The complex classical Lie algebras and groups

Notation:

- $\mathfrak{gl}(n,\mathbb{C})$ is the Lie algebra of $n\times n$ complex matrices.
- For $i,j=1,\ldots,n$, $e_{ij}\in\mathfrak{gl}(n,\mathbb{C})$ is the matrix which is 1 in the $(i,j)^{\text{th}}$ position, and 0 elsewhere.
- $GL(n,\mathbb{C})$ is the group of invertible matrices in $\mathfrak{gl}(n,\mathbb{C})$.
- If $A\in\mathfrak{gl}(n,\mathbb{C})$, A^t is the transpose of A, and $\mathrm{Tr}(A)$ the trace of A. Write I for the identity matrix.

1. The special linear group

Define $\mathfrak{sl}(n,\mathbb{C})=\{x\in\mathfrak{gl}(n,\mathbb{C}):\mathrm{Tr}(x)=0\}$ and $SL(n,\mathbb{C})=\{A\in GL(n,\mathbb{C}):\det(A)=1\}$.
Then $SL(n,\mathbb{C})$ is a connected, simply-connected, simple Lie group, of dimension n^2-1 and rank $n-1$. A Cartan subalgebra for $\mathfrak{sl}(n,\mathbb{C})$ is the subspace of diagonal matrices.

2. The orthogonal and special orthogonal groups

Define $\mathfrak{o}(n,\mathbb{C})=\mathfrak{so}(n,\mathbb{C})=\{x\in\mathfrak{gl}(n,\mathbb{C}):x+x^t=0\}$,
$O(n,\mathbb{C})=\{A\in GL(n,\mathbb{C}):AA^t=I\}$, and $SO(n,\mathbb{C})=\{A\in O(n,\mathbb{C}):\det(A)=1\}$.
The orthogonal group $O(n,\mathbb{C})$ has 2 connected components, and the connected component of the identity is the special orthogonal group $SO(n,\mathbb{C})$. For $n>2$, $SO(n,\mathbb{C})$ is a connected, semisimple Lie group with fundamental group $\mathbb{Z}_2$, which is simple when $n\neq 4$. The double cover of $SO(n,\mathbb{C})$ is $Spin(n,\mathbb{C})$, and it is a connected, simply-connected, semisimple Lie group. The dimension of $SO(n,\mathbb{C})$ is $n(n-1)/2$, and the rank is $n/2$ for n even, and $(n-1)/2$ for n odd.

3. The symplectic groups

Define a matrix $L_n\in\mathfrak{gl}(2n,\mathbb{C})$ by $L_n=\sum_{j=1}^n(e_{j(j+n)}-e_{(j+n)j})$. Define $\mathfrak{sp}(n,\mathbb{C})=\{x\in\mathfrak{gl}(2n,\mathbb{C}):xL_n+L_nx^t=0\}$ and $Sp(n,\mathbb{C})=\{A\in GL(n,\mathbb{C}):ALA^t=L\}$.
The symplectic group $Sp(n,\mathbb{C})$ is a connected, simply-connected, simple Lie group of dimension $n(2n+1)$ and rank n, with $Sp(n,\mathbb{C})\subset SL(2n,\mathbb{C})$.
Warning: Some authors write $Sp(2n,\mathbb{C})$ or $Sp_{2n}(\mathbb{C})$ instead of $Sp(n,\mathbb{C})$.

3. Dynkin diagrams of classical groups

The classical Lie algebras are related to the Lie algebras A_l,B_l,C_l and D_l by:

$$A_l=\mathfrak{sl}(l+1,\mathbb{C}),\qquad B_l=\mathfrak{so}(2l+1,\mathbb{C}),\qquad C_l=\mathfrak{sp}(l,\mathbb{C}),\qquad\text{and}\qquad D_l=\mathfrak{so}(2l,\mathbb{C}).$$

Note the low-dimensional coincidences

$$A_1=B_1=C_1,\qquad D_2=A_1\oplus A_1,\qquad B_2=C_2\qquad\text{and}\qquad A_3=D_3,$$

which give the isomorphisms $\mathfrak{sl}(2,\mathbb{C})\cong\mathfrak{so}(3,\mathbb{C})\cong\mathfrak{sp}(1,\mathbb{C})$,

$$\mathfrak{so}(4,\mathbb{C})\cong\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}),\qquad\mathfrak{so}(5,\mathbb{C})\cong\mathfrak{sp}(2,\mathbb{C}),\quad\text{and}\quad\mathfrak{sl}(4,\mathbb{C})\cong\mathfrak{so}(6,\mathbb{C}).$$

Real forms of the complex classical Lie groups

We use the notation overleaf, and in addition, $\mathbb{H}$ will represent the *quaternions*, a noncommutative division algebra with $\mathbb{H} \cong \mathbb{R}^4$ as a vector space.

1. The linear groups over $\mathbb{R}$ and $\mathbb{H}$

Define $GL(n,\mathbb{R})$ and $GL(n,\mathbb{H})$ to be the Lie groups of $n \times n$ invertible matrices over $\mathbb{R}$ and $\mathbb{H}$ respectively. Define $SL(n,\mathbb{R})$ to be the subgroup of $GL(n,\mathbb{R})$ of matrices with determinant 1. Then $SL(n,\mathbb{R})$ is a real, simple Lie group with rank $n-1$ and dimension n^2-1, a real form of $SL(n,\mathbb{C})$.

Now $GL(n,\mathbb{H})$ has dimension $4n^2$ and rank $2n$, and it can be regarded as a subgroup of $GL(4n,\mathbb{R})$. Define $SL(n,\mathbb{H})$ to be $GL(n,\mathbb{H}) \cap SL(4n,\mathbb{R})$. Then $SL(n,\mathbb{H})$ is a real, simple Lie group of rank $2n-1$ and dimension $4n^2-1$, a real form of $SL(2n,\mathbb{C})$.

2. The unitary groups

Let $(z_1,\ldots,z_n)$ be coordinates on $\mathbb{C}^n$. Define $U(n)$ to be the subgroup of $GL(n,\mathbb{C})$ preserving the hermitian metric $|dz_1|^2+\cdots+|dz_n|^2$ on $\mathbb{C}^n$. Then $U(n)$ is a compact, connected Lie group of rank n and dimension n^2, which is not semisimple. It is a real form of $GL(n,\mathbb{C})$.

Define $SU(n)$ to be the subgroup of $SL(n,\mathbb{C})$ preserving the same hermitian metric. Then $SU(n)$ is a compact, connected, simply-connected, simple Lie group of rank $n-1$ and dimension n^2-1, a real form of $SL(n,\mathbb{C})$.

3. The orthogonal groups

Let $(x_1,\ldots,x_n)$ be coordinates on $\mathbb{R}^n$. Define $O(n)$ and $SO(n)$ to be the subgroups of $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$ preserving the metric $dx_1^2+\cdots+dx_n^2$ on $\mathbb{R}^n$. These groups are compact, with dimension $n(n-1)/2$ and rank $n/2$ (n even) or $(n-1)/2$ (n odd). Note $O(n)$ has 2 connected components, and $SO(n)$ is the identity component of $O(n)$. For $n>2$, $SO(n)$ has fundamental group $\mathbb{Z}_2$, and $Spin(n)$ is the simply-connected double cover. For $n=3$ and $n>4$ $SO(n)$ and $Spin(n)$ are simple, and $Spin(4) \cong Spin(3) \times Spin(3)$. The groups $O(n), SO(n)$ and $Spin(n)$ are real forms of $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$ and $Spin(n,\mathbb{C})$ respectively.

4. The symplectic groups

Define $Sp(n)$ to be the intersection of $GL(n,\mathbb{H})$ and $SO(4n)$. Then $Sp(n)$ acts on $\mathbb{R}^{4n}$ preserving a metric and an action of $\mathbb{H}$. It has dimension $n(2n+1)$ and rank n, and is a compact, connected, simply-connected, simple Lie group, a real form of $Sp(n,\mathbb{C})$.

5. Groups with mixed signature

For $p,q>1$, let $U(p,q)$, $SU(p,q)$ be the subgroups of $GL(p+q,\mathbb{C})$, $SL(p+q,\mathbb{C})$ respectively preserving the pseudo-hermitian metric $|dz_1|^2+\cdots+|dz_p|^2-|dz_{p+1}|^2+\cdots+|dz_{p+q}|^2$. These are noncompact real forms of $GL(n,\mathbb{C})$ and $SL(n,\mathbb{C})$. In the same way, we may define groups $O(p,q)$, $SO(p,q)$ and $Sp(p,q)$. Note $O(p,q)$ has 4 connected components and $SO(p,q)$ has 2.