The complex classical Lie algebras and groups

Notation:

- \( \mathfrak{gl}(n, \mathbb{C}) \) is the Lie algebra of \( n \times n \) complex matrices.
- For \( i, j = 1, \ldots, n \), \( e_{ij} \in \mathfrak{gl}(n, \mathbb{C}) \) is the matrix which is 1 in the \((i, j)^{th}\) position, and 0 elsewhere.
- \( \text{GL}(n, \mathbb{C}) \) is the group of invertible matrices in \( \mathfrak{gl}(n, \mathbb{C}) \).
- If \( A \in \mathfrak{gl}(n, \mathbb{C}) \), \( A^t \) is the transpose of \( A \), and \( \text{Tr}(A) \) the trace of \( A \). Write \( I \) for the identity matrix.

1. The special linear group
Define \( \mathfrak{sl}(n, \mathbb{C}) = \{ x \in \mathfrak{gl}(n, \mathbb{C}) : \text{Tr}(x) = 0 \} \) and \( \text{SL}(n, \mathbb{C}) = \{ A \in \text{GL}(n, \mathbb{C}) : \det(A) = 1 \} \).
Then \( \text{SL}(n, \mathbb{C}) \) is a connected, simply-connected, simple Lie group, of dimension \( n^2 - 1 \) and rank \( n - 1 \). A Cartan subalgebra for \( \mathfrak{sl}(n, \mathbb{C}) \) is the subspace of diagonal matrices.

2. The orthogonal and special orthogonal groups
Define \( \mathfrak{o}(n, \mathbb{C}) = \mathfrak{so}(n, \mathbb{C}) = \{ x \in \mathfrak{gl}(n, \mathbb{C}) : x + x^t = 0 \} \), \( O(n, \mathbb{C}) = \{ A \in \text{GL}(n, \mathbb{C}) : AA^t = I \} \), and \( \text{SO}(n, \mathbb{C}) = \{ A \in \text{O}(n, \mathbb{C}) : \det(A) = 1 \} \).
The orthogonal group \( O(n, \mathbb{C}) \) has 2 connected components, and the connected component of the identity is the special orthogonal group \( \text{SO}(n, \mathbb{C}) \). For \( n > 2 \), \( \text{SO}(n, \mathbb{C}) \) is a connected, semisimple Lie group with fundamental group \( \mathbb{Z}_2 \), which is simple when \( n \neq 4 \). The double cover of \( \text{SO}(n, \mathbb{C}) \) is \( \text{Spin}(n, \mathbb{C}) \), and it is a connected, simply-connected, semisimple Lie group. The dimension of \( \text{SO}(n, \mathbb{C}) \) is \( n(n-1)/2 \), and the rank is \( n/2 \) for \( n \) even, and \((n-1)/2\) for \( n \) odd.

3. The symplectic groups
Define a matrix \( L_n \in \mathfrak{gl}(2n, \mathbb{C}) \) by \( L_n = \sum_{j=1}^n (e_{j(j+n)} - e_{(j+n)j}) \). Define \( \mathfrak{sp}(n, \mathbb{C}) = \{ x \in \mathfrak{gl}(2n, \mathbb{C}) : xL_n + L_n x^t = 0 \} \) and \( \text{Sp}(n, \mathbb{C}) = \{ A \in \text{GL}(n, \mathbb{C}) : ALA^t = L \} \).
The symplectic group \( \text{Sp}(n, \mathbb{C}) \) is a connected, simply-connected, simple Lie group of dimension \( n(2n+1) \) and rank \( n \), with \( \text{Sp}(n, \mathbb{C}) \subset \text{SL}(2n, \mathbb{C}) \).

Warning: Some authors write \( \text{Sp}(2n, \mathbb{C}) \) or \( \text{Sp}_{2n}(\mathbb{C}) \) instead of \( \text{Sp}(n, \mathbb{C}) \).

3. Dynkin diagrams of classical groups
The classical Lie algebras are related to the Lie algebras \( A_l, B_l, C_l \) and \( D_l \) by:
\[
A_l = \mathfrak{sl}(l+1, \mathbb{C}), \quad B_l = \mathfrak{so}(2l+1, \mathbb{C}), \quad C_l = \mathfrak{sp}(l, \mathbb{C}), \quad \text{and} \quad D_l = \mathfrak{so}(2l, \mathbb{C}).
\]
Note the low-dimensional coincidences
\[
A_1 = B_1 = C_1, \quad D_2 = A_1 \oplus A_1, \quad B_2 = C_2 \quad \text{and} \quad A_3 = D_3,
\]
which give the isomorphisms \( \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sp}(1, \mathbb{C}) \),
\( \mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \), \( \mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C}) \), and \( \mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C}) \).
Real forms of the complex classical Lie groups

We use the notation overleaf, and in addition, $\mathbb{H}$ will represent the quaternions, a noncommutative division algebra with $\mathbb{H} \cong \mathbb{R}^4$ as a vector space.

1. The linear groups over $\mathbb{R}$ and $\mathbb{H}$

Define $GL(n, \mathbb{R})$ and $GL(n, \mathbb{H})$ to be the Lie groups of $n \times n$ invertible matrices over $\mathbb{R}$ and $\mathbb{H}$ respectively. Define $SL(n, \mathbb{R})$ to be the subgroup of $GL(n, \mathbb{R})$ of matrices with determinant 1. Then $SL(n, \mathbb{R})$ is a real, simple Lie group with rank $n - 1$ and dimension $n^2 - 1$, a real form of $SL(n, \mathbb{C})$.

Now $GL(n, \mathbb{H})$ has dimension $4n^2$ and rank $2n$, and it can be regarded as a subgroup of $GL(4n, \mathbb{R})$. Define $SL(n, \mathbb{H})$ to be $GL(n, \mathbb{H}) \cap SL(4n, \mathbb{R})$. Then $SL(n, \mathbb{H})$ is a real, simple Lie group of rank $2n - 1$ and dimension $4n^2 - 1$, a real form of $SL(2n, \mathbb{C})$.

2. The unitary groups

Let $(z_1, \ldots, z_n)$ be coordinates on $\mathbb{C}^n$. Define $U(n)$ to be the subgroup of $GL(n, \mathbb{C})$ preserving the hermitian metric $|dz_1|^2 + \cdots + |dz_n|^2$ on $\mathbb{C}^n$. Then $U(n)$ is a compact, connected Lie group of rank $n$ and dimension $n^2$, which is not semisimple. It is a real form of $GL(n, \mathbb{C})$.

Define $SU(n)$ to be the subgroup of $SL(n, \mathbb{C})$ preserving the same hermitian metric. Then $SU(n)$ is a compact, connected, simply-connected, simple Lie group of rank $n - 1$ and dimension $n^2 - 1$, a real form of $SL(n, \mathbb{C})$.

3. The orthogonal groups

Let $(x_1, \ldots, x_n)$ be coordinates on $\mathbb{R}^n$. Define $O(n)$ and $SO(n)$ to be the subgroups of $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$ preserving the metric $dx_1^2 + \cdots + dx_n^2$ on $\mathbb{R}^n$. These groups are compact, with dimension $n(n - 1)/2$ and rank $n/2$ (n even) or $(n - 1)/2$ (n odd). Note $O(n)$ has 2 connected components, and $SO(n)$ is the identity component of $O(n)$. For $n > 2$, $SO(n)$ has fundamental group $\mathbb{Z}_2$, and $Spin(n)$ is the simply-connected double cover. For $n = 3$ and $n > 4$ $SO(n)$ and $Spin(n)$ are simple, and $Spin(4) \cong Spin(3) \times Spin(3)$.

The groups $O(n), SO(n)$ and Spin(n) are real forms of $O(n, \mathbb{C}), SO(n, \mathbb{C})$ and $Spin(n, \mathbb{C})$ respectively.

4. The symplectic groups

Define $Sp(n)$ to be the intersection of $GL(n, \mathbb{H})$ and $SO(4n)$. Then $Sp(n)$ acts on $\mathbb{R}^{4n}$ preserving a metric and an action of $\mathbb{H}$. It has dimension $n(2n + 1)$ and rank $n$, and is a compact, connected, simply-connected, simple Lie group, a real form of $Sp(n, \mathbb{C})$.

5. Groups with mixed signature

For $p, q > 1$, let $U(p, q), SU(p, q)$ be the subgroups of $GL(p + q, \mathbb{C}), SL(p + q, \mathbb{C})$ respectively preserving the pseudo-hermitian metric $|dz_1|^2 + \cdots + |dz_p|^2 - |dz_{p+1}|^2 + \cdots + |dz_{p+q}|^2$. These are noncompact real forms of $GL(n, \mathbb{C})$ and $SL(n, \mathbb{C})$. In the same way, we may define groups $O(p, q), SO(p, q)$ and $Sp(p, q)$. Note $O(p, q)$ has 4 connected components and $SO(p, q)$ has 2.