Introduction to Differential Geometry

Lecture 7 of 10: Riemannian manifolds

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These slides available at
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Plan of talk:

1. Riemannian manifolds
2. Riemannian metrics
3. The Levi-Civita connection
4. The Riemann curvature tensor
5. Volume forms and integrating functions
7. Riemannian manifolds

7.1. Riemannian metrics

In Euclidean geometry on $\mathbb{R}^n$, by Pythagoras’ Theorem the distance between two points $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ is

$$d_{\mathbb{R}^n}(x, y) = \left( (x^1 - y^1)^2 + \cdots + (x^n - y^n)^2 \right)^{1/2}.$$ 

Note that squares of distances, rather than distances, behave nicely, algebraically. If $\gamma = (\gamma^1, \ldots, \gamma^n) : [0, 1] \to \mathbb{R}^n$ is a smooth path in $\mathbb{R}^n$, then the length of $\gamma$ is

$$l(\gamma) = \int_0^1 \left( \left( \frac{d\gamma^1}{dt}(t) \right)^2 + \cdots + \left( \frac{d\gamma^n}{dt}(t) \right)^2 \right)^{1/2} dt.$$ 

Note that this is unchanged under reparametrizations of $[0, 1]$; replacing $t$ by an alternative coordinate $\tilde{t}$ multiplies $\frac{d\gamma^i}{dt}$ by $\frac{dt}{d\tilde{t}}$ and $dt$ by $\frac{d\tilde{t}}{dt}$, which cancel.

Regarding $\gamma : [0, 1] \to \mathbb{R}^n$ as a smooth map of manifolds, we have

$$l(\gamma) = \int_0^1 g_{\mathbb{R}^n}|_{\gamma(t)} \left( \frac{d\gamma}{dt}(t), \frac{d\gamma}{dt}(t) \right)^{1/2} dt,$$

where $\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} \mathbb{R}^n \cong \mathbb{R}^n$, and $g_{\mathbb{R}^n}|_{x} = (dx^1)^2 + \cdots + (dx^n)^2$ in $S^2 T_x^* \mathbb{R}^n \cong S^2(\mathbb{R}^n)^*$ for $x \in \mathbb{R}^n$, so that $g_{\mathbb{R}^n} \in C^\infty(S^2 T^* \mathbb{R}^n)$. This is a simple example of a Riemannian metric on a manifold, being used to define lengths of curves.
Definition

Let $X$ be a manifold. A Riemannian metric $g$ (or just metric) is a smooth section of $S^2 T^*X$ such that $g|_x \in S^2 T^*_x X$ is a positive definite quadratic form on $T_x X$ for all $x \in X$. In index notation we write $g = g_{ab}$, with $g_{ab} = g_{ba}$. We call $(X, g)$ a Riemannian manifold. Let $\gamma : [0, 1] \to X$ be a smooth map, considered as a curve in $X$. The length of $\gamma$ is

$$l(\gamma) = \int_0^1 g|_{\gamma(t)} \left( \frac{d\gamma}{dt}(t) \right) \frac{1}{2} dt.$$ 

If $X$ is (path-)connected, we can define a metric $d_g$ on $X$, in the sense of metric spaces, by

$$d_g(x, y) = \inf_{\gamma : [0, 1] \to X : C^\infty, \gamma(0) = x, \gamma(1) = y} l(\gamma).$$

Roughly, $d_g(x, y)$ is the length of the shortest curve $\gamma$ from $x$ to $y$.

Restricting metrics to submanifolds

Let $i : X \to Y$ be an immersion or an embedding, so that $X$ is a submanifold of $Y$, and $g \in C^\infty(S^2 T^*Y) \subseteq C^\infty(\bigotimes^2 T^*Y)$ be a Riemannian metric on $Y$. Pulling back gives $i^\#(g) \in C^\infty(S^2 i^*(T^*Y)).$ We have a vector bundle morphism $di : TX \to i^*(TY)$ on $X$, which is injective as $i$ is an immersion, and a dual surjective morphism $(di)^* : i^*(T^*Y) \to T^*X$. Symmetrizing gives $S^2(di)^* : S^2 i^*(T^*Y) \to S^2 T^*X.$ Define $i^*(g) = (S^2(di)^*)(i^\#(g)) \in C^\infty(S^2 T^*X).$ This is defined for any smooth map $i : X \to Y$. But if $i$ is an immersion, so that $di : TX \to i^*(TY)$ is injective, then $g$ positive definite implies $i^*(g)$ positive definite, so $i^*(g)$ is a Riemannian metric on $X$. We call it the pullback or restriction of $g$ to $X$, and write it as $g|_X$. 


Submanifolds of Euclidean space

**Example**

Define $g_{\mathbb{R}^n} = (dx^1)^2 + \cdots + (dx^n)^2$ in $C^\infty(S^2(\mathbb{R}^n)^*)$. This is a Riemannian metric on $\mathbb{R}^n$, which induces the usual notions of lengths of curves and distance in Euclidean geometry. We call $g_{\mathbb{R}^n}$ the *Euclidean metric* on $\mathbb{R}^n$.

**Example**

Let $X$ be any submanifold of $\mathbb{R}^n$. Then $g_{\mathbb{R}^n}|_X$ is a Riemannian metric on $X$.

Since any manifold $X$ can be embedded in $\mathbb{R}^n$ for $n \gg 0$ (Whitney Embedding Theorem), this implies

**Corollary**

*Any manifold $X$ admits a Riemannian metric.*

These ideas are important even in really basic applied mathematics, physics, geography, etc:

**Example**

Model the surface of the earth as a sphere $S^2_R$ of radius $R = 6,371$ km about 0 in $\mathbb{R}^3$. Then the Riemannian metric $g_E = g_{\mathbb{R}^3}|_{S^2_R}$ determines lengths of paths on the earth. Define spherical polar coordinates $(\theta, \varphi)$ (latitude and longitude) on $S^2_R \setminus \{N, S\}$ by $x(\theta, \varphi) = (R \sin \theta \cos \varphi, R \sin \theta \sin \varphi, R \cos \theta)$. Then

$$
g_E = \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right)|_{S^2}$$

$$= (d(R \sin \theta \cos \varphi))^2 + (d(R \sin \theta \sin \varphi))^2 + (d(R \cos \theta))^2$$

$$= R^2[(\cos \theta \cos \varphi \, d\theta - \sin \theta \sin \varphi \, d\varphi)^2$$

$$+ (\cos \theta \sin \varphi \, d\theta + \sin \theta \cos \varphi \, d\varphi)^2 + (-\sin \theta \, d\theta)^2]$$

$$= R^2[d\theta^2 + \sin^2 \theta \, d\varphi^2].$$
7.2. The Levi-Civita connection

Any Riemannian manifold \((X, g)\) has a natural connection \(\nabla\) on \(TX\), called the \textit{Levi-Civita connection}. This is known as the ‘Fundamental Theorem of Riemannian Geometry’.

\begin{theorem}[The Fundamental Theorem of Riemannian Geometry]
Let \((X, g)\) be a Riemannian manifold. Then there exists a unique connection \(\nabla\) on \(TX\), such that \(\nabla\) is torsion-free, and the induced connection \(\nabla'\) on \(\bigotimes^2 T^*X\) satisfies \(\nabla' g = 0\). We call \(\nabla\) the \textit{Levi-Civita connection} of \(g\).
\end{theorem}

Usually we write \(\nabla\) for all the induced connections on \(\bigotimes^k TX \otimes \bigotimes^l T^*X\), without comment. So \(\nabla\) allows us to differentiate all tensors \(T\) on a Riemannian manifold \((X, g)\), without making any arbitrary choices.

\begin{proof}[Proof of the Fundamental Theorem]
The theorem is local in \(X\), so it is enough to prove it in coordinates \((x^1, \ldots, x^n)\) defined on open \(U \subseteq X\). Let \(\nabla\) be a connection on \(TX\), with Christoffel symbols \(\Gamma^a_{bc} : U \to \mathbb{R}\), as in \(\S 6.5\), so \(g = \sum_{a,b=1}^n g_{ab} \, dx^a \otimes dx^b\) with \(g_{ab} = g_{ba}\). Then \((g_{ab})_{a,b=1}^n\) is a symmetric, positive-definite, invertible matrix of functions on \(U\).

Write \((g^{ab})_{a,b=1}^n\) for the inverse matrix of functions. Then \(\nabla\) torsion-free is equivalent to
\[
\Gamma^a_{bc} = \Gamma^a_{cb},
\]
(7.1)

and \(\nabla'_c g_{ab} = 0\) is equivalent to
\[
\frac{\partial}{\partial x^c} g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = 0.
\]
(7.2)

Calculation shows that (7.1)–(7.2) have a unique solution for \(\Gamma^a_{bc}\):
\[
\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left( \frac{\partial}{\partial x^c} g_{db} + \frac{\partial}{\partial x^b} g_{dc} - \frac{\partial}{\partial x^d} g_{bc} \right).
\]
(7.3)

This gives the unique connection \(\nabla\) we want. \(\square\)
7.3. The Riemann curvature tensor

Let $(X, g)$ be a Riemannian manifold. Then by the FTRG we have a natural connection $\nabla$ on $TX$. As in §6.3, the curvature of $\nabla$ is $R \in C^\infty(TX \otimes T^*X \otimes \Lambda^2 T^*X)$, which is called the Riemann curvature tensor of $g$. In index notation $R = R^a_{bcd}$, and it is characterized by the formula for all vector fields $u, v, w \in C^\infty(TX)$

$$R^a_{bcd} u^b v^c w^d = v^c \nabla_c (w^d \nabla_d u^a) - w^c \nabla_c (v^d \nabla_d u^a) - (v^c \nabla_c w^d - w^c \nabla_c v^d) \nabla_d u^a$$

(7.4)

using $[v, w]^d = v^c \nabla_c w^d - w^c \nabla_c v^d$ as $\nabla$ is torsion-free. Thus

$$R^a_{bcd} u^b = (\nabla_c \nabla_d - \nabla_d \nabla_c) u^a. \tag{7.5}$$

Riemann curvature in coordinates

Let $(X, g)$ be a Riemannian manifold with Riemann curvature $R$, and $(x^1, \ldots, x^n)$ be coordinates on $U \subseteq X$. From (6.7) we have

$$R^a_{bcd} = \frac{\partial}{\partial x^\epsilon} \Gamma^a_{bd} - \frac{\partial}{\partial x^d} \Gamma^a_{bc} + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc}.$$

Substituting in (7.3) gives an expression for $R$ in coordinates. This is rather long, but we expand the first part:

$$R^a_{bcd} = \frac{1}{2} g^{ae} \left( \frac{\partial^2 g_{ed}}{\partial x^b \partial x^c} + \frac{\partial^2 g_{bc}}{\partial x^d \partial x^a} - \frac{\partial^2 g_{ec}}{\partial x^b \partial x^d} - \frac{\partial^2 g_{bd}}{\partial x^c \partial x^a} \right) + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc}.$$

As $\Gamma^a_{bc}$ involves $g_{ab}, g^{ab}$ and $\frac{\partial}{\partial x^c} g_{ab}$, we see that $R^a_{bcd}$ depends on $g_{ab}$, its first and second derivatives, and its inverse $g^{ab}$.
Flat and locally Euclidean metrics

A Riemannian metric \( g \) is called flat if it has Riemann curvature \( R^a_{bcd} = 0 \). In a similar way to Theorem 6.2, one can prove:

**Theorem**

Let \((X, g)\) be a flat Riemannian manifold. Then for each \( x \in X \), there exist coordinates \((x^1, \ldots, x^n)\) on an open neighbourhood \( U \) of \( x \) in \( X \) with \( g|_U = (dx^1)^2 + \cdots + (dx^n)^2 \).

That is, a flat Riemannian manifold \((X, g)\) is locally isometric to Euclidean space \((\mathbb{R}^n, g_{\mathbb{R}^n})\). Here an isometry of Riemannian manifolds \((X, g), (Y, h)\) is a diffeomorphism \( f : X \to Y \) with \( f^*(h) = g \). (‘Iso-metry’ from Greek ‘same distance’.)

Symmetries of Riemann curvature

It is often more convenient to work with \( R_{abcd} = g_{ae}R_{ebcd} \) (also called the Riemann curvature) rather than \( R^a_{bcd} \). Since \( R^a_{bcd} = g^{ae}R_{abce} \), the two are equivalent.

**Theorem**

Let \((X, g)\) be a Riemannian manifold, with Riemann curvature \( R_{abcd} \). Then \( R_{abcd} \) and \( \nabla_e R_{abcd} \) satisfy the equations

\[
R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab},
\]

\[
R_{abcd} + R_{adbc} + R_{acdb} = 0,
\]

and \( \nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0 \).

This can be proved using the coordinate expressions for \( R_{abcd}, \nabla_e R_{abcd} \). Here (7.7) and (7.8) are the first and second Bianchi identities, as in §6.5 with torsion \( T^a_{bc} = 0 \).
Ricci curvature and scalar curvature

Let \((X, g)\) be a Riemannian manifold. The Riemann curvature tensor \(R^a_{bcd}\) of \(g\) is a complicated object. Often it is helpful to work with components of \(R^a_{bcd}\) which are simpler.

**Definition**

The **Ricci curvature** of \(g\) is \(R_{ab} = R_{acb}^c = g^{cd} R_{cadb}\). By (7.6) it satisfies \(R_{ab} = R_{ba}\), so \(R_{ab} \in C^\infty(S^2 T^*X)\). The **scalar curvature** of \(g\) is \(s = g^{ab} R_{ab} = g^{ab} R_{acb}^c\), so that \(s : X \to \mathbb{R}\) is smooth.

Here \(R_{ab}\) is the trace of \(R^a_{bcd}\), and \(s\) the trace of \(R_{ab}\).

We say that \(g\) is **Einstein** if \(R_{ab} = \lambda g_{ab}\) for \(\lambda \in \mathbb{R}\), and **Ricci-flat** if \(R_{ab} = 0\). Einstein and Ricci-flat metrics are important for many reasons; they arise in Einstein’s General Relativity.

### 7.4. Volume forms and integrating functions

Let \((X, g)\) be a Riemannian manifold, of dimension \(n\). Then \(g\) induces norms \(|\cdot|_g\) on the bundles of \(k\)-forms \(\Lambda^k T^*X\), and in particular on \(\Lambda^n T^*X\).

If \(X\) is also oriented (§5.2) then \(\Lambda^n T^*X \setminus 0\) is divided into positive forms and negative forms, where positive forms are all proportional by positive constants.

Therefore there is a unique positive \(n\)-form \(dV_g \in C^\infty(\Lambda^n T^*X)\) with \(|dV_g|_g = 1\). We call \(dV_g\) the **volume form** of \(g\).

It can be characterized as follows: if \(x \in X\) and \((v_1, \ldots, v_n)\) is an oriented basis of \(T_x X\) which is orthonormal w.r.t. \(g\big|_x\), then \(dV_g \cdot (v_1 \wedge \cdots \wedge v_n) = 1\). In local coordinates \((x^1, \ldots, x^n)\) we have

\[
dV_g = \pm \left[ \det(g_{ab})_{a,b=1}^n \right]^{1/2} dx^1 \wedge \cdots \wedge dx^n,
\]

where the sign depends on whether \(\frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n}\) is an oriented or anti-oriented basis of \(T_x X\).
Let \((X, g)\) be an oriented Riemannian manifold (say compact, for simplicity), and \(f : X \to \mathbb{R}\) a smooth function. Then \(f \, dV_g\) is an \(n\)-form on \(X\), with \(X\) oriented, so as in \(\S 5.3\) we have the integral
\[
\int_X f \, dV_g.
\]

**Remark**

Changing the orientation of \(X\) changes the sign of both the operator \(\int_X\), and of the \(n\)-form \(dV_g\), so \(\int_X f \, dV_g\) is unchanged. Thus the orientation on \(X\) is not really important. We ignore the orientation issue from now on.

Thus, we can integrate functions on Riemannian manifolds.

We can use these ideas to define *Lebesgue spaces* and *Sobolev spaces*, Banach spaces of functions (or tensors, etc.) on Riemannian manifolds, which are important in many p.d.e. problems. Let \((X, g)\) be a Riemannian manifold, not necessarily compact. We say a smooth function \(f : X \to \mathbb{R}\) is *compactly-supported* if \(\text{supp} \, f = \{x \in X : f(x) \neq 0\}\) is contained in a compact subset of \(X\). Write \(C^\infty_{cs}(X)\) for the vector space of compactly supported functions \(f : X \to \mathbb{R}\).

For real \(p \geq 1\) and integer \(k \geq 0\), define the *Lebesgue norm* \(\| \cdot \|_{L^p}\) and *Sobolev norm* \(\| \cdot \|_{L^p_k}\) on \(C^\infty_{cs}(X)\) by
\[
\| f \|_{L^p} = \left( \int_X |f|^p dV_g \right)^{1/p}, \quad \| f \|_{L^p_k} = \left( \sum_{j=0}^k \int_X |\nabla^j f|^p dV_g \right)^{1/p}.
\]

Then define the Banach spaces \(L^p(X)\) and \(L^p_k(X)\) to be the completions of \(C^\infty_{cs}(X)\) w.r.t. the norms \(\| \cdot \|_{L^p}\) and \(\| \cdot \|_{L^p_k}\). Note that \(L^p(X) = L^p_0(X)\). \(L^2(X)\) and \(L^2_k(X)\) are Hilbert spaces.
We define Banach spaces of tensors such as $L^p_k(\bigotimes^l T X \otimes \bigotimes^m T^* X)$ by completion of $C^\infty_{cs}(\bigotimes^l T X \otimes \bigotimes^m T^* X)$ in the same way.

**Example**

Let $(X, g)$ be a compact, connected Riemannian manifold. The *Laplacian* $\Delta : C^\infty(X) \to C^\infty(X)$ may be defined by $\Delta f = -g^{ab}\nabla_a \nabla_b f$. Then $\Delta$ extends uniquely to a bounded linear operator on Banach spaces $\Delta : L^p_{k+2}(X) \to L^p_k(X)$.

It is known that if $p > 1$ and $k \geq 0$ then $\Delta : \{ f \in L^p_{k+2}(X) : \int_X f \, dV_g = 0 \} \to \{ h \in L^p_k(X) : \int_X h \, dV_g = 0 \}$ is an isomorphism of topological vector spaces. That is, if $h \in L^p_k(X)$ then the linear elliptic p.d.e. $\Delta f = h$ has a solution $f$ in $L^p_{k+2}(X)$ iff $\int_X h \, dV_g = 0$, and if $\int_X f \, dV_g = 0$ then the solution $f$ is unique.
Plan of talk:

8. More about Riemannian manifolds
   8.1. Examples: spheres and hyperbolic spaces
   8.2. Riemannian 2-manifolds and surfaces in $\mathbb{R}^3$
   8.3. Geodesics

Let $g, h$ be Riemannian metrics on a manifold $X$. We call $g, h$ conformally equivalent if $g = f \cdot h$ for smooth $f : X \to (0, \infty)$. Then $g, h$ define the same notion of angles between vectors in $T_xX$, since angles depend only on ratios between distances. We will show that the complement of a point in the sphere $(S^n_R, g_R)$ of radius $R$ in $\mathbb{R}^{n+1}$ is conformally equivalent to Euclidean space $(\mathbb{R}^n, h_{\text{Euc}})$. Define a bijection between points $(y_0, y_1, \ldots, y_n)$ in $S^n_R \setminus (R, 0, \ldots, 0)$ and $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $(R, 0, \ldots, 0), (y_0, y_1, \ldots, y_n)$ and $(0, x_1, \ldots, x_n)$ are collinear in $\mathbb{R}^{n+1}$. An easy calculation shows that

$$(y_0, y_1, \ldots, y_n) = \left( \frac{R^3}{R^2 + r^2}, \frac{R^2 x_1}{R^2 + r^2}, \ldots, \frac{R^2 x_n}{R^2 + r^2} \right),$$

where $r^2 = x_1^2 + \cdots + x_n^2$. 


Regarding \((x_1, \ldots, x_n)\) as coordinates on \(S^n_R \setminus (R,0,\ldots,0)\), this enables us to compute \(g_R\) in the coordinates \((x_1, \ldots, x_n)\): we have

\[
\begin{align*}
    g_R &= dy_0^2 + dy_1^2 + \cdots + dy_n^2 \\
    &= \left(\frac{R^3 \cdot 2r dr}{(R^2 + r^2)^2}\right)^2 + \sum_{i=1}^{n} \left(\frac{R^2 dx_i}{R^2 + r^2} - \frac{R^2 x_i \cdot 2r dr}{(R^2 + r^2)^2}\right)^2 \\
    &= \frac{R^4}{(R^2 + r^2)^2} (dx_1^2 + \cdots + dx_n^2) = \frac{R^4}{(R^2 + x_1^2 + \cdots + x_n^2)^2} \cdot h_{\text{Euc}}.
\end{align*}
\]

Equation (8.1) has an interesting feature: we can replace \(R\) by an imaginary number \(iR\), and get a new Euclidean metric on \(\mathbb{R}^n\):

\[
\frac{R^4}{(R^2 - x_1^2 - \cdots - x_n^2)^2} \cdot h_{\text{Euc}}
\]

which is defined except on the sphere of radius \(R\) in \(\mathbb{R}^n\). Taking \(R = 1\), define \(n\)-dimensional hyperbolic space \((\mathcal{H}^n, g_{\mathcal{H}^n})\) by

\[
\mathcal{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1\},
\]

\[
g_{\mathcal{H}^n} = \frac{1}{(1 - x_1^2 - \cdots - x_n^2)^2} \cdot (dx_1^2 + \cdots + dx_n^2).
\]

Morally this is a ‘sphere of radius \(\sqrt{-1}\)’. It has a large isometry group \(O_+(n,1)\), of dimension \(\frac{1}{2} n(n+1)\). Whereas spheres \(S^n_R\) are Einstein with positive scalar curvature, hyperbolic spaces are Einstein with negative scalar curvature. Hyperbolic spaces were historically important in the development of non-Euclidean geometry.
8.2. Riemannian 2-manifolds and surfaces in $\mathbb{R}^3$

For a Riemannian 2-manifold $(X, g)$, the Ricci curvature $R_{ab}$ and Riemann curvature $R^a_{bcd}$ are determined by the scalar curvature $s$ and $g$ by $R_{ab} = \frac{1}{2} sg_{ab}$ and $R^a_{bcd} = \frac{1}{2} s(\delta^a_c g_{bd} - \delta^a_d g_{bc})$. The scalar curvature $s$ is often called the Gaussian curvature, and written $\kappa$. Suppose $X$ is a 2-submanifold of $\mathbb{R}^3$, $(s, t)$ are coordinates on $X$, and the embedding $X \hookrightarrow \mathbb{R}^3$ is $r(s, t) = (x(s, t), y(s, t), z(s, t))$. Then the Riemann metric $g = g_{\mathbb{R}^3}|_X$ on $X$ (often called the first fundamental form) is

$$g = Eds^2 + F(dsdt + dtds) + Gdt^2,$$

with

$$E = \frac{|\partial r/\partial s|^2}{|\partial r/\partial s \times \partial r/\partial t|}, \quad F = \langle \partial r/\partial s, \partial r/\partial t \rangle, \quad G = |\partial r/\partial t|^2.$$

Define $n$ to be the unit normal vector to $X$ in $\mathbb{R}^3$, that is,

$$n = \frac{\partial r/\partial s \times \partial r/\partial t}{|\partial r/\partial s \times \partial r/\partial t|}.$$

Then the second fundamental form is

$$I = Lds^2 + M(dsdt + dtds) + Ndt^2,$$

with

$$L = n \cdot \frac{\partial^2 r}{\partial s^2}, \quad M = n \cdot \frac{\partial^2 r}{\partial s \partial t}, \quad N = n \cdot \frac{\partial^2 r}{\partial t^2}.$$

The principal curvatures $\kappa_1, \kappa_2$ are the solutions $\lambda$ of

$$\det \left[ \begin{array}{cc} \lambda & E \\ F & G \end{array} \right] = 0.$$

The Gaussian curvature (= scalar curvature) is

$$\kappa = \kappa_1 \kappa_2 = (LN - M^2)/(EG - F^2).$$

Although $L, M, N, \kappa_1, \kappa_2$ depend on the embedding of $X$ in $\mathbb{R}^3$, the Gaussian curvature $\kappa = \kappa_1 \kappa_2$ depends only on $(X, g)$. A sphere $S^2_R$ of radius $R$ in $\mathbb{R}^3$ has principal curvatures $\kappa_1 = \kappa_2 = R^{-1}$ everywhere, so $\kappa = R^{-2}$. 
The Gauss–Bonnet Theorem

Recall that if $X$ is a compact $n$-manifold it has finite-dimensional de Rham cohomology groups $H^k_{dR}(X, \mathbb{R})$ for $k = 0, \ldots, n$. The Betti numbers are $b^k(X) = \dim H^k_{dR}(X, \mathbb{R})$, and the Euler characteristic is $\chi(X) = \sum_{k=0}^{n} (-1)^k b^k(X)$. If $n = 2$ and $X$ is a surface of genus $g$ then $\chi(X) = 2 - 2g$.

**Theorem (Gauss–Bonnet)**

Let $(X, g)$ be a compact Riemannian 2-manifold, with Gauss curvature $\kappa$. Then

$$\int_X \kappa \, dV_g = 2\pi \chi(X).$$

This is an avatar of a lot of important geometry in higher dimensions – index theorems, characteristic classes. For a simpler analogy, let $\gamma : S^1 \to \mathbb{R}^2$ be an immersed curve, and $\kappa : S^1 \to \mathbb{R}$ be the curvature (rate of change of angle of tangent direction). Then $\int_{S^1} \kappa \, ds = 2\pi W(\gamma)$, where $\int \cdots \, ds$ is integration w.r.t arc-length, and $W(\gamma)$ is the winding number of $\gamma$.

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Minimal surfaces in $\mathbb{R}^3$

Let $X \hookrightarrow \mathbb{R}^3$ be an (oriented) embedded surface in $\mathbb{R}^3$. The mean curvature $H : X \to \mathbb{R}^3$ is $H = \frac{1}{2}(\kappa_1 + \kappa_2)$, the average of the principal curvatures of $X$. The mean curvature vector is $Hn$. (The sign of $H$ depends on the orientation of $X$, but $Hn$ is independent of orientation.) We call $X$ a minimal surface if $H = 0$. It turns out that $X$ is minimal if and only if $X$ is locally volume-minimizing in $\mathbb{R}^3$ (the equation $H = 0$ is the Euler–Lagrange equation for the volume functional on surfaces in $\mathbb{R}^3$).

Minimal surfaces are important in physical problems – if you dip a twisted loop of wire in the washing up and it is spanned by a bubble, this will be a minimal surface (to first approximation), as the surface tension in the bubble tries to minimize its area. Finding a minimal surface with given boundary is called Plateau’s problem. More generally, a bubble separating two regions in $\mathbb{R}^3$ with different air pressures should satisfy the p.d.e. $H = \text{constant}$ (e.g. a sphere).
Isothermal coordinates

Let \((X, g)\) be a Riemannian 2-manifold. Then near each point \(x \in X\) there exists a local coordinate system \((x_1, x_2)\) such that

\[
g = f(x_1, x_2) \cdot (dx_1^2 + dx_2^2),
\]

for \(f(x_1, x_2)\) a smooth positive function. That is, in 2 dimensions any Riemannian metric is locally conformally equivalent to the Euclidean plane \((\mathbb{R}^2, g_{\text{Euc}})\). Such coordinates \((x_1, x_2)\) are called isothermal coordinates. This is false in dimension \(> 2\).

If also \(X\) is oriented, and we take \((x_1, x_2)\) to be oriented coordinates, we can take \(x_1 + ix_2\) to be a complex local coordinate on \(X\). Such complex coordinates have holomorphic transition functions, and make \(X\) into a Riemann surface.

Basically, a conformal structure (Riemannian metric modulo conformal equivalence) on an oriented 2-manifold is equivalent to the data of how to rotate vectors 90° in each tangent space \(T_xX\) (i.e. multiply by \(i\) in \(\mathbb{C}\)), and this is equivalent to a complex structure.

8.3. Geodesics

Let \((X, g)\) be a Riemannian manifold. Consider a smooth immersed curve \(\gamma: [a, b] \to X\). The length of \(\gamma\) is

\[
l(\gamma) = \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} \, dt.
\]

To a first approximation, a geodesic is a locally length-minimizing curve \(\gamma\), that is, it satisfies the Euler–Lagrange equations for the length functional \(l\) on curves \(\gamma\). Actually, this turns out not to be well behaved. If \(F: [a', b'] \to [a, b]\) is any diffeomorphism then \(\gamma\) is locally length-minimizing iff \(\gamma \circ F\) is locally length-minimizing, as length is independent of parametrization. Thus, geodesics defined this way would come in infinite-dimensional families.
Instead, we define the energy of a curve $\gamma$ in $(X, g)$ by

$$E(\gamma) = \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t)) \, dt.$$  

We define a geodesic $\gamma : [a, b] \to X$ or $\gamma : \mathbb{R} \to X$ to satisfy the Euler–Lagrange equations for the energy functional $E$ on curves $\gamma$. Then $\gamma$ is a geodesic iff:

- $\gamma$ is locally length-minimizing, i.e. $\gamma$ satisfies the Euler–Lagrange equation for the length functional $l$; and
- $\gamma$ is parametrized with constant speed, that is, $g(\dot{\gamma}(t), \dot{\gamma}(t))$ is (locally) constant along $\gamma$.

**Example**

Take $(X, g)$ to be Euclidean $n$-space $(\mathbb{R}^n, h_{\text{Euc}})$. Then $\gamma = (\gamma_1, \ldots, \gamma_n) : [a, b] \to \mathbb{R}^n$ satisfies the geodesic equations iff

$$\frac{d^2 \gamma_i}{dt^2} = 0$$

for $i = 1, \ldots, n$. Hence geodesics are of the form $\gamma(t) = at + b$ for $a, b \in \mathbb{R}^n$. That is, they are straight lines in $\mathbb{R}^n$ traversed with constant speed.

**The geodesic equations in local coordinates**

Let $(x_1, \ldots, x_n)$ be local coordinates on $X$. Write $g = g^{ij}(x_1, \ldots, x_n)$, and let $g_{ij}$ be the inverse matrix of functions. Write a smooth map $\gamma : [a, b] \to X$ as $\gamma = (x_1(t), \ldots, x_n(t))$ in coordinates. Then $\gamma$ satisfies the geodesic equations iff we can extend $\gamma$ to a $2n$-tuple $(x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t))$ satisfying the o.d.e.s

$$\frac{dx_j}{dt} = \sum_{i=1}^n g_{ij}(x_1(t), \ldots, x_n(t)) \cdot y_i(t),$$

$$\frac{dy_k}{dt} = -\frac{1}{2} \sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial x_k}(x_1(t), \ldots, x_n(t)) \cdot y_i(t)y_j(t).$$

(8.2)

Here $D\gamma = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t))$ is naturally a curve in the cotangent bundle $T^*X$, and $\gamma = (x_1(t), \ldots, x_n(t))$ is its projection to $X$. We can think of $D\gamma$ as a flowline of a fixed vector field $v$ on $T^*X$ depending on $g$, called the geodesic flow.
From the geodesic equations (8.2) and standard results about o.d.e.s we see that for any $x \in X$ and any vector $v \in T_x X$, there exists a unique solution $\gamma : I \rightarrow X$ to the geodesic equations with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$, where $0 \in I \subseteq \mathbb{R}$ is an open interval, which we can take to be maximal.

A Riemannian manifold $(X, g)$ is called complete if we can take $I = \mathbb{R}$ for all such $x, v$. If $X$ is compact then any $g$ is complete, but many noncompact Riemannian manifolds such as $(\mathbb{R}^n, g_{\text{Euc}})$ and $(\mathcal{H}^n, h_{\mathcal{H}^n})$ are complete. Roughly, to be complete means that the boundary/edge of $(X, g)$ is at infinite distance from the interior of $X$.

**Example**

In the 2-sphere $S^2_R$ of radius $R$ in $\mathbb{R}^3$, the geodesics are the great circles, that is, intersections of $S^2_R$ with a plane $\mathbb{R}^2$ in $\mathbb{R}^3$ passing through the centre $(0, 0, 0)$ of $S^2_R$. So for example on the earth, the equator is a closed geodesic.

Note that geodesics need not globally be a shortest path: you can make the equator shorter by deforming it through lines of latitude. But geodesics have stationary length, and a geodesic $\gamma$ gives the shortest path between points $x, y$ on $\gamma$ if $x, y$ are sufficiently close.

**Example**

Take $(\mathcal{H}^2, g_{\mathcal{H}^2})$ to be the hyperbolic plane $$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$ with $$g_{\mathcal{H}^2} = (1 - x^2 - y^2)^{-1}(dx^2 + dy^2).$$ Then geodesics in $\mathcal{H}^2$ are the intersection of $\mathcal{H}^2$ with circles and straight lines in $\mathbb{R}^2$ which intersect the unit circle $x^2 + y^2 = 1$ at right angles.
Geodesic triangles in a Riemannian 2-manifold

Let \((X, g)\) be a Riemannian 2-manifold. Suppose we are given points \(A, B, C \in X\), and geodesic segments \(AB, BC, CA\) in \(X\) with endpoints \(A, B, C\), which enclose a triangle \(ABC\) homeomorphic to a disc \(D^2\). Let \(\alpha, \beta, \gamma\) be the internal angles of the triangle at \(A, B, C\) computed using \(g\). (That is, \(\alpha\) is the angle in \((T_A X, g|_A)\) between the tangent vectors to \(AB, AC\) at \(A\), etc.)

Then one can show that

\[
\alpha + \beta + \gamma - \pi = \int_{\Delta} \kappa \, dV_g,
\]

(8.3)

where \(\kappa : X \to \mathbb{R}\) is the Gaussian curvature of \(g\).

If \((X, g)\) is \((\mathbb{R}^2, g_{\text{Euc}})\) then \(\kappa = 0\) and (8.3) becomes

\[
\alpha + \beta + \gamma = \pi,
\]

that is, the angles in a triangle in \(\mathbb{R}^2\) add up to \(\pi\).

If \((X, g)\) is the unit sphere \(S^2\) then \(\kappa = 1\), so (8.3) becomes

\[
\alpha + \beta + \gamma = \pi + \text{area}(ABC).
\]

Thus, the angles in a triangle on \(S^2\) add up to more than \(\pi\).

If \((X, g)\) is the hyperbolic plane \((\mathcal{H}^2, g_{\mathcal{H}^2})\) then \(\kappa = -1\), so (8.3) becomes \(\alpha + \beta + \gamma = \pi - \text{area}(ABC)\). Thus, the angles in a triangle on \(S^2\) add up to less than \(\pi\). Also, all triangles have area less than \(\pi\), however long their sides.

We can use (8.3) to prove the Gauss–Bonnet Theorem. Suppose \((X, g)\) is a compact Riemannian 2-manifold. Choose a division of \(X\) into small triangles \(\Delta_1, \ldots, \Delta_N\) with geodesic sides, and sum (8.3) over \(1, \ldots, N\). We get

\[
2\pi(\text{#vertices}) - \pi(\text{#triangles}) = \int_X \kappa \, dV_g.
\]

Since \(2\text{#edges} = 3\text{#triangles}\) we have

\[
\text{#triangles} = 2(\text{edges} - \text{#triangles}).
\]

Then using \(\chi(X) = \text{#vertices} - \text{#edges} + \text{#triangles}\) proves G–B.