A universal theory of enumerative invariants and wall-crossing formulae

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Workshop 'Quivers, Calabi–Yau 3-folds and Donaldson–Thomas invariants', Paris, April 2022.

Based on arXiv:2111.04694 (302 pages).

(See also arXiv:2005.05637 with Jacob Gross and Yuuji Tanaka).

Funded by the Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics.

These slides available at
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1. Introduction

An *enumerative invariant theory* in Algebraic or Differential Geometry is the study of invariants $I_\alpha(\tau)$ which 'count' τ-semistable objects E with fixed topological invariants $[\![E]\!]=\alpha$ in some geometric problem, usually by means of a virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ for the moduli space $\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-semistable objects in some homology theory, with $I_\alpha(\tau)=\int_{[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}}\mu_\alpha$ for some natural cohomology class μ_α. We call the theory $\mathbb{C}$-*linear* if the objects E live in a $\mathbb{C}$-linear additive category $\mathcal{A}$. For example:

- Invariants counting semistable vector bundles on curves.
- Mochizuki-style invariants counting coherent sheaves on surfaces. (Think of as algebraic Donaldson invariants.)
- Donaldson–Thomas invariants of Calabi–Yau or Fano 3-folds.
- Donaldson–Thomas type invariants of Calabi–Yau 4-folds.
- Invariants counting representations of quivers Q.
- $\mathrm{U}(m)$ Donaldson invariants of 4-manifolds.

I have proved that many such theories in Algebraic Geometry, in which either the moduli spaces are automatically smooth (e.g. coherent sheaves on curves, quiver representations), or the invariants are defined using Behrend–Fantechi obstruction theories and virtual classes, share a common universal structure.

I expect this universal structure also to extend to Calabi–Yau 4-fold invariants defined using Borisov–Joyce / Oh–Thomas virtual classes, and to Donaldson invariants in Differential Geometry.

Here is an outline of this structure:

(a) We form two moduli stacks $\mathcal{M},\mathcal{M}^{\rm pl}$ of all objects E in $\mathcal{A}$, where $\mathcal{M}$ is the usual moduli stack, and $\mathcal{M}^{\rm pl}$ the 'projective linear' moduli stack of objects E modulo 'projective isomorphisms', i.e. quotient by $\lambda\,\mathrm{id}_E$ for $\lambda\in\mathbb{G}_m$.

(b) We are given a quotient $K_0(\mathcal{A})\twoheadrightarrow K(\mathcal{A})$, where $K(\mathcal{A})$ is the lattice of topological invariants $[\![E]\!]$ of E (e.g. fixed Chern classes). We split $\mathcal{M}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha$, $\mathcal{M}^{\rm pl}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha^{\rm pl}$.

(c) There is a symmetric biadditive *Euler form* $\chi:K(\mathcal{A})\times K(\mathcal{A})\rightarrow\mathbb{Z}$.

(d) We can form the homology $H_*(\mathcal{M}), H_*(\mathcal{M}^{\rm pl})$ over $\mathbb{Q}$, with $H_*(\mathcal{M})=\bigoplus_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}_\alpha)$, $H_*(\mathcal{M}^{\rm pl})=\bigoplus_{\alpha\in K(\mathcal{A})}H_*(\mathcal{M}_\alpha^{\rm pl})$. Define shifted versions $\hat H_*(\mathcal{M}), \check H_*(\mathcal{M}^{\rm pl})$ by $\hat H_n(\mathcal{M}_\alpha)=H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$, $\check H_n(\mathcal{M}_\alpha^{\rm pl})=H_{n+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl})$. Then previous work by me (later) makes $\hat H_*(\mathcal{M})$ into a *graded vertex algebra*, and $\check H_*(\mathcal{M}^{\rm pl})$ into a *graded Lie algebra*.

(e) There is a notion of *stability condition* τ on $\mathcal{A}$. When $\mathcal{A}=\mathrm{coh}(X)$, this can be Gieseker stability for a polarization on X. For each $\alpha\in K(\mathcal{A})$ we can form moduli spaces $\mathcal{M}_\alpha^{\rm st}(\tau)\subseteq\mathcal{M}_\alpha^{\rm ss}(\tau)$ of τ-(semi)stable objects in class α. Here $\mathcal{M}_\alpha^{\rm st}(\tau)$ is a substack of $\mathcal{M}_\alpha^{\rm pl}$, and is a $\mathbb{C}$-scheme with perfect obstruction theory. Also $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is proper. Thus, if $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ we have a virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$, which we regard as an element of $H_*(\mathcal{M}_\alpha^{\rm pl})$. The virtual dimension is $\mathrm{vdim}_{\mathbb{R}}[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}=2-\chi(\alpha,\alpha)$, so $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ lies in $\check H_0(\mathcal{M}_\alpha^{\rm pl})\subset\check H_0(\mathcal{M}^{\rm pl})$, which is a Lie algebra by (d).

(f) For many theories, there is a problem defining the invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)\neq\mathcal{M}_\alpha^{\rm ss}(\tau)$, i.e. when the moduli spaces $\mathcal{M}_\alpha^{\rm ss}(\tau)$ contain *strictly τ-semistable points*.
I give a systematic way to define $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in homology over $\mathbb{Q}$ (not $\mathbb{Z}$) in these cases, using auxiliary pair invariants. (This method is well known, e.g. in Joyce–Song D–T theory.)
I prove the $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ are independent of the choices used in the pair invariant method.

(g) If $\tau,\tilde\tau$ are stability conditions and $\alpha\in K(\mathcal{A})$, I prove a wall crossing formula

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm virt}=\sum_{\alpha_1+\cdots+\alpha_n=\alpha}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm virt},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm virt}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm virt}\bigr], \tag{1}$$

where $\tilde U(-)$ are combinatorial coefficients defined in my previous work on wall-crossing formulae for motivic invariants, and [,] is the Lie bracket on $\check H_0(\mathcal{M}^{\rm pl})$ from (d).

(h) In some theories the natural obstruction theory on $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ has a trivial summand $\mathbb{C}^{o_\alpha}$ in its obstruction sheaf for $o_\alpha>0$, and so the virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ is zero. In these cases one defines a *reduced* obstruction theory on $\mathcal{M}_\alpha^{\rm st}(\tau)$ by deleting the $\mathbb{C}^{o_\alpha}$ factor, and obtains *reduced* virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$. For example, this holds for coherent sheaves on surfaces X with geometric genus $p_g>0$, with $o_\alpha=p_g$ when $\operatorname{rank}\alpha>0$.
My theory extends to 'reduced' invariants, allowing o_α to depend on $\alpha\in K(\mathcal{A})$ with $o_\alpha+o_\beta\geqslant o_{\alpha+\beta}$, giving invariants $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$ in $\check H_{2o_\alpha}(\mathcal{M}_\alpha^{\rm pl})$. Generalizing (1), they satisfy the wall crossing formula

$$[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm red}=\sum_{\substack{\alpha_1+\cdots+\alpha_n=\alpha:\\ o_{\alpha_1}+\cdots+o_{\alpha_n}=o_\alpha}}\tilde U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\bigl[\bigl[\ldots\bigl[[\mathcal{M}_{\alpha_1}^{\rm ss}(\tau)]_{\rm red},[\mathcal{M}_{\alpha_2}^{\rm ss}(\tau)]_{\rm red}\bigr],\ldots\bigr],[\mathcal{M}_{\alpha_n}^{\rm ss}(\tau)]_{\rm red}\bigr]. \qquad (2)$$

If $o_\alpha=o>0$ for all α this reduces to $[\mathcal{M}_\alpha^{\rm ss}(\tilde\tau)]_{\rm red}=[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm red}$, that is, the invariants are independent of stability condition.

(i) The next part is not written up in detail yet.
When $\mathcal{A} = \mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$ for X a Calabi–Yau 3-fold, the natural obstruction theory on $\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$ has terms in degree -2 from $\mathrm{Ext}^3(E, E)$. We can remove these by taking trace-free Ext to define Donaldson–Thomas invariants, changing the real virtual dimension by 2.
To include these in the theory, for $\mathcal{A}$ odd Calabi–Yau we can modify (d) above to make $\hat{H}_*(\mathcal{M})$ into a *graded vertex Lie algebra* (with grading changed by 2) and $\check{H}_*(\mathcal{M}^{\mathrm{pl}})$ into a *graded Lie algebra* (with grading changed by 2), as before.
So we can include Donaldson–Thomas theory in our picture.
For ordinary D–T invariants this does not add much to the Joyce–Song / Kontsevich–Soibelman picture. However, for a local Calabi–Yau 3-fold with an action of a group G (e.g. $\mathbb{G}_m$ acting on K_X for X a surface) we can do Donaldson–Thomas theory in G-equivariant homology, giving non-motivic invariants, with applications to Thomas' equivariant Vafa–Witten theory.

2. Vertex and Lie algebras on homology of moduli stacks
2.1. Vertex algebras (don't try to understand this slide.)

Let R be a commutative ring. A *vertex algebra* over R is an R-module V equipped with morphisms $D^{(n)}: V \to V$ for $n = 0, 1, 2, \ldots$ with $D^{(0)} = \mathrm{id}_V$ and $v_n : V \to V$ for all $v \in V$ and $n \in \mathbb{Z}$, with v_n R-linear in v, and a distinguished element $\mathbb{1} \in V$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $v \in V$ then $\mathbb{1}_{-1}(v) = v$ and $\mathbb{1}_n(v) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(iii) If $v \in V$ then $v_n(\mathbb{1}) = D^{(-n-1)}(v)$ for $n < 0$ and $v_n(\mathbb{1}) = 0$ for $n \geqslant 0$.

(iv) $u_n(v) = \sum_{k\geqslant 0}(-1)^{k+n+1}D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.

(v) $$(u_l(v))_m(w) = \sum_{n\geqslant 0}(-1)^n \binom{l}{n} \bigl(u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w))\bigr)$$

for all $u, v, w \in V$ and $l, m \in \mathbb{Z}$, where the sum makes sense by (i).

We can also define *graded vertex algebras* and *vertex superalgebras*.

It is usual to encode the maps $u_n : V \to V$ for $n \in \mathbb{Z}$ in generating function form as R-linear maps for each $u \in V$

$$Y(u,z) : V \longrightarrow V[[z,z^{-1}]], \quad Y(u,z) : v \longmapsto \textstyle\sum_{n\in\mathbb{Z}} u_n(v) z^{-n-1},$$

where z is a formal variable. The $Y(u,z)$ are called *fields*, and have a meaning in Physics. Parts (i)–(v) may be rewritten as properties of the $Y(u,z)$. One interesting property is this: for all $u,v,w \in V$ there exist $N \gg 0$ depending on u,v such that

$$(y-z)^N Y(u,y) Y(v,z) w = (y-z)^N Y(v,z) Y(u,y) w. \qquad (2)$$

There may be a V-valued rational function $R(y,z)$ with poles when $y=0$, $z=0$ and $y=z$, such that the l.h.s. of (2) is a formal Laurent series convergent to $R(y,z)$ when $0<|y|<|z|$, and the r.h.s. converges to $R(y,z)$ when $0<|z|<|y|$.

Think of $u *_z v = Y(u,z)v$ as a multiplication on V depending on a complex variable z, with poles at $z=0$. Very roughly, V is a commutative associative algebra under $*_z$, with identity $\mathbb{1}$, except the formal power series and poles make everything more complicated.

Any commutative algebra $(V, \mathbb{1}, \cdot)$ with derivation D is a vertex algebra, with $Y(u,z)v = (e^{zD}u)\cdot v$, so no poles, where $u_n(v) = \bigl(\frac{1}{(n+1)!}D^{n+1}u\bigr)\cdot v$ for $n \geqslant -1$, and $u_n(v) = 0$ for $n < -1$. We call such V a *commutative vertex algebra*. All non-commutative vertex algebras are infinite-dimensional, so even the simplest nontrivial examples are large, complicated objects, which are difficult to write down.

Let R be a field of characteristic zero. A *vertex operator algebra* (*VOA*) over R is a vertex algebra V over R, with a distinguished *conformal element* $\omega \in V$ and a *central charge* $c_V \in R$, such that writing $L_n = \omega_{n+1} : V_* \to V_*$, the L_n define an action of the *Virasoro algebra* on V_*, with central charge c_V, and $L_{-1} = D^{(1)}$. VOAs are important in Physics. We will give a geometric construction of vertex algebras, but often they will *not* be VOAs.

If V is a (graded/super) vertex algebra then $V/\langle D^{(k)}(V),\ k\geqslant 1\rangle$ is a (graded/super) Lie algebra, with Lie bracket

$$\bigl[u+\langle D^{(k)}(V),\ k\geqslant 1\rangle, v+\langle D^{(k)}(V),\ k\geqslant 1\rangle\bigr]=u_0(v)+\langle D^{(k)}(V),\ k\geqslant 1\rangle.$$

Vertex algebras were introduced in mathematics by Borcherds, who noticed that certain infinite-dimensional Lie algebras important in Representation Theory were constructed as $V/\langle D^{(k)}(V),\ k\geqslant 1\rangle$. For example, Kac–Moody Lie algebras are (Lie subalgebras of) the Lie algebras associated to lattice vertex algebras.
Vertex algebras are used in Representation Theory, both of infinite-dimensional Lie algebras, and in Moonshine – the Monster may be characterized as the symmetry group of a certain infinite-dimensional vertex algebra.

2.2. Vertex and Lie algebras on homology of moduli stacks

Let $\mathcal{A}$ be a $\mathbb{C}$-linear abelian or triangulated category from Algebraic Geometry or Representation Theory, e.g. $\mathcal{A} = \mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme, or $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q$ or $D^b \mathrm{mod}\text{-}\mathbb{C}Q$. Write $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A}$, which is an Artin $\mathbb{C}$-stack in the abelian case, and a higher $\mathbb{C}$-stack in the triangulated case. There is a morphism $\Phi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ acting by $([E],[F]) \to [E \oplus F]$ on $\mathbb{C}$-points.

Now $\mathbb{G}_m$ acts on objects E in $\mathcal{A}$ with $\lambda \in \mathbb{G}_m$ acting as $\lambda\,\mathrm{id}_E : E \to E$. This induces an action $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ of the group stack $[*/\mathbb{G}_m]$ on $\mathcal{M}$. We write $\mathcal{M}^{\mathrm{pl}} = \mathcal{M}/[*/\mathbb{G}_m]$ for the quotient, called the 'projective linear' moduli stack. There is a morphism $\mathcal{M} \to \mathcal{M}^{\mathrm{pl}}$ which is a $[*/\mathbb{G}_m]$-fibration on $\mathcal{M} \setminus \{[0]\}$.

We need some extra data:

- A quotient $K_0(\mathcal{A}) \twoheadrightarrow K(\mathcal{A})$ giving splittings $\mathcal{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha$, $\mathcal{M}^{\rm pl} = \coprod_{\alpha \in K(\mathcal{A})} \mathcal{M}_\alpha^{\rm pl}$.
- A symmetric biadditive *Euler form* $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$.
- A perfect complex $\Theta^\bullet$ on $\mathcal{M} \times \mathcal{M}$ satisfying some assumptions, including $\operatorname{rank} \Theta^\bullet|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta} = \chi(\alpha, \beta)$.
  If $\mathcal{A}$ is a 4-Calabi–Yau category, and we will use Borisov–Joyce virtual classes, we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, where $\mathcal{E}xt^\bullet \to \mathcal{M} \times \mathcal{M}$ is the *Ext complex*. Otherwise we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)$, where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ swaps the factors.
- Signs $\epsilon_{\alpha,\beta} \in \{\pm 1\}$ for $\alpha, \beta \in K(\mathcal{A})$ with $\epsilon_{\alpha,\beta} \cdot \epsilon_{\alpha+\beta,\gamma} = \epsilon_{\alpha,\beta+\gamma} \cdot \epsilon_{\beta,\gamma}$ and $\epsilon_{\alpha,\beta} \cdot \epsilon_{\beta,\alpha} = (-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)}$.
  (These compare orientations on $\mathcal{M}_\alpha, \mathcal{M}_\beta, \mathcal{M}_{\alpha+\beta}$.)

Then we can make the homology $H_*(\mathcal{M})$, with grading shifted to $\hat{H}_*(\mathcal{M})$ as above, into a *graded vertex algebra*.

Writing $H_*([*/\mathbb{G}_m])=\mathbb{Q}[t]$ with $\deg t=2$, the state-field correspondence $Y(z)$ is given by, for $u\in H_a(\mathcal{M}_\alpha)$, $v\in H_b(\mathcal{M}_\beta)$

$$Y(u,z)v=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}z^{\chi(\alpha,\beta)}\cdot H_*\bigl(\Phi\circ(\Psi\times\mathrm{id})\bigr) \tag{3}$$
$$\Bigl\{\bigl(\sum_{i\geqslant 0}z^it^i\bigr)\boxtimes\bigl[(u\boxtimes v)\cap\exp\bigl(\sum_{j\geqslant 1}(-1)^{j-1}(j-1)!z^{-j}\,\mathrm{ch}_j([\Theta^\bullet])\bigr)\bigr]\Bigr\}.$$

The identity $\mathbb{1}$ is $1\in H_0(\mathcal{M}_0)$. Define $e^{zD}:\check H_*(\mathcal{M})\rightarrow\check H_*(\mathcal{M})[[z]]$ by $Y(v,z)\mathbb{1}=e^{zD}v$. Then $(\check H_*(\mathcal{M}),\mathbb{1},e^{zD},Y)$ is a graded vertex algebra, so $\check H_{*+2}(\mathcal{M})/D(\check H_*(\mathcal{M}))$ is a graded Lie algebra. In the abelian category case at least, there is a canonical isomorphism $\check H_*(\mathcal{M}^{\mathrm{pl}})\cong\check H_{*+2}(\mathcal{M})/D(\check H_*(\mathcal{M}))$. This makes $\check H_*(\mathcal{M}^{\mathrm{pl}})$ into a graded Lie algebra, and $\check H_0(\mathcal{M}^{\mathrm{pl}})$ into a Lie algebra.

Remarks

• One can often write down $\check H_*(\mathcal{M})$ and $\check H_*(\mathcal{M}^{\rm pl})$ with their algebraic structures explicitly. The answer is usually simpler in the derived category case. For example, my student Jacob Gross showed that if a smooth projective $\mathbb{C}$-scheme X is a curve, surface, or toric variety, and $\mathcal{M}$ is the moduli stack of $D^b\operatorname{coh}(X)$, then

$$\hat H_*(\mathcal{M},\mathbb{Q})\cong \mathbb{Q}[K^0_{\rm sst}(X)]\otimes_{\mathbb{Q}}\mathrm{Sym}^*\bigl(K^0(X^{\rm an})\otimes_{\mathbb{Z}}t^2\mathbb{Q}[t^2]\bigr)\otimes_{\mathbb{Q}}\textstyle\bigwedge^*\bigl(K^1(X^{\rm an})\otimes_{\mathbb{Z}}t\mathbb{Q}[t^2]\bigr), \qquad (4)$$

with a super-lattice vertex algebra structure. Thus we can use this for explicit computations in examples, as well as for abstract theory.

• It helps to study $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ in $\operatorname{coh}(X)$ using $H_*(\mathcal{M})$, $H_*(\mathcal{M}^{\rm pl})$ for $D^b\operatorname{coh}(X)$, so we can use the presentation (4).

• Although Lie algebras are much simpler than vertex algebras, it is difficult to write down the Lie bracket on $\check H_*(\mathcal{M}^{\rm pl})$ explicitly: the best way seems to be via the vertex algebra structure on $\hat H_*(\mathcal{M})$.

3. Enumerative invariants

3.1. Virtual classes of moduli spaces

The vertex and Lie algebras $\hat{H}_*(\mathcal{M})$, $\check{H}_*(\mathcal{M}^{\rm pl})$ above work for $\mathcal{M}$ the moduli stack of objects in $\mathrm{coh}(X)$ or $D^b\,\mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme of any dimension. However, defining virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ when $\mathcal{M}_\alpha^{\rm st}(\tau)=\mathcal{M}_\alpha^{\rm ss}(\tau)$ is much more restrictive:

- If $\dim \mathcal{A}=1$, say if $\mathcal{A}=\text{mod-}\mathbb{C}Q$ or $\mathcal{A}=\mathrm{coh}(X)$ for X a curve, then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is a smooth projective $\mathbb{C}$-scheme, and has a fundamental class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm fund}$.
- If $\dim \mathcal{A}=2$, say if $\mathcal{A}=\text{mod-}\mathbb{C}Q/I$ or $\mathcal{A}=\mathrm{coh}(X)$ for X a surface, then $\mathcal{M}_\alpha^{\rm ss}(\tau)$ is a projective $\mathbb{C}$-scheme with obstruction theory, and has a Behrend–Fantechi virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$.
- If $\mathcal{A}=\mathrm{coh}(X)$ for X a Calabi–Yau or Fano 3-fold, one can also define Behrend–Fantechi virtual classes $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$.
- If $\mathcal{A}=\mathrm{coh}(X)$ for X a Calabi–Yau 4-fold, Borisov–Joyce define a different kind of virtual class $[\mathcal{M}_\alpha^{\rm ss}(\tau)]_{\rm virt}$, with *half the expected dimension* of the Behrend–Fantechi class.

On moduli stacks and moduli schemes

There are two main ways of forming moduli spaces in Algebraic Geometry: as *schemes* or *stacks*. An important difference is that if $\mathcal{M}$ is a moduli stack of objects E, then automorphism groups are remembered in the isotropy groups of $\mathcal{M}$ by $\mathrm{Iso}_{\mathcal{M}}([E])=\mathrm{Aut}(E)$, but moduli schemes forget automorphism groups.

Our moduli stacks $\mathcal{M},\mathcal{M}^{\rm pl}$ differ in that their isotropy groups are $\mathrm{Iso}_{\mathcal{M}}([E])=\mathrm{Aut}(E)$, but $\mathrm{Iso}_{\mathcal{M}^{\rm pl}}([E])=\mathrm{Aut}(E)/(\mathbb{G}_m\cdot\mathrm{id}_E)$.

If E is τ-stable then $\mathrm{Aut}(E)=\mathbb{G}_m\cdot\mathrm{id}_E$, so $\mathrm{Iso}_{\mathcal{M}^{\rm pl}}([E])=\{1\}$.

Because of this, the τ-stable moduli scheme $\mathcal{M}^{\rm st}_\alpha(\tau)$ is actually an *open substack* in $\mathcal{M}^{\rm pl}$ (but not $\mathcal{M}$). This makes $\mathcal{M}^{\rm pl}$ useful for us.

The τ-semistable moduli scheme $\mathcal{M}^{\rm ss}_\alpha(\tau)$ has the *good property* that it is usually compact (proper). But it has the *bad properties* that it does not map to $\mathcal{M}^{\rm pl}$ or $\mathcal{M}$, and the obstruction theory (or other nice structure) on $\mathcal{M}^{\rm st}_\alpha(\tau)$ does not extend to $\mathcal{M}^{\rm ss}_\alpha(\tau)$, so we cannot define a virtual class $[\mathcal{M}^{\rm ss}_\alpha(\tau)]_{\rm virt}$ unless $\mathcal{M}^{\rm st}_\alpha(\tau)=\mathcal{M}^{\rm ss}_\alpha(\tau)$.

3.2. The case of quivers

Let $Q=(Q_0,Q_1,h,t)$ be a quiver, with finite sets Q_0 of vertices and Q_1 of edges, and head and tail maps $h,t:Q_1\to Q_0$. Then we have a $\mathbb{C}$-linear abelian category $\text{mod-}\mathbb{C}Q$ of *representations* (V_v,ρ_e) of Q, comprising a finite-dimensional $\mathbb{C}$-vector space V_v for each $v\in Q_0$ and a linear map $\rho_e:V_{t(e)}\to V_{h(e)}$ for each $e\in Q_1$. The *dimension vector* of (V_v,ρ_e) is $\boldsymbol{d}\in\mathbb{N}^{Q_0}$, where $\boldsymbol{d}(v)=\dim V_v$.

We can work out our theory very explicitly for $\mathcal{A}=\text{mod-}\mathbb{C}Q$. We take $K(\mathcal{A})=\mathbb{Z}^{Q_0}$. Then $\mathcal{M}=\coprod_{\boldsymbol{d}\in\mathbb{N}^{Q_0}}\mathcal{M}_{\boldsymbol{d}}$, $\mathcal{M}^{\rm pl}=\coprod_{\boldsymbol{d}\in\mathbb{N}^{Q_0}}\mathcal{M}_{\boldsymbol{d}}^{\rm pl}$, where $\mathcal{M}_{\boldsymbol{d}}=[R_{\boldsymbol{d}}/\mathrm{GL}_{\boldsymbol{d}}]$, $\mathcal{M}_{\boldsymbol{d}}^{\rm pl}=[R_{\boldsymbol{d}}/\mathrm{PGL}_{\boldsymbol{d}}]$ with

$$R_{\boldsymbol{d}}=\textstyle\prod_{e\in Q_1}\mathrm{Hom}(\mathbb{C}^{t(\boldsymbol{d}(e))},\mathbb{C}^{h(\boldsymbol{d}(e))}),\;\; \mathrm{GL}_{\boldsymbol{d}}=\textstyle\prod_{v\in Q_0}\mathrm{GL}(\boldsymbol{d}(v)),$$

and $\mathrm{PGL}_{\boldsymbol{d}}=\mathrm{GL}_{\boldsymbol{d}}/\mathbb{G}_m$. Hence $H_*(\mathcal{M}_{\boldsymbol{d}})\cong H_*(B\,\mathrm{GL}_{\boldsymbol{d}})$ and $H_*(\mathcal{M}_{\boldsymbol{d}}^{\rm pl})\cong H_*(B\,\mathrm{PGL}_{\boldsymbol{d}})$, which we can write explicitly.

Slope stability conditions

Fix $\mu_v \in \mathbb{R}$ for all $v \in Q_0$. Define $\mu : \mathbb{N}^{Q_0} \setminus \{0\} \to \mathbb{R}$ by

$$\mu(\boldsymbol{d}) = \left(\sum_{v\in Q_0} \mu_v \boldsymbol{d}(v)\right) / \left(\sum_{v\in Q_0} \boldsymbol{d}(v)\right).$$

We call μ a *slope function*. An object $0 \neq E \in \text{mod-}\mathbb{C}Q$ is called *μ-semistable* (or *μ-stable*) if whenever $0 \neq E' \subsetneq E$ is a subobject we have $\mu(\mathbf{dim} E') \geqslant \mu(\mathbf{dim} E)$ (or $\mu(\mathbf{dim} E') > \mu(\mathbf{dim} E)$).

Recall that $\mathcal{M}^{\mathrm{pl}}_{\boldsymbol{d}} = [R_{\boldsymbol{d}}/\operatorname{PGL}_{\boldsymbol{d}}]$ as a quotient stack. King (1994) showed that there is a linearization θ of the action of $\operatorname{PGL}_{\boldsymbol{d}}$ on $R_{\boldsymbol{d}}$, such that a $\mathbb{C}$-point $[E] \in [R_{\boldsymbol{d}}/\operatorname{PGL}_{\boldsymbol{d}}]$ is μ-(semi)stable in $\text{mod-}\mathbb{C}Q$ iff the corresponding point in $R_{\boldsymbol{d}}$ is GIT (semi)stable.

Hence there are moduli schemes $\mathcal{M}^{\mathrm{st}}_{\boldsymbol{d}}(\mu) \subseteq \mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu)$ which are the GIT (semi)stable quotients $R_{\boldsymbol{d}}/\!/^{\mathrm{st}}_{\theta} \operatorname{PGL}_{\boldsymbol{d}} \subseteq R_{\boldsymbol{d}}/\!/^{\mathrm{ss}}_{\theta} \operatorname{PGL}_{\boldsymbol{d}}$.

If Q has *no oriented cycles* then a $\mathbb{G}_m$ subgroup of $\operatorname{PGL}_{\boldsymbol{d}}$ acts on $R_{\boldsymbol{d}}$ with positive weights, so $\mathcal{M}^{\mathrm{ss}}_{\boldsymbol{d}}(\mu) = R_{\boldsymbol{d}}/\!/^{\mathrm{ss}}_{\theta} \operatorname{PGL}_{\boldsymbol{d}}$ is a projective $\mathbb{C}$-scheme. Also $\mathcal{M}^{\mathrm{st}}_{\boldsymbol{d}}(\mu) = R_{\boldsymbol{d}}/\!/^{\mathrm{st}}_{\theta} \operatorname{PGL}_{\boldsymbol{d}}$ is a smooth quasi-projective $\mathbb{C}$-scheme, an open substack of $\mathcal{M}^{\mathrm{pl}}_{\boldsymbol{d}} = [R_{\boldsymbol{d}}/\operatorname{PGL}_{\boldsymbol{d}}]$.

Thus, if Q has no oriented cycles, and μ is a slope function on $\text{mod-}\mathbb{C}Q$, and $\boldsymbol{d}\in\mathbb{N}^{Q_0}\setminus\{0\}$ with $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu)=\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)$, then $\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)$ is a smooth projective $\mathbb{C}$-scheme and an open substack of $\mathcal{M}^{\rm pl}_{\boldsymbol{d}}$, and has a fundamental class $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm fund}$ in $H_*(\mathcal{M}^{\rm pl}_{\boldsymbol{d}})$. It has dimension $2-\chi(\boldsymbol{d},\boldsymbol{d})$, where $\chi:\mathbb{Z}^{Q_0}\times\mathbb{Z}^{Q_0}\to\mathbb{Z}$ is

$$\chi(\boldsymbol{d},\boldsymbol{e})=2\sum_{v\in Q_0}\boldsymbol{d}(v)\boldsymbol{e}(v)-\sum_{e\in Q_1}\bigl(\boldsymbol{d}(h(e))\boldsymbol{e}(t(e))+\boldsymbol{d}(t(e))\boldsymbol{e}(h(e))\bigr).$$

Theorem 1 (Gross–Joyce–Tanaka arXiv:2005.05637.)

Let Q be a quiver with no oriented cycles. Then for all slope functions μ on $\text{mod-}\mathbb{C}Q$ and $\boldsymbol{d}\in\mathbb{N}^{Q_0}\setminus\{0\}$, there exist unique classes $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}\in H_{2-\chi(\boldsymbol{d},\boldsymbol{d})}(\mathcal{M}^{\rm pl}_{\boldsymbol{d}})=\check H_0(\mathcal{M}^{\rm pl}_{\boldsymbol{d}})$ such that:

- (i) *If $\mathcal{M}^{\rm st}_{\boldsymbol{d}}(\mu)=\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)$ then $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}=[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm fund}$.*
- (ii) *The $[\mathcal{M}^{\rm ss}_{\boldsymbol{d}}(\mu)]_{\rm virt}$ transform according to the wall-crossing formula (1) above in the Lie algebra $\check H_0(\mathcal{M}^{\rm pl})$ under change of stability condition.*

In arXiv:2111.04694 I generalize Theorem 1 to many other situations in Algebraic Geometry. The general format is that I prove that, given an abelian category $\mathcal{A}$ (e.g. $\mathcal{A}=\text{mod-}\mathbb{C}Q$ or $\mathcal{A}=\text{coh}(X)$) and a lot of extra data (stability conditions, moduli stacks, Behrend–Fantechi obstruction theories, ...) satisfying a long list of assumptions, then I can define enumerative invariants and prove they satisfy wall-crossing formulae (1) and (2). Then I define the data and verify the assumptions in my favourite examples (e.g. $\mathcal{A}=\text{coh}(X)$ for X a curve, surface or Fano 3-fold). The proofs are very long and complicated (the paper is 302 pages). The rough idea for the WCF proof is that for 'simple' wall-crossings, involving splittings $E=E_1\oplus\cdots\oplus E_n$ in $\mathcal{A}$ for n at most 2, I can prove the WCF (with splittings $\alpha=\alpha_1+\cdots+\alpha_n$ for $n\leqslant 2$) by $\mathbb{G}_m$-localization on a master space. Then I show that complicated wall-crossings in $\mathcal{A}$ can be reduced to a sequence of simple wall-crossings in an auxiliary category $\mathcal{B}$ in an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \text{mod-}\mathbb{C}Q \longrightarrow 0.$$

3.3. Counting coherent sheaves on surfaces

This section is work in progress.

Let X be a complex projective surface, with geometric genus $p_g = \frac{1}{2}(b^2_+(X) - 1) \geqslant 0$. Write $K(\mathrm{coh}(X))$ for the image of $K^0(\mathrm{coh}(X))$ in the topological K-theory $K^0_{\mathrm{top}}(X)$. Consider stability conditions $(\tau, T, \leqslant)$ on $\mathrm{coh}(X)$ which are either Gieseker or μ-stability with respect to a real Kähler class ω on X. Then my theory defines invariants $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ for $\alpha \in K(\mathrm{coh}(X))$. When $p_g > 0$ we take these to be *reduced* invariants.

Write $\bar{\mathfrak{M}}$ for the (higher) moduli stack of objects in $D^b \mathrm{coh}(X)$, and $\bar{\mathfrak{M}}^{\mathrm{pl}}$ for its projective linear version. They split as $\bar{\mathfrak{M}} = \coprod_{\alpha \in K(\mathrm{coh}(X))} \bar{\mathfrak{M}}_\alpha$ and $\bar{\mathfrak{M}}^{\mathrm{pl}} = \coprod_{\alpha \in K(\mathrm{coh}(X))} \bar{\mathfrak{M}}^{\mathrm{pl}}_\alpha$. We consider the invariants $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ to lie in $H_*(\bar{\mathfrak{M}}^{\mathrm{pl}}_\alpha) \cong H_*(\bar{\mathfrak{M}}_\alpha)/D(H_*(\bar{\mathfrak{M}}_\alpha))$. For $\mathrm{rank}\,\alpha > 0$ there is a systematic way to lift the invariants (and the Lie bracket) from $H_*(\bar{\mathfrak{M}}^{\mathrm{pl}}_\alpha)$ to $H_*(\bar{\mathfrak{M}}_\alpha)$. So we take $[\mathcal{M}^{\mathrm{ss}}_\alpha(\tau)]_{\mathrm{virt}}$ to lie in $H_*(\bar{\mathfrak{M}}_\alpha)$.

Work of Jacob Gross gives explicit isomorphisms with polynomial superalgebras

$$H_*(\bar{\mathfrak{M}}_\alpha) \cong e^\alpha \mathbb{Q}[s_{jkl} : k = 0, \ldots, 4,\ j = 1, \ldots, b^k(X),\ l > k/2],$$
$$H_*(\bar{\mathfrak{M}}) \cong \bigoplus_{\alpha \in K(\mathrm{coh}(X))} e^\alpha \mathbb{Q}[s_{jkl},\ \forall j, k, l],$$

where e^α is a formal symbol, and s_{jkl} is a graded formal variable with $\deg s_{jkl} = 2l - k$, and variables of odd degree anticommute. Roughly speaking, s_{jkl} is a dual variable to $S_{jkl} = \mathrm{ch}_l(\mathcal{U}_\alpha^\bullet) \backslash e_{jk}$ in $H^{2l-k}(\bar{\mathfrak{M}}_\alpha)$, where $\mathcal{U}_\alpha^\bullet \to X \times \bar{\mathfrak{M}}_\alpha$ is the universal complex and $(e_{jk})_{j=1}^{b^k(X)}$ is a basis for $H^k(X, \mathbb{Q})$.
We can also write the vertex algebra structure explicitly in this representation.
Thus we may write $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}} = e^\alpha P_\alpha(s_{jkl})$, for P_α a superpolynomial homogeneous of degree $2\,\mathrm{vdim}_{\mathbb{C}}\,\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)$.
I aim to compute the invariants $[\mathcal{M}_\alpha^{\mathrm{ss}}(\tau)]_{\mathrm{virt}}$ as explicitly as I can, as functions of the formal variables s_{jkl}.

Applying my WCF in an abelian category of 'pairs' $\rho : V \otimes_{\mathbb{C}} L \to E$, for $V \in \mathrm{Vect}_{\mathbb{C}}$, $E \in \mathrm{coh}(X)$ and $L \to X$ a fixed line bundle, there is an algorithm to compute $[\mathcal{M}^{\mathrm{ss}}_{\alpha}(\tau)]_{\mathrm{virt}}$ for $\mathrm{rank}\,\alpha = r > 0$ in terms of rank 1 pair and sheaf invariants, by induction on r. For example, if $p_g > 0$ and $r > 0$ I can prove that

$$\sum_{k\in\frac{1}{2}\mathbb{Z}:\ 2k-\int_X \alpha^2\in 2\mathbb{Z}} q^{\mathrm{vdim}_{\mathbb{C}}\,\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)} \frac{[\mathcal{M}^{\mathrm{ss}}_{(r,\alpha,k)}(\tau)]_{\mathrm{virt}}}{e^{(r,\alpha,k)}}$$

$$= \sum_{\beta_1,\ldots,\beta_{r-1}\in H^2(X,\mathbb{Z})} \prod_{a=1}^{r-1} \mathrm{SW}(\beta_a)\cdot F_r(\alpha,\beta_1,\ldots,\beta_{r-1},q,s_{jkl}\ \forall j,k,l),$$

where $\mathrm{SW}(\beta_a)$ are Seiberg–Witten invariants and F_r is a universal function, and I can say a lot about the form of F_r. Using this I hope to be able to prove a number of conjectures in the literature.