and $s \in C^\infty(E)$ be a section (e.g. s is a tensor).
What is meant by the 'derivative' of s / 'differentiating s'?

We have defined several operations involving differentiation:

- Lie bracket $[v, w]$ of vector fields v, w.
- Lie derivative $\mathcal{L}_v T$ of tensors T, for vector fields v.
- de Rham differential $\mathrm{d}\alpha$ of k-forms α.

These all make sense on X just as a manifold, without making any additional choices (e.g. choosing coordinates). But they are not really derivatives of s.

It turns out that to differentiate sections s of a nontrivial vector bundle $E \to X$ (even if E is a tensor bundle), you have to make an arbitrary choice. This choice is called a 'connection', written ∇ (pronounced 'nabla'). The derivative of s is then $\nabla s \in C^\infty(E \otimes T^*X)$. Alternatively, if $v \in C^\infty(TX)$ is a vector field, we write $\nabla_v s \in C^\infty(E)$ for $v \cdot \nabla s$, the derivative of s in direction v. To see why we need to make an arbitrary choice, note that heuristically we want

$$\nabla_v s|_x = \lim_{t \to 0} \frac{s|_{x+tv} - s|_x}{t}. \tag{6.1}$$

If $s : X \to \mathbb{R}$ were smooth, this would make sense (more-or-less). But as s is a section of a vector bundle $E \to X$, we have $s|_{x+tv} \in E_{x+tv}$ and $s|_x \in E_x$, so $s|_{x+tv}$ and $s|_x$ lie in different vector spaces, and $s|_{x+tv} - s|_x$ does not make sense.
Roughly, the job a connection ∇ does is identify fibres $E_x \cong E_y$ for x and y (infinitesimally) close in X, so we can make sense of (6.1).

6.2. The definition of connections

There are several equivalent ways to define connections.

Definition (First definition of connections)

Let X be a manifold and $E \to X$ a vector bundle. A *connection* ∇ on E is an $\mathbb{R}$-linear map $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*X)$ satisfying

$$\nabla(fs) = f\nabla s + s \otimes \mathrm{d}f \tag{6.2}$$

for all sections $s \in C^\infty(E)$ and smooth $f : X \to \mathbb{R}$, where $\mathrm{d}f \in C^\infty(T^*X)$ is the de Rham differential. In local coordinates $(x^1, \ldots, x^n)$ on X, we have $\mathrm{d}f = \frac{\partial f}{\partial x^1}\mathrm{d}x^1 + \cdots + \frac{\partial f}{\partial x^n}\mathrm{d}x^n$.
For $v \in C^\infty(TX)$, $s \in C^\infty(E)$ we write $\nabla_v s = v \cdot \nabla s \in C^\infty(E)$.

This definition is based on two ideas:

- We know how to differentiate smooth $f : X \to \mathbb{R}$, by $\mathrm{d}f$.
- Differentiation should satisfy the product rule, hence (6.2).

Nonuniqueness of connections

Suppose ∇, ∇' are both connections on $E \to X$. Then subtracting (6.2) for ∇' and ∇ gives

$$(\nabla' - \nabla)(fs) = f(\nabla' - \nabla)s.$$

That is, $\nabla' - \nabla$ is linear not just over $\mathbb{R}$, but over all smooth functions $f : X \to \mathbb{R}$. So there is a unique $C \in C^\infty(E \otimes E^* \otimes T^*X)$ such that $(\nabla' - \nabla)s = C \cdot s$, where $C \cdot s \in C^\infty(E \otimes T^*X)$ pairs the E^* factor in $E \otimes E^* \otimes T^*X \ni C$ with $E \ni s$. Thus

$$\nabla' s = \nabla s + C \cdot s. \qquad (6.3)$$

Conversely, if ∇ is a connection on E and $C \in C^\infty(E \otimes E^* \otimes T^*X)$ then ∇' defined by (6.3) is a connection. It turns out that connections ∇ exist on any $E \to X$. Thus, the family of all connections ∇' on X is an affine space modelled on the (infinite-dimensional) vector space $C^\infty(E \otimes E^* \otimes T^*X)$.

Connections in coordinates

Let $E \to X$ be a vector bundle of rank r over an n-manifold X. Choose coordinates $(x^1, \ldots, x^n)$ over an open set $U \subseteq X$. Making U smaller, we can suppose $E|_U$ is trivial, so we can choose a basis of sections $e_1, \ldots, e_r$ of $E|_U$. Define smooth functions $\Gamma^\alpha_{\beta c} : U \to \mathbb{R}$ for $\alpha, \beta = 1, \ldots, r$ and $c = 1, \ldots, n$ by

$$\nabla e_\beta|_U = \sum_{\alpha=1}^r \sum_{c=1}^n \Gamma^\alpha_{\beta c} \cdot e_\alpha \otimes \mathrm{d}x^c.$$

The $\Gamma^\alpha_{\beta c}$ are called *Christoffel symbols*. Then (6.2) gives

$$\nabla\Bigl(\sum_{\alpha=1}^r s^\alpha e_\alpha\Bigr) = \sum_{\alpha=1}^r \sum_{c=1}^n \Bigl(\frac{\partial s^\alpha}{\partial x^c} + \sum_{\beta=1}^r \Gamma^\alpha_{\beta c} s^\beta\Bigr) \cdot e_\alpha \otimes \mathrm{d}x^c. \qquad (6.4)$$

For any smooth functions $\Gamma^\alpha_{\beta c}$, equation (6.4) defines a connection on $E|_U \to U$, and any connection on $E|_U$ is of this form for unique $\Gamma^\alpha_{\beta c}$.

Connections under change of coordinates

Let ∇ be a connection on $E \to X$, and let $(x^1,\ldots,x^n)$ be coordinates on $U \subseteq X$, $e_1,\ldots,e_r$ a basis of sections of $E|_U$, and $\Gamma^\alpha_{\beta c}$ the Christoffel symbols. Suppose $(\tilde x^1,\ldots,\tilde x^n)$, $\tilde U \subseteq X$, $\tilde e_1,\ldots,\tilde e_r$, $\tilde\Gamma^\alpha_{\beta c}$ are alternative choices.
Then on $U \cap \tilde U$ we may write $\tilde e_\alpha = \sum_{\beta=1}^r A^\beta_\alpha e_\beta$ for $\bigl(A^\beta_\alpha\bigr)^r_{\alpha,\beta=1}$ an invertible $r \times r$ matrix of smooth functions $A^\beta_\alpha : U \cap \tilde U \to \mathbb{R}$. Write $\bigl(B^\beta_\alpha\bigr)^r_{\alpha,\beta=1}$ for the inverse matrix, so that $e_\alpha = \sum_{\beta=1}^r B^\beta_\alpha \tilde e_\beta$. Then calculation using (6.4) shows that

$$\tilde\Gamma^\alpha_{\beta c} = \sum_{\gamma,\delta=1}^r \sum_{d=1}^n A^\gamma_\beta B^\alpha_\delta \frac{\partial x^d}{\partial \tilde x^c} \Gamma^\delta_{\gamma d} + \sum_{\gamma=1}^r \Bigl(\frac{\partial}{\partial \tilde x^c} A^\gamma_\beta\Bigr) B^\alpha_\gamma. \qquad (6.5)$$

The first term is the transformation rule for a section of $E \otimes E^* \otimes T^*X$, but the second term is extra.

This gives us an alternative, coordinate-based definition of connections:

Definition (Second definition of connections)

A connection ∇ on a vector bundle $E \to X$ assigns 'Christoffel symbols', smooth functions $\Gamma^\alpha_{\beta c} : U \to \mathbb{R}$ for $\alpha,\beta = 1,\ldots,r$ and $c = 1,\ldots,n$ whenever $(x^1,\ldots,x^n)$ are coordinates on open $U \subseteq X$ and $e_1,\ldots,e_r$ a basis of $E|_U$, which must transform according to (6.5) under change of $U,(x^1,\ldots,x^n),e_1,\ldots,e_r$.
Then ∇s for $s \in C^\infty(E)$ is defined in coordinates by (6.4).

Definition (Third definition of connections)

Let $\pi : E \to X$ be a vector bundle over X. Then E is a manifold, and $\mathrm{d}\pi : TE \to \pi^*(TX)$ is a surjective morphism of vector bundles. Define $V = \mathrm{Ker}\,\mathrm{d}\pi$, a vector subbundle of TE isomorphic to $\pi^*(E)$. We call V the 'vertical subbundle' of TE.
A *connection* ∇ on E is a choice of vector subbundle $H \subset TE$ called the 'horizontal subbundle', such that $TE = V \oplus H$, which implies that $\mathrm{d}\pi|_H : H \to \pi^*(TX)$ is an isomorphism, and H satisfies a compatibility with the vector bundle structure on E.
For $s \in C^\infty(E)$, we define $\nabla s \in C^\infty(E \otimes T^*X)$ by the composition

$$TX \xrightarrow{\mathrm{d}s} s^*(TE) = s^*(V) \oplus s^*(H) \xrightarrow{\pi_{s^*(V)}} s^*(V) \xrightarrow{\cong} s^*(\pi^*(E)) = E.$$

This is related to the other definitions as follows: if $s \in C^\infty(E)$ then the graph of s, $\Gamma_s = s(X)$, is a submanifold of E diffeomorphic to X. The subbundle H is characterized by: if $\nabla s|_x = 0$ then $T(\Gamma_s)|_{s(x)} = H|_{s(x)}$.

Connections on associated vector bundles

Let ∇^E be a connection on $E \to X$. Then there is a unique connection ∇^{E^*} on the dual bundle $E^* \to X$ with the property that

$$\mathrm{d}(\sigma \cdot s) = \sigma \cdot \nabla^E s + (\nabla^{E^*}\sigma) \cdot s \in C^\infty(T^*X)$$

for all $s \in C^\infty(E)$ and $\sigma \in C^\infty(E^*)$, as one would expect from the product rule. If ∇^E has Christoffel symbols $\Gamma^\alpha_{\beta c}$ then ∇^{E^*} has Christoffel symbols $-\Gamma^\beta_{\alpha c}$ w.r.t. the dual basis of E^*.
Similarly, if ∇^F is a connection on $F \to X$, there is a unique connection $\nabla^{E\otimes F}$ on $E \otimes F \to X$ such that

$$\nabla^{E\otimes F}(s \otimes t) = (\nabla^E s) \otimes t + s \otimes (\nabla^F t)$$

for all $s \in C^\infty(E)$ and $t \in C^\infty(F)$, from the product rule.
Thus a connection on $E \to X$ induces connections on $\bigotimes^k E \otimes \bigotimes^l E^*$, $S^k E$, $\Lambda^k E$, and so on.

Pullbacks of connections

Let $f : X \to Y$ be a smooth map of manifolds, $E \to Y$ a vector bundle, and ∇ a connection on E. Then we have a pullback vector bundle $f^*(E) \to X$. It turns out that there is a unique *pullback connection* $\nabla' = f^*(\nabla)$ on $f^*(E)$, with the property that if $s \in C^\infty(E)$ then

$$\nabla'\big(f^*(s)\big) = (\mathrm{d}f)^* \cdot f^*(\nabla s),$$

where $(\mathrm{d}f)^*$ maps $f^*(T^*Y) \to T^*X$, so $(\mathrm{d}f)^*\cdot$ maps $f^*(E) \otimes f^*(T^*Y) \to f^*(E) \otimes T^*X$.

6.3. Curvature of connections

If $f(x_1, \ldots, x_n)$ is a smooth function, we know that $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$. That is, the partial derivatives $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ commute on f. Should we expect partial derivatives to commute in Differential Geometry? Thinking of the action of $\frac{\partial}{\partial x^i}$ on functions as the Lie derivative $\mathcal{L}_v$ for $v = \frac{\partial}{\partial x^i}$, we have

Lemma 6.1

Let $f : X \to \mathbb{R}$ be smooth, and $v, w \in C^\infty(TX)$. Then

$$\mathcal{L}_v(\mathcal{L}_w f) - \mathcal{L}_w(\mathcal{L}_v f) = \mathcal{L}_{[v,w]} f.$$

Proof. In local coordinates $(x^1, \ldots, x^n)$ on X we have

$$\mathcal{L}_v(\mathcal{L}_w f) - \mathcal{L}_w(\mathcal{L}_v f) = v^b \frac{\partial}{\partial x^b}\Big[w^a \frac{\partial f}{\partial x^a}\Big] - w^b \frac{\partial}{\partial x^b}\Big[v^a \frac{\partial f}{\partial x^a}\Big]$$
$$= v^a w^b \Big[\frac{\partial^2 f}{\partial x^a \partial x^b} - \frac{\partial^2 f}{\partial x^b \partial x^a}\Big] + \Big(v^b \frac{\partial}{\partial x^b} w^a - w^b \frac{\partial}{\partial x^b} v^a\Big)\frac{\partial f}{\partial x^a} = 0 + \mathcal{L}_{[v,w]} f. \quad \square$$

The lemma tells us that the Lie bracket $[v, w]$ measures the extent to which derivatives by v, w on functions commute, and $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ holds because $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$.
Now let $E \to X$ be a vector bundle, and ∇ a connection on E. Then for $v, w \in C^\infty(TX)$ we can consider whether ∇_v and ∇_w commute on sections $s \in C^\infty(E)$. Motivated by Lemma 6.1, a better question is whether

$$\nabla_v(\nabla_w s) - \nabla_w(\nabla_v s) = \nabla_{[v,w]} s$$

for all $v, w \in C^\infty(TX)$ and $s \in C^\infty(E)$.

Orientations and integration
Connections and curvature
Differentiation in Differential Geometry
The definition of connections
Curvature of connections
Flat and locally trivial connections
Connections on TX and torsion

Proposition (Definition of curvature)

Let ∇ be a connection on a vector bundle $E \to X$. Then there is a unique $R \in C^\infty(E \otimes E^* \otimes \Lambda^2 T^*X)$ called the **curvature** *with the property that*

$$\nabla_v(\nabla_w s) - \nabla_w(\nabla_v s) - \nabla_{[v,w]} s = R \cdot (s \otimes v \otimes w) \in C^\infty(E) \quad (6.6)$$

for all $v, w \in C^\infty(TX)$ and $s \in C^\infty(E)$. In coordinates $(x^1, \ldots, x^n)$ on $U \subseteq X$ and a basis $e_1, \ldots, e_r$ for $E|_U$ and dual basis $e^1, \ldots, e^r$ for $E^*|_U$ we have

$$R = \sum_{\alpha,\beta=1,\ldots,r} \sum_{c,d=1}^{n} R^\alpha_{\beta cd} e_\alpha \otimes e^\beta \otimes \mathrm{d}x^c \otimes \mathrm{d}x^d, \quad \text{where}$$
$$R^\alpha_{\beta cd} = \frac{\partial}{\partial x^c} \Gamma^\alpha_{\beta d} - \frac{\partial}{\partial x^d} \Gamma^\alpha_{\beta c} + \sum_{\epsilon=1}^{r} \left(\Gamma^\alpha_{\epsilon c} \Gamma^\epsilon_{\beta d} - \Gamma^\alpha_{\epsilon d} \Gamma^\epsilon_{\beta c} \right), \quad (6.7)$$

with $\Gamma^\alpha_{\beta c}$ the Christoffel symbols of ∇.

Proof (exercise to complete). Check using (6.4) that in coordinates, the $R, R^\alpha_{\beta cd}$ defined in (6.7) satisfy (6.6) for all $v = v^a, w = w^b, s = s^\alpha$. □

The curvature is an important differential-geometric invariant of a connection ∇, that measures the extent to which partial derivatives using ∇ commute. In particular, if $(x^1, \ldots, x^n)$ are local coordinates and $s \in C^\infty(E)$ then the analogue of $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$,

$$\nabla_{\frac{\partial}{\partial x^i}}\bigl(\nabla_{\frac{\partial}{\partial x^j}} s\bigr) = \nabla_{\frac{\partial}{\partial x^j}}\bigl(\nabla_{\frac{\partial}{\partial x^i}} s\bigr),$$

holds for all $i, j = 1, \ldots, n$ and $s \in C^\infty(E)$ if and only if $R = 0$.

6.4. Flat and locally trivial connections

Definition

Let ∇ be a connection on a vector bundle $E \to X$, with curvature R. We call ∇ *flat* if $R = 0$.
We call ∇ *locally trivial* if every $x \in X$ has an open neighbourhood $U \subseteq X$ such that $E|_U$ has a basis of sections $e_1, \ldots, e_r$ with $\nabla e_i = 0$ for $i = 1, \ldots, r$. That is, over U we can identify E with the trivial vector bundle $U \times \mathbb{R}^r \to U$ and ∇ with the trivial connection $\sum_{a=1}^n \frac{\partial}{\partial x^a} \otimes \mathrm{d}x^a$ on $U \times \mathbb{R}^r \to U$.

Theorem 6.2 (Consequence of the Frobenius Theorem)

A connection ∇ on $E \to X$ is flat if and only if it is locally trivial.

This is a theorem about p.d.e.s — it says that if $R = 0$, we can find $\operatorname{rank} E$ local solutions of the p.d.e. $\nabla s = 0$ for $s \in C^\infty(E)$.

6.5. Connections on TX and torsion

An important case is when E is the tangent bundle TX.
Note that a connection ∇ on TX also induces connections on the tensor bundles $\bigotimes^k TX \otimes \bigotimes^l T^*X$, and exterior forms $\Lambda^k T^*X$.
Given coordinates $(x^1, \ldots, x^n)$ on $U \subseteq X$, we have a natural basis of sections $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ of $TX|_U$, and we take this to be $e^1, \ldots, e^n$.
We write Christoffel symbols as Γ^a_{bc} rather than $\Gamma^\alpha_{\beta c}$, defined by

$$\nabla \frac{\partial}{\partial x^b} = \sum_{a,c=1}^n \Gamma^a_{bc} \cdot \frac{\partial}{\partial x^a} \otimes \mathrm{d}x^c.$$

Then in index notation, equations (6.4)–(6.5) become

$$\nabla_c v^a = \frac{\partial v^a}{\partial x^c} + \Gamma^a_{bc} v^b, \tag{6.8}$$

$$\tilde{\Gamma}^a_{bc} = \frac{\partial \tilde{x}^a}{\partial x^d} \frac{\partial x^e}{\partial \tilde{x}^b} \frac{\partial x^f}{\partial \tilde{x}^c} \Gamma^d_{ef} + \frac{\partial^2 x^d}{\partial \tilde{x}^b \partial \tilde{x}^c} \frac{\partial \tilde{x}^a}{\partial x^d}. \tag{6.9}$$

Orientations and integration
Connections and curvature
Differentiation in Differential Geometry
The definition of connections
Curvature of connections
Flat and locally trivial connections
Connections on TX and torsion

Definition

Let ∇ be a connection on $TX \to X$. Then there is a unique tensor $T = T^a_{bc} \in C^\infty(TX \otimes \Lambda^2 T^*X)$ called the *torsion* of ∇, with $T^a_{bc} = -T^a_{cb}$, with the property that

$$T \cdot (v \otimes w) = \nabla_v w - \nabla_w v - [v, w] \in C^\infty(TX) \tag{6.10}$$

for all vector fields $v, w \in C^\infty(TX)$. If $T = 0$, then ∇ is called *torsion-free*, and $\nabla_v w - \nabla_w v = [v, w]$ for all vector fields v, w.
In coordinates $(x^1, \ldots, x^n)$ on $U \subseteq X$, the torsion is given by

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb},$$

where Γ^a_{bc} are the Christoffel symbols. Note that (6.9) implies that T transforms under change of coordinates by

$$\tilde{T}^a_{bc} = \frac{\partial \tilde{x}^a}{\partial x^d} \frac{\partial x^e}{\partial \tilde{x}^b} \frac{\partial x^f}{\partial \tilde{x}^c} T^d_{ef},$$

the correct transformation rule for a tensor.

Torsion-free connections are the 'best' kind of connections on TX. For connections ∇ on TX we have two similar invariants: the torsion $T\in C^\infty(TX\otimes\Lambda^2T^*X)$, and the curvature $R\in C^\infty(TX\otimes T^*X\otimes\Lambda^2T^*X)$. They satisfy the *Bianchi identities*

$$\begin{aligned} R^a_{bcd}+R^a_{cdb}+R^a_{dbc}&=T^a_{ed}T^e_{bc}+T^a_{eb}T^e_{cd}+T^a_{ec}T^e_{db}\\ &\quad+\nabla_bT^a_{cd}+\nabla_cT^a_{db}+\nabla_dT^a_{bc},\\ \nabla_cR^a_{bde}+\nabla_dR^a_{bec}+\nabla_eR^a_{bcd}&+T^f_{cd}R^a_{bfe}+T^f_{de}R^a_{bfc}+T^f_{ec}R^a_{bfd}=0. \end{aligned}$$

Here torsion is a *first-order invariant*: its definition (6.10) involves one derivative (and no derivatives of Γ^a_{bc}). And curvature is a *second-order invariant*: its definition (6.6) involves two derivatives (and one derivative of Γ^a_{bc}).