Questions on Lie Groups. Sheet 1

A1. Let $M$ be a manifold and $u, v, w$ be vector fields on $M$. The Jacobi identity is

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0.$$ 

Prove the Jacobi identity in coordinates $(x^1, \ldots, x^n)$ on a coordinate patch $U$. Use the coordinate expression for the Lie bracket of vector fields.

In the following, $M_n(\mathbb{R})$ is the algebra of $n \times n$ real matrices, $e_{ij}$ is the matrix in $M_n(\mathbb{R})$ which is 1 in position $(i, j)$ and 0 elsewhere, and $I$ is the identity matrix. If $A \in M_n(\mathbb{R})$ then $A^t$ is the transpose of $A$, and $GL(n, \mathbb{R}) \subset M_n(\mathbb{R})$ is the Lie group of invertible matrices in $M_n(\mathbb{R})$.

A2. Define $O(n)$ to be the subset of matrices $A$ in $GL(n, \mathbb{R})$ such that $AA^t = I$.

(i)* Show that $O(n)$ is a compact Lie subgroup of $GL(n, \mathbb{R})$.

(ii) Show that the Lie algebra $\mathfrak{o}(n)$ is $\{ A \in M_n(\mathbb{R}) : A + A^t = 0 \}$.

(iii) For $1 \leq i < j \leq n$, define $f_{ij} = e_{ij} - e_{ji}$. Find an expression for the Lie bracket $[f_{ij}, f_{kl}]$.

(iv) Show that the $f_{ij}$ form a basis for $\mathfrak{o}(n)$, and hence find the dimension of $O(n)$.

A3* (a) Let $G, H$ be Lie groups, and let $\Phi : G \to H$ be a Lie group homomorphism. Prove carefully that $\text{Ker} \, \Phi$ is a Lie subgroup of $G$, and that $\text{Im} \, \Phi$ is a Lie subgroup of $H$ if and only if it is closed.

(b) Let $G$ be the Lie group $\mathbb{R}$, and let $H$ be the Lie group $\mathbb{R}^2 / \mathbb{Z}^2$, that is, the torus $T^2$. Define a Lie group homomorphism $\Phi : G \to H$ by $\Phi(t) = (t + \mathbb{Z}, \sqrt{2}t + \mathbb{Z})$. Show that $\Phi(G)$ is a subgroup of $H$, but it is not closed, and thus is not a submanifold of $H$.

A4* (a) Prove that the vector subspace $\langle f_{12} + f_{34}, f_{13} + f_{42}, f_{14} + f_{23} \rangle$ of $\mathfrak{o}(4)$ is a Lie subalgebra of $\mathfrak{o}(4)$ isomorphic to $\mathfrak{o}(3)$.

(b) Hence show that $\mathfrak{o}(4)$ is isomorphic as a Lie algebra to $\mathfrak{o}(3) \oplus \mathfrak{o}(3)$.

(c) Let $SO(n)$ be the connected component of $O(n)$ containing the identity. For $n > 2$, define $\text{Spin}(n)$ to be the universal cover of $SO(n)$. Deduce that $\text{Spin}(4) \cong \text{Spin}(3) \times \text{Spin}(3)$ as Lie groups.
Note: Here in part (b), if \( g \) and \( h \) are Lie algebras, then the Lie algebra \( g \oplus h \) is the vector space \( g \oplus h \) with the Lie bracket \([x_1, y_1], (x_2, y_2)\] = \([x_1, x_2], [y_1, y_2] \), for \( x_1, x_2 \in g \) and \( y_1, y_2 \in h \).

Questions for practice

B1. By definition \( GL(n, \mathbb{C}) \) is the group of invertible \( n \times n \) matrices over \( \mathbb{C} \). It is a complex Lie group of complex dimension \( n^2 \), but we may also regard it as a real Lie group of real dimension \( 2n^2 \). Define \( U(n) \) by \( U(n) = \{ A \in GL(n, \mathbb{C}) : A\overline{A}^t = I \} \), where \( \overline{A}^t \) is the transpose of the complex conjugate of \( A \).

(i)* Show that \( U(n) \) is a compact, real Lie group.

(ii) Show that the Lie algebra \( u(n) \) of \( U(n) \) is \( u(n) = \{ A \in M_n(\mathbb{C}) : A + \overline{A}^t = 0 \} \), and calculate its dimension.

(iii) Show that \( u(2) \) is isomorphic to \( \mathbb{R} \oplus o(3) \) as a Lie algebra.