Configurations in abelian categories: stability conditions, and invariants counting semistable objects

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1. Introduction

Let $\mathcal{A}$ be an abelian category, and $\mathfrak{Obj}_{\mathcal{A}}$ the moduli $\mathbb{K}$-stack of objects in $\mathcal{A}$, as in the previous talks. We shall define a very general notion of (*weak*) *stability condition* $(\tau, T, \leqslant)$ on $\mathcal{A}$. When $(\tau, T, \leqslant)$ is *permissible* the moduli spaces $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}, \mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ of τ-(semi)stable objects in a class $\alpha \in K(\mathcal{A})$ are *constructible sets* in $\mathfrak{Obj}_{\mathcal{A}}$.

We define interesting *algebras* $\mathcal{H}_{\tau}^{\mathrm{to}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ of *constructible functions* and *stack functions*, generated by the characteristic function of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ for $\alpha \in K(\mathcal{A})$, and have interesting *Lie subalgebras* $\mathcal{L}_{\tau}^{\mathrm{to}}, \bar{\mathcal{L}}_{\tau}^{\mathrm{to}}$. These turn out to be independent of $(\tau, T, \leqslant)$.

Given a *motivic invariant* Υ of $\mathbb{K}$-varieties, we extend it to Υ' on constructible sets in $\mathbb{K}$-stacks, and define *invariants* $I_{\mathrm{ss}}^{\alpha}(\tau) = \Upsilon'(\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau))$ which 'count' τ-semistable objects in class α, and other more general invariants 'counting' τ-semistable configurations. These satisfy *additive identities*. If $\mathrm{Ext}^i(X,Y)=0$ for all $i>1$ and $X,Y\in\mathcal{A}$, or under other conditions, we prove extra *multiplicative identities* on some classes of invariants. This happens if $\mathcal{A}=\text{mod-}\mathbb{K}Q$, or if $\mathcal{A}=\mathrm{coh}(P)$ for P a smooth curve, or a surface with K_P^{-1} semiample, or a Calabi–Yau 3-fold. The identities come from (Lie) algebra morphisms from $\bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ or $\bar{\mathcal{L}}_{\tau}^{\mathrm{to}}$ to some explicit (Lie) algebra.

2. (Weak) stability conditions

Let $\mathcal{A}$ be an abelian category, and $K(\mathcal{A})$ the quotient of the Grothendieck group $K_0(\mathcal{A})$ by some fixed subgroup, such that if $X \in \mathcal{A}$ and $[X] = 0$ in $K(\mathcal{A})$ then $X \cong 0$. Define the *positive cone* in $K(\mathcal{A})$:

$C(\mathcal{A}) = \big\{[X] \in K(\mathcal{A}) : X \in \mathcal{A},\ \ X \ncong 0\big\}$.

Suppose $(T, \leqslant)$ is a totally ordered set, and $\tau : C(\mathcal{A}) \rightarrow T$ a map. Call $(\tau, T, \leqslant)$ a *stability condition* on $\mathcal{A}$ if whenever α, β, γ lie in $C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$, or $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$, or $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$. This definition is modelled on Rudakov's stability conditions.

Call $(\tau, T, \leqslant)$ a *weak stability condition* if $\tau(\alpha) \leqslant \tau(\beta) \leqslant \tau(\gamma)$ or $\tau(\alpha) \geqslant \tau(\beta) \geqslant \tau(\gamma)$.

Call $X \in \mathcal{A}$ *τ-semistable* (or *τ-stable*) if for all subobjects $S \subset X$ with $S \neq 0, X$ we have $\tau([S]) \leqslant \tau([X/S])$ (or $\tau([S]) < \tau([X/S])$).

If $(\tau, T, \leqslant)$ is a weak stability condition and $\mathcal{A}$ is noetherian and τ-artinian then every $X \in \mathcal{A}$ has a unique *Harder–Narasimhan filtration* $0 = A_0 \subset A_1 \subset \cdots A_n = X$ with all quotients $S_i = A_i/A_{i-1}$ τ-semistable and $\tau([S_1]) > \cdots > \tau([S_n])$.

If $(\tau, T, \leqslant)$ is a stability condition, every τ-semistable X also has such a (nonunique) filtration with S_i τ-stable and $\tau([S_i]) = \tau([X])$ for all i, S_i unique up to order, iso.

So *τ-semistability is well-behaved for weak stability conditions*, and *τ-stability is well-behaved for stability conditions*.

Examples. **(a)** Let $Q=(Q_0,Q_1,b,e)$ be a quiver, $\mathcal{A}=\text{mod-}\mathbb{K}Q$ and $K(\mathcal{A})=\mathbb{Z}^{Q_0}$. Then $C(\mathcal{A})=\mathbb{N}^{Q_0}\setminus\{0\}$. Choose maps $c:Q_0\to\mathbb{Z}$ and $r:Q_0\to\mathbb{Z}_+$ and define the *slope* $\mu:C(\mathcal{A})\to\mathbb{R}$ by
$\mu(\alpha)=\bigl(\sum_{v\in Q_0}c(v)\alpha(v)\bigr)/\bigl(\sum_{v\in Q_0}r(v)\alpha(v)\bigr)$.
Then $(\mu,\mathbb{R},\leqslant)$ is a stability condition.

(b) Let P be a smooth projective $\mathbb{K}$-scheme, $\mathcal{A}=\mathrm{coh}(P)$ and $K(\mathcal{A})=K^{\mathrm{num}}(\mathcal{A})$ the *numerical Grothendieck group*, a subgroup of $H^{\mathrm{even}}(P,\mathbb{Q})$. Set $D=\{-\dim P,1-\dim P,\ldots,0\}$, and define $\delta:C(\mathcal{A})\to D$ by $\delta([X])=-\dim\operatorname{supp}X$. Then $(\delta,D,\leqslant)$ is a *weak stability condition* on $\mathcal{A}$, and $X\in\mathcal{A}$ is τ-semistable if X is *pure*. The δ Harder–Narasimhan filtration of X in $\mathcal{A}$ is its *torsion filtration*.

(c) For $\mathcal{A} = \mathrm{coh}(P)$ and $K(\mathcal{A})$ as in (b), define G to be the set of monic real polynomials $t^d + a_{d-1}t^{d-1} + \cdots + a_0$ of degree $d \leqslant \dim P$. Define a total order '$\leqslant$' on G by $p \leqslant q$ if either $\deg p > \deg q$, or $\deg p = \deg q$ and $p(t) \leqslant q(t)$ for all $t \gg 0$.

Let L be an ample line bundle on P, and define $\gamma : C(\mathcal{A}) \to G$ by $\gamma([X]) = P_X(t)/l_X$, where $P_X(t)$ is the Hilbert polynomial of X w.r.t. L, with leading coefficient l_X.

Then $(\gamma, G, \leqslant)$ is a *stability condition* on $\mathcal{A}$, and $X \in \mathcal{A}$ is τ-(semi)stable if and only if it is *Gieseker (semi)stable*. Note that X τ-semistable implies X *pure*, we don't need purity as an extra assumption.

Let $(\tau, T \leqslant)$ be a weak stability condition, and for $\alpha \in C(\mathcal{A})$ define $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}, \mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$ to be the sets of $[X] \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ with X τ-(semi)stable. Call $(\tau, T, \leqslant)$ *permissible* if $\mathcal{A}$ is noetherian and τ-artinian and $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ is constructible for all $\alpha \in C(\mathcal{A})$.

Examples: any weak stability condition on mod-$\mathbb{K}Q$ is permissible. Gieseker stability $(\gamma, G, \leqslant)$ on coh(P) is permissible.

For $(I, \preceq)$ a poset and $\kappa : I \to K(\mathcal{A})$ a map, define $\mathcal{M}_{\mathrm{ss}}, \mathcal{M}_{\mathrm{st}}(I, \preceq, \kappa, \tau)_{\mathcal{A}}$ to be the subsets of $[(\sigma, \iota, \pi)]$ in $\mathfrak{M}(I, \preceq)(\mathbb{K})$ with $\sigma(\{i\})$ τ-*(semi)stable* and $[\sigma(\{i\})] = \kappa(i)$ in $K(\mathcal{A})$ for all $i \in I$. They are constructible.

3. Algebras of constructible functions

Recall that $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is an *algebra*, with associative, noncommutative multiplication $*$. For permissible $(\tau, T, \leqslant)$, let $\delta_{\mathrm{ss}}^{\alpha}(\tau)$ in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\delta_{\mathrm{ss}}(I, \preceq, \kappa, \tau) \in \mathrm{CF}(\mathfrak{M}(I, \preceq)_{\mathcal{A}})$ be the *characteristic functions* of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ and $\mathcal{M}_{\mathrm{ss}}(I, \preceq, \kappa, \tau)_{\mathcal{A}}$. Define $\mathcal{H}_{\tau}^{\mathrm{pa}}, \mathcal{H}_{\tau}^{\mathrm{to}}$ to be the subspaces of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ spanned by $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{ss}}(I, \preceq, \kappa, \tau)$ for all $(I, \preceq, \kappa)$, with $\preceq$ a total order for $\mathcal{H}_{\tau}^{\mathrm{to}}$.

Then $\mathcal{H}_{\tau}^{\mathrm{to}} \subseteq \mathcal{H}_{\tau}^{\mathrm{pa}}$ are *subalgebras* of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, and $\mathcal{H}_{\tau}^{\mathrm{to}}$ is generated as an algebra by the $\delta_{\mathrm{ss}}^{\alpha}(\tau)$ for $\alpha \in C(\mathcal{A})$.

There are also *stack function* versions $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau), \bar{\delta}_{\mathrm{ss}}(I, \preceq, \kappa, \tau), \bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$.

Define $\mathcal{L}_\tau^{\mathrm{pa}}, \mathcal{L}_\tau^{\mathrm{to}}$ to be the intersections of $\mathcal{H}_\tau^{\mathrm{pa}}, \mathcal{H}_\tau^{\mathrm{to}}$ with the Lie algebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ *supported on indecomposables*. They are *Lie algebras*. For $\alpha \in C(\mathcal{A})$, define

$$\epsilon^\alpha(\tau) = \sum_{\substack{\alpha_1,\dots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\dots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha),\,\forall i}} \frac{(-1)^{n-1}}{n}\cdot \delta_{\mathrm{ss}}^{\alpha_1}(\tau)*\dots*\delta_{\mathrm{ss}}^{\alpha_n}(\tau). \quad (1)$$

This is *invertible* combinatorially: we have

$$\delta_{\mathrm{ss}}^\alpha(\tau) = \sum_{\substack{\alpha_1,\dots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\dots+\alpha_n=\alpha,\\ \tau(\alpha_i)=\tau(\alpha),\,\forall i}} \frac{1}{n!}\cdot \epsilon^{\alpha_1}(\tau)*\dots*\epsilon^{\alpha_n}(\tau). \quad (2)$$

For $[X]\in\mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{K})$ we have

- $\epsilon^\alpha(\tau)([X])=1$ if X is τ-*stable*,
- $\epsilon^\alpha(\tau)([X])=0$ is X is τ-*unstable* or *decomposable*,
- $\epsilon^\alpha(\tau)([X])\in\mathbb{Q}$ if X is *strictly* τ-*semistable* and *indecomposable*.

Therefore $\epsilon^\alpha(\tau) \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, so $\epsilon^\alpha(\tau) \in \mathcal{L}_\tau^{\mathrm{to}}$. By (1), (2) the $\delta_{\mathrm{ss}}^\alpha(\tau), \epsilon^\alpha(\tau)$ generate the same subalgebra $\mathcal{H}_\tau^{\mathrm{to}}$ of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, so the $\epsilon^\alpha(\tau)$ are *alternative generators* for $\mathcal{H}_\tau^{\mathrm{to}}$. It follows that $\mathcal{L}_\tau^{\mathrm{to}}$ is the Lie subalgebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\epsilon^\alpha(\tau)$ for $\alpha \in C(\mathcal{A})$, and $\mathcal{H}_\tau^{\mathrm{to}} \cong U(\mathcal{L}_\tau^{\mathrm{to}})$.

Similarly, we can construct a spanning set for $\mathcal{L}_\tau^{\mathrm{pa}}$ and show $\mathcal{H}_\tau^{\mathrm{pa}} \cong U(\mathcal{L}_\tau^{\mathrm{pa}})$. We can also define alternative spanning sets for $\mathcal{H}_\tau^{\mathrm{pa}}$ in terms of τ-stable or indecomposable τ-semistable objects, with change of basis formulae relating the spanning sets. There are stack function analogues $\bar{\epsilon}^\alpha(\tau)$ in $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}), \ldots$ of all this.

4. Change of weak stability condition

Let $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be different weak stability conditions on $\mathcal{A}$, e.g. Gieseker stability on coh(P) w.r.t. different ample line bundles $L,\tilde L$ on P. Then we prove a *universal formula*

$$\delta_{\mathrm{ss}}^{\alpha}(\tilde\tau)=\sum_{\substack{\alpha_1,\ldots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha}} \begin{array}{l} S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\\ \delta_{\mathrm{ss}}^{\alpha_1}(\tau)*\cdots*\delta_{\mathrm{ss}}^{\alpha_n}(\tau).\end{array} \tag{3}$$

Here $S(\cdots)$ are *explicit combinatorial coefficients* equal to $1,0$ or -1, depending on the orderings of $\tau(\alpha_i)$ and $\tilde\tau(\alpha_i)$. There are problems with whether (3) has *finitely many nonzero terms*. This is true if $\mathcal{A}=$ mod-$\mathbb{K}Q$ or $\mathcal{A}=$ coh(P) for $\dim P\leqslant 2$.

Sketch proof: Say $\tilde\tau$ *dominates* τ if $\tau(\alpha)\leqslant\tau(\beta)$ implies $\tilde\tau(\alpha)\leqslant\tilde\tau(\beta)$ for $\alpha,\beta\in C(\mathcal{A})$. Then for $\alpha\in C(\mathcal{A})$ we have

$$\delta_{\rm ss}^\alpha(\tilde\tau)=\sum_{\substack{\alpha_1,\ldots,\alpha_n\in C(\mathcal{A}):\,\alpha_1+\cdots+\alpha_n=\alpha,\\ \tilde\tau(\alpha_i)=\tilde\tau(\alpha)\,\forall i,\,\tau(\alpha_1)>\cdots>\tau(\alpha_n)}}\delta_{\rm ss}^{\alpha_1}(\tau)*\cdots*\delta_{\rm ss}^{\alpha_n}(\tau). \tag{4}$$

To prove (4), let $X\in\mathcal{A}$ have τ Harder–Narasimhan filtration $0=A_0\subset\cdots\subset A_n=X$ with τ-semistable factors $S_i=A_i/A_{i-1}$, and set $\alpha_i=[S_i]$ in $C(\mathcal{A})$. Then X is $\tilde\tau$-semistable iff $\tilde\tau(\alpha_i)=\tilde\tau(\alpha)$ for all i, and $\delta_{\rm ss}^{\alpha_1}(\tau)*\cdots*\delta_{\rm ss}^{\alpha_n}(\tau)$ is the characteristic function of all $[X]$ with τ Harder–Narasimhan filtrations with these $\alpha_1,\ldots,\alpha_n$.

We can combinatorially invert (4) to write $\delta_{\rm ss}^\alpha(\tau)$ in terms of $\delta_{\rm ss}^{\alpha_i}(\tilde\tau)$. This gives two special cases of (3). For the general case, we find a weak stability condition $(\hat\tau,\hat T,\leqslant)$ dominating both $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$ and use (4) to write $\delta_{\rm ss}^\beta(\hat\tau)$ in terms of $\delta_{\rm ss}^\gamma(\tau)$ and its inverse to write $\delta_{\rm ss}^\alpha(\tilde\tau)$ in terms of $\delta_{\rm ss}^\beta(\hat\tau)$. The argument uses associativity of $*$. The stack function analogue also holds. $\square$

Now (3) shows $\delta^{\alpha}_{\mathrm{ss}}(\tilde\tau)$ lies in the subalgebra of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\delta^{\beta}_{\mathrm{ss}}(\tau)$, and vice versa. Thus $\mathcal{H}^{\mathrm{to}}_{\tau}=\mathcal{H}^{\mathrm{to}}_{\tilde\tau}$. Similarly, the (Lie) algebras $\mathcal{H}^{\mathrm{pa}}_{\tau},\mathcal{H}^{\mathrm{to}}_{\tau},\mathcal{L}^{\mathrm{pa}}_{\tau},\mathcal{L}^{\mathrm{to}}_{\tau}$ and $\bar{\mathcal{H}}^{\mathrm{pa}}_{\tau},\bar{\mathcal{H}}^{\mathrm{to}}_{\tau},\bar{\mathcal{L}}^{\mathrm{pa}}_{\tau},\bar{\mathcal{L}}^{\mathrm{to}}_{\tau}$ are *independent of the choice of* $(\tau,T,\leqslant)$.

Combining (1), (2) and (3) gives

$$\epsilon^{\alpha}(\tilde\tau)=\sum_{\substack{\alpha_1,\ldots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha}} \begin{array}{l} U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\\ \epsilon^{\alpha_1}(\tau)*\cdots*\epsilon^{\alpha_n}(\tau),\end{array} \tag{5}$$

for combinatorial coefficients $U(\cdots)\in\mathbb{Q}$. We rewrite (5) as a *Lie algebra identity*

$$\begin{array}{l}\epsilon^{\alpha}(\tilde\tau)=\epsilon^{\alpha}(\tau)+\mathbb{Q}\text{-linear combination}\\ \text{of commutators of }\epsilon^{\alpha_1}(\tau),\ldots,\epsilon^{\alpha_n}(\tau),\end{array} \tag{6}$$

where a *commutator* is
$[\epsilon^{\alpha}(\tau),\epsilon^{\beta}(\tau)]=\epsilon^{\alpha}(\tau)*\epsilon^{\beta}(\tau)-\epsilon^{\beta}(\tau)*\epsilon^{\alpha}(\tau)$, $[\epsilon^{\alpha}(\tau),[\epsilon^{\beta}(\tau),\epsilon^{\gamma}(\tau)]]$, and so on.

5. Invariants counting τ-semistables

Recall from first seminar: let Υ be a *motivic invariant* of $\mathbb{K}$-varieties with values in a $\mathbb{Q}$-algebra Λ, $\ell = \Upsilon(\mathbb{K})$, ℓ and $\ell^k - 1$, $k \geqslant 1$ invertible in Λ. We extend Υ uniquely to $\Upsilon'(\mathfrak{F})$ for finite type $\mathbb{K}$-stacks $\mathfrak{F}$, such that $\Upsilon'([X/G]) = \Upsilon(X)\Upsilon(G)^{-1}$ for X a variety and G a special $\mathbb{K}$-group.

Example: $\Upsilon(X)$ can be the *virtual Poincaré polynomial* $P_X(z)$, Λ the $\mathbb{Q}$-algebra of rational functions in z.

For such Υ, Λ, define a $\mathbb{Q}$-linear map
$\Pi_\Lambda : \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}) \to \Lambda$ by
$\Pi_\Lambda : [(\mathfrak{R}, \rho)] \mapsto \Upsilon'(\mathfrak{R})$.

If $(\tau, T, \leqslant)$ is a permissible weak stability condition and $\alpha \in C(\mathcal{A})$, define *invariants* $I_{\rm ss}^\alpha(\tau) = \Pi_\Lambda(\bar\delta_{\rm ss}^\alpha(\tau)) = \Upsilon'(\mathrm{Obj}_{\rm ss}^\alpha(\tau))$ and $J^\alpha(\tau)^\Lambda = (\ell - 1)\Pi_\Lambda(\bar\epsilon^\alpha(\tau))$ in Λ. Since $\bar\epsilon^\alpha(\tau) \in \mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$, can show $J^\alpha(\tau)^\Lambda$ lies in a certain subalgebra Λ° of Λ in which $\ell - 1$ is not invertible.

There is a $\mathbb{Q}$-algebra morphism $\pi : \Lambda^\circ \to \Omega$ with $\pi(\ell) = 1$, which projects virtual Poincaré polynomials to Euler characteristics. Set $J^\alpha(\tau)^\Omega = \pi\bigl(J^\alpha(\tau)^\Lambda\bigr)$.

Interpret $I_{\rm ss}^\alpha(\tau), J^\alpha(\tau)^\Lambda, J^\alpha(\tau)^\Omega$ as different invariants 'counting' τ-semistables in class α in $C(\mathcal{A})$.

From second seminar: if $\mathrm{Ext}^i(X,Y)=0$ for all $i>1$ and $X,Y\in\mathcal{A}$ then there is a biadditive $\chi: K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$ with

$$\dim\mathrm{Hom}(X,Y)-\dim\mathrm{Ext}^1(X,Y)=\chi([X],[Y])$$

for all $X,Y\in\mathcal{A}$. This holds for $\mathcal{A}=$ mod-$\mathbb{K}Q$ and $\mathcal{A}=\mathrm{coh}(P)$, P smooth curve. Then we construct an *algebra morphism* $\Phi^\Lambda:\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})\to A(\mathcal{A},\Lambda,\chi)$ to an explicit algebra $A(\mathcal{A},\Lambda,\chi)$. Suppose $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$ are permissible weak stability conditions on $\mathcal{A}$. Applying Φ^Λ to the stack function analogue of (3) above gives:

$$I_{\mathrm{ss}}^\alpha(\tilde\tau)=\sum_{\substack{\alpha_1,\ldots,\alpha_n\in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha}} \begin{array}{l} S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\\ \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\alpha_j,\alpha_i)}\cdot \\ \prod_{i=1}^n I_{\mathrm{ss}}^{\alpha_i}(\tau).\end{array} \qquad (7)$$

We can also prove that (7) holds if $\mathcal{A} = \mathrm{coh}(P)$ for P a *smooth projective surface with K_P^{-1} semiample*, even though Φ^Λ is *not* a morphism in this case. If $\tilde\tau$ dominates τ then applying Φ^Λ to the stack function analogue of (4) above yields:

$$I_{\mathrm{ss}}^{\alpha}(\tilde\tau) = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\ \alpha_1+\cdots+\alpha_n=\alpha, \\ \tilde\tau(\alpha_i)=\tilde\tau(\alpha)\ \forall i,\ \tau(\alpha_1)>\cdots>\tau(\alpha_n)}} \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\alpha_j,\alpha_i)}. \prod_{i=1}^{n} I_{\mathrm{ss}}^{\alpha_i}(\tau). \tag{8}$$

This is because we can show using Serre duality and $\tau(\alpha_1) > \cdots > \tau(\alpha_n)$ that all the relevant Ext^2 groups between terms in (4) vanish, so we reduce to the case $\mathrm{Ext}^i(X,Y) = 0$ for all $i > 1$ and $X, Y \in \mathcal{A}$. We can then prove (7) from (8) in the same way that we proved (3) from (4).

6. Sheaves on Calabi–Yau 3-folds

Recall from second seminar: if $\mathcal{A}=\mathrm{coh}(P)$ for P a *Calabi–Yau* 3-*fold* then for biadditive $\bar\chi:K(\mathcal{A})\times K(\mathcal{A})\rightarrow\mathbb{Z}$ and all X,Y in $\mathcal{A}$ we have

$$\begin{aligned}&\dim\mathrm{Hom}(X,Y)-\dim\mathrm{Ext}^1(X,Y)-\\&\dim\mathrm{Hom}(Y,X)+\dim\mathrm{Ext}^1(Y,X)=\bar\chi([X],[Y]).\end{aligned}\tag{9}$$

We construct $\Psi^\Omega:\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow C(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$, a *Lie algebra morphism* to an explicit algebra. Let $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be permissible weak stability conditions on $\mathcal{A}$. If $\alpha\in C(\mathcal{A})$ then $\bar\epsilon^\alpha(\tau)\in\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, and $\Psi^\Omega(\bar\epsilon^\alpha(\tau))=J^\alpha(\tau)^\Omega c^\alpha$. Applying Ψ^Ω to (5), which is a Lie algebra identity as in (6), yields:

$$\begin{aligned}J^\alpha(\tilde\tau)^\Omega=\sum_{\substack{\text{iso. classes}\\\text{of }\Gamma,I,\kappa}}&V(\Gamma,I,\kappa,\tau,\tilde\tau)\cdot\\&\prod_{i\in I}J^{\kappa(i)}(\tau)^\Omega\cdot\\&\prod_{\substack{\text{edges}\\ i\rightarrow j\text{ in }\Gamma}}\bar\chi(\kappa(i),\kappa(j)).\end{aligned}\tag{10}$$

Here Γ is a *connected, simply-connected digraph* with *vertices* I, $\kappa:I\rightarrow C(\mathcal{A})$ has $\sum_{i\in I}\kappa(i)=\alpha$, and $V(\cdots)\in\mathbb{Q}$ are *explicit combinatorial coefficients*, depending on orientation of Γ only up to sign.

Remarks: • I haven't proved (10) has *only finitely many nonzero terms*. But can find $\tau=\tau_0,\tau_1,...,\tau_n=\tilde\tau$ with finitely many terms going from τ_{i-1} to τ_i, $i=1,\ldots,n$.

• (10) expresses $J^\alpha(\tilde\tau)^\Omega$ in terms of invariants $J^\beta(\tau)^\Omega$ of the *same type*. This is a special feature of the C–Y 3-fold case. In general, we can only write $I_{\rm ss}(I,\preceq,\kappa,\tilde\tau)$ as a linear combination of $I_{\rm ss}(J,\lesssim,\lambda,\tau)$ for posets $(J,\lesssim)$ larger than $(I,\preceq)$.

• The $J^\alpha(\tau)^\Omega$ are *not* expected to be *unchanged by deformations of X*, as Donaldson–Thomas invariants are.

Conjecture: there exists an *extension* of D–T invariants to the stable≠semistable case, which are deformation-invariant, and transform according to (10).

• The form of (10) as a sum over graphs Γ emerges combinatorially in a bizarre way. But it is natural in the *mirror picture* of counting SL 3-folds, when one SL 3-fold decays into a tree of intersecting SL 3-folds as the complex structure deforms.

Conjecture: there exist invariants counting SL 3-folds in class $\alpha \in H_3(M, \mathbb{Z})$ in a C–Y 3-fold M, which are independent of the Kähler class, and transform according to (10) under deformation of complex structure.

• The sum over Γ in (10) looks like a sum of *Feynman diagrams*. I think there is some *new physics* behind this, to do with Π-*stability*.