Problem Sheet 3

1. Take $\alpha \in \Lambda^k V$ where $\dim V = n$ and consider the linear map $A_\alpha : \Lambda^{n-k} V \to \Lambda^n V$ defined by $A_\alpha(\beta) = \alpha \wedge \beta$.

(i) Show that if $\alpha \neq 0$, then $A_\alpha \neq 0$.

(ii) Prove that the map $\alpha \mapsto A_\alpha$ is an isomorphism from $\Lambda^k V$ to the vector space $\text{Hom}(\Lambda^{n-k} V, \Lambda^n V)$ of linear maps from $\Lambda^{n-k} V$ to $\Lambda^n V$. Thus if we choose an isomorphism $\Lambda^n V \cong \mathbb{R}$ we get isomorphisms $\Lambda^k V \cong (\Lambda^{n-k} V)^*$.

2. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (xy, yz, zx)$. Calculate $F^*(x \, dy \wedge dz)$ and $F^*(x \, dy + y \, dz)$.

3. Show that the product $X \times Y$ of two orientable manifolds is orientable.

4. Is $S^2 \times \mathbb{RP}^2$ orientable? What about $\mathbb{RP}^2 \times \mathbb{RP}^2$?

5. A Riemann surface is defined as a 2-dimensional manifold $X$ with an atlas $\{(U_i, \phi_i) : i \in I\}$ whose transition maps $\phi_j^{-1} \circ \phi_i$ for $i, j \in I$ are maps from an open set $\phi_i^{-1}(\phi_j(U_j))$ of $\mathbb{C} = \mathbb{R}^2$ to another open set $\phi_j^{-1}(\phi_i(U_i))$ which are holomorphic and invertible. By considering the Jacobian of $\phi_j^{-1} \circ \phi_i$, show that a Riemann surface is orientable.

P.T.O.
The objective of this question is to prove that for all \( n > 0 \)
\[
H^k(S^n) \cong \begin{cases} \mathbb{R}^2, & k = 0, \\ \mathbb{R}, & k = 0, n, \\ 0, & \text{otherwise}, \end{cases}
\]
You may assume the following facts from lectures:

- \( H^0(X) \cong \mathbb{R}^N \), where \( N \) is the number of connected components of \( X \).
- \( H^0(\mathbb{R}^n) = \mathbb{R} \) and \( H^k(\mathbb{R}^n) = 0 \), \( k > 0 \).
- \( H^k(X \times \mathbb{R}^n) \cong H^k(X) \) for all manifolds \( X \) and \( k, n \geq 0 \).

Define \( U = S^n \setminus \{(1,0,\ldots,0)\} \), \( V = S^n \setminus \{(-1,0,\ldots,0)\} \) and \( W = U \cap V \). Then \( U, V, W \) are open in \( S^n \) with \( S^n = U \cup V \), and we have diffeomorphisms
\[
U \cong \mathbb{R}^n, \quad V \cong \mathbb{R}^n, \quad W \cong S^{n-1} \times \mathbb{R}.
\]
If \( B \subseteq A \subseteq S^n \) are open, write \( \rho_{AB} : \Omega^k(A) \to \Omega^k(B) \) for the restriction map. Then we have an exact sequence
\[
0 \to \Omega^k(S^n) \xrightarrow{\rho_{S^n U} \oplus \rho_{S^n V}} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\rho_{U W} \oplus -\rho_{V W}} \Omega^k(W) \to 0.
\]

(a) Suppose \( \alpha \in \Omega^k(S^n) \) for \( k > 1 \) with \( d\alpha = 0 \). Show that there exist \( \beta \in \Omega^{k-1}(U) \) and \( \gamma \in \Omega^{k-1}(V) \) with \( \alpha|_U = d\beta \) and \( \alpha|_V = d\gamma \). Set \( \delta = \beta|_W - \gamma|_W \in \Omega^{k-1}(W) \). Show that \( d\delta = 0 \).

We have cohomology classes \([\alpha] \in H^k(S^n)\) and \([\delta] \in H^{k-1}(W)\). Show that \([\delta]\) depends only on \([\alpha]\), not on the choices of \( \alpha, \beta, \gamma \). Thus we may define a linear map \( \Phi : H^k(S^n) \to H^{k-1}(W) \), \( \Phi : [\alpha] \mapsto [\delta] \).

(b) Suppose \([\delta] = \Phi([\alpha]) = 0\). Then \( \delta = d\epsilon \) for \( \epsilon \in \Omega^{k-2}(W) \). Prove that \([\alpha] = 0\) in \( H^k(S^n) \), so that \( \Phi \) is injective.

**Hint:** Let \( \{\eta_U, \eta_V\} \) be a partition of unity on \( S^n \) subordinate to \( \{U, V\} \), and consider \( \beta|_W - d(\eta_U \epsilon) \) and \( \gamma|_W + d(\eta_V \epsilon) \) in \( \Omega^{k-1}(W) \).

(c) Suppose \( \delta \in \Omega^{k-1}(W) \) with \( d\delta = 0 \). Show that we can choose \( \alpha, \beta, \gamma \) in \( \Omega^k \) giving this \( \delta \). Then \( \Phi([\alpha]) = [\delta] \), so that \( \Phi \) is surjective.

**Hint:** Choose \( \alpha, \beta, \gamma \) such that \( \alpha|_W = d\eta_U \wedge \delta = -d\eta_V \wedge \delta \).

(d) Use (a)–(c) and the facts above to show \( H^k(S^n) \cong H^{k-1}(S^{n-1}) \) if \( k > 1 \).

(e) What goes wrong in part (a) if \( k = 1 \)? Adapt your arguments to show that \( H^1(S^1) \cong \mathbb{R} \), and \( H^1(S^n) = 0 \) for \( n > 1 \).

(f) Deduce (1) by induction on \( n \).