Problem Sheet -1

1(a) Let X be the sphere $\mathcal{S}^n = \bigl\{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\bigr\}$. Explain why we may identify

$$T_{(x_0,\ldots,x_n)}\mathcal{S}^n \cong \bigl\{(y_0,\ldots,y_n) \in \mathbb{R}^{n+1} : x_0 y_0 + \cdots + x_n y_n = 0\bigr\}.$$

(b) By identifying $\mathbb{R}^{2k+2} \cong \mathbb{C}^{k+1}$, show that any odd-dimensional sphere $\mathcal{S}^{2k+1}$ has a nonvanishing vector field $v \in C^\infty(T\mathcal{S}^{2k+1})$ (i.e. $v \neq 0$ at every point).

For discussion: can the same thing hold for even-dimensional spheres $\mathcal{S}^{2k}$?

2(a) Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x,y,z) = (xy, yz, zx)$. Calculate $F^*(x\,\mathrm{d}y \wedge \mathrm{d}z)$ and $F^*(x\,\mathrm{d}y + y\,\mathrm{d}z)$.

(b) Let X be the circle $\mathbb{R}/\mathbb{Z}$. Then we may write

$$\begin{aligned} C^\infty(\Lambda^0 T^*X) &= \bigl\{f : \ f : \mathbb{R} \to \mathbb{R}\ C^\infty,\ f(x) = f(x+1),\ x \in \mathbb{R}\bigr\}, \\ C^\infty(\Lambda^1 T^*X) &= \bigl\{g\,\mathrm{d}x : \ g : \mathbb{R} \to \mathbb{R}\ C^\infty,\ g(x) = g(x+1),\ x \in \mathbb{R}\bigr\}, \end{aligned}$$

and $\mathrm{d} : C^\infty(\Lambda^0 T^*X) \to C^\infty(\Lambda^1 T^*X)$ maps $f \mapsto \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x$. Use these to compute the kernel and cokernel of d, and so find the de Rham cohomology $H^k_{\mathrm{dR}}(X,\mathbb{R})$ for $k = 0, 1$.

3. Prove that the product $X \times Y$ of two oriented manifolds X, Y is orientable.

PTO

4. The Lie group SU(2) is the group of 2×2 complex matrices A satisfying $A\bar{A}^t = \mathrm{id}$ and $\det_{\mathbb{C}} A = 1$. Its Lie algebra $\mathfrak{su}(2)$ is the real vector space of 2×2 complex matrices B with $B + \bar{B}^t = 0$ and $\mathrm{Trace}\, B = 0$.

The Lie group SO(3) is the group of 3×3 real matrices C satisfying $CC^t = \mathrm{id}$ and $\det C = 1$. Its Lie algebra $\mathfrak{so}(3)$ is the real vector space of 3×3 real matrices D with $D + D^t = 0$.

(a) Show that we may write

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

Deduce that SU(2) is diffeomorphic to the 3-sphere $\mathcal{S}^3$.

(b) Define a basis e_1, e_2, e_3 of $\mathfrak{su}(2)$ over $\mathbb{R}$ by

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Write down the Lie bracket $[\,,\,]$ on $\mathfrak{su}(2)$ by computing $[e_i, e_j]$.

(c) Define a basis f_1, f_2, f_3 of $\mathfrak{so}(3)$ over $\mathbb{R}$ by

$$f_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Write down the Lie bracket $[\,,\,]$ on $\mathfrak{so}(3)$ by computing $[f_i, f_j]$. Deduce that $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic Lie algebras.

(d) By considering the action of SU(2) on $\mathfrak{su}(2) \cong \mathbb{R}^3$ by conjugation (you may assume this preserves the inner product for which e_1, e_2, e_3 are an orthonormal basis), define a Lie group morphism $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$. Is this an isomorphism?