**Dominic Joyce** **Differential Geometry** **Nairobi 2019**

## Questions on Lie Groups. Sheet 2

**A1.** Let $\mathfrak{g}$ be a Lie algebra, and let $\langle\, ,\, \rangle$ be the Killing form of $\mathfrak{g}$. Prove that for $x, y, z \in \mathfrak{g}$,

$$\big\langle [x,y], z \big\rangle = \big\langle [y,z], x \big\rangle = \big\langle [z,x], y \big\rangle.$$

**A2.** Consider the following Lie algebras $\mathfrak{g}$. They are written in terms of a basis $x_1, \ldots, x_n$ for $\mathfrak{g}$. To define the Lie bracket on $\mathfrak{g}$, it is enough to define $[x_i, x_j]$ for $1 \le i < j \le n$.

(*a*) $\mathfrak{g} = \langle x_1, x_2 \rangle$, with $[x_1, x_2] = x_1$.
(*b*) $\mathfrak{g} = \langle x_1, x_2, x_3 \rangle$, with $[x_1, x_2] = x_3$, $[x_1, x_3] = 0$, $[x_2, x_3] = 0$.
(*c*) $\mathfrak{g} = \langle x_1, x_2, x_3 \rangle$, with $[x_1, x_2] = x_3$, $[x_1, x_3] = -x_2$, $[x_2, x_3] = -x_1$.

**(i)** In each case, write down the adjoint map 'ad' in matrix form. Hence find the Killing form $\langle\, ,\, \rangle$ of $\mathfrak{g}$ by direct calculation.

**(ii)** Which of the Lie algebras (*a*)-(*c*) are solvable?

**(iii)** Which of the Lie algebras (*a*)-(*c*) are semisimple?

**A3.** Let $\{f_{ij} : 1 \le i < j \le n\}$ be the basis of the Lie algebra $\mathfrak{o}(n)$ of $O(n)$ defined on the first question sheet.

**(i)** Show that if $A = (a_{ij})_{i,j=1}^n$ is a matrix in $\mathfrak{o}(n)$, then $[A, f_{ij}] = 0$ if and only if $a_{ik} = a_{ki} = 0$ for all $k$ with $j \ne k$, and $a_{jk} = a_{kj} = 0$ for all $k$ with $i \ne k$.

**(ii)** Deduce that if $n > 2$ and $[A, B] = 0$ for all $B \in \mathfrak{o}(n)$ then $A = 0$, so that the centre $Z\big(\mathfrak{o}(n)\big)$ is zero.

**(iii)** Prove that if $n > 2$, then $\mathfrak{o}(n)$ is a semisimple Lie algebra with negative definite Killing form. You may suppose that $O(n)$ is compact.

**A4.** Let $\mathfrak{g}$ be a semisimple Lie algebra. Define $\mathfrak{h} = \mathrm{span}\big([x,y] : x, y \in \mathfrak{g}\big)$, written $\mathfrak{h} = \mathrm{span}\big([\mathfrak{g},\mathfrak{g}]\big)$ for short. Clearly, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Define $\mathfrak{m}$ by $\mathfrak{m} = \big\{x \in \mathfrak{g} : \langle x, y \rangle = 0$ for all $y \in \mathfrak{h}\big\}$. Prove that $\mathfrak{m}$ is an abelian ideal in $\mathfrak{g}$. Deduce that $\mathfrak{m} = \{0\}$ and that $\mathfrak{h} = \mathfrak{g}$. This proves that if $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \mathrm{span}\big([\mathfrak{g},\mathfrak{g}]\big)$.

## Questions for practice

**B1*.** Show that any compact, connected, abelian Lie group is isomorphic to $T^n$.

**B2*.** Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$.

- **(a)** Show that $G$ is generated by $\exp_{\mathfrak{g}}(\mathfrak{g})$.
- **(b)** Prove that Ker(Ad) is equal to the centre $Z(G)$. (It may help to use **(a)**).
- **(c)** Prove that $Z(G)$ is a (closed) Lie subgroup of $G$.

**B3*.** Now suppose that $G$ is compact and connected. Write $\mathbb{R}^n$ for the centre of $\mathfrak{g}$ and $\mathfrak{h}$ for $[\mathfrak{g}, \mathfrak{g}]$. From lectures, we have $\mathfrak{g} \cong \mathbb{R}^n \oplus \mathfrak{h}$ as a Lie algebra. Also, $\mathfrak{h}$ is semisimple.

- **(a)** Show that the connected component of the identity in $Z(G)$ is isomorphic to $T^n$. (You may assume questions B1 and B2).
- **(b)** Let $H$ be the unique connected, simply-connected Lie group with Lie algebra $\mathfrak{h}$. Show that $\mathbb{R}^n \times H$ is the universal cover of $G$.
- **(c)** It is a fact that $H$ is compact. Using this, show that every compact connected Lie group $G$ has a finite cover isomorphic to $T^n \times H$, where $H$ is a connected, simply-connected, compact semisimple Lie group.

**B4*.** Let $G$ be a *complex* Lie group of complex dimension $n$, with Lie algebra $\mathfrak{g}$. We may also regard $G$ as a *real* Lie group, of real dimension $2n$. Let $\langle\, , \rangle_{\mathbb{C}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the Killing form of $\mathfrak{g}$ viewed as a complex Lie algebra, and let $\langle\, , \rangle_{\mathbb{R}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the Killing form of $\mathfrak{g}$ viewed as a real Lie algebra.

- **(i)** Show that for $x, y \in \mathfrak{g}$, we have $\langle x, y\rangle_{\mathbb{R}} = 2\,\mathrm{Re}\big(\langle x, y\rangle_{\mathbb{C}}\big)$.
- **(ii)** Deduce that $\langle x, x\rangle_{\mathbb{R}} = -\langle ix, ix\rangle_{\mathbb{R}}$ for $x \in \mathfrak{g}$.
- **(iii)** Now suppose $G$ is *compact*. Prove that $\langle\, , \rangle_{\mathbb{R}} = 0$, and hence that $\mathfrak{g}$ is abelian.

Combined with question B1, this question shows that every compact, connected complex Lie group is isomorphic to $\mathbb{C}^n/\mathbb{Z}^{2n}$, for some lattice $\mathbb{Z}^{2n} \subset \mathbb{C}^n$.