Kuranishi (co)homology: a new tool in symplectic geometry. II. Kuranishi (co)homology

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These slides available at

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II.1. Introduction

Let Y be an orbifold and R a $\mathbb{Q}$-algebra. We will define *Kuranishi homology* $KH_*(Y;R)$, a homology theory of Y with coefficients in R. It is the homology of a chain complex $\bigl(KC_*(Y;R),\partial\bigr)$, spanned by isomorphism classes $[X,\boldsymbol{f},\boldsymbol{G}]$ of triples $(X,\boldsymbol{f},\boldsymbol{G})$, for X a compact, oriented Kuranishi space with boundary and corners, $\boldsymbol{f}:X\rightarrow Y$ a strongly smooth map, and $\boldsymbol{G}$ some *gauge-fixing data* for $(X,\boldsymbol{f})$. The boundary operator is

$$\partial:[X,\boldsymbol{f},\boldsymbol{G}]\mapsto[\partial X,\boldsymbol{f}|_{\partial X},\boldsymbol{G}|_{\partial X}].$$

For Kuranishi cohomology, we add *co-gauge-fixing data* $\boldsymbol{C}$.

II.2. Why we need gauge-fixing data

A naïve guess for how to define Kuranishi homology is to take chains $KC_k^{\text{na}}(Y;R)$ spanned over R by $[X,\boldsymbol{f}]$, where X is a compact, oriented Kuranishi space with boundary and corners, $\operatorname{vdim} X=k$, and $\partial: KC_k^{\text{na}}(Y;R)\rightarrow KC_{k-1}^{\text{na}}(Y;R)$ to be $\partial : [X,\boldsymbol{f}] \mapsto [\partial X,\boldsymbol{f}|_{\partial X}]$, and $KH_*^{\text{na}}(Y;R)$ to be the homology of $\bigl(KC_*^{\text{na}}(Y;R),\partial\bigr)$. Unfortunately this yields $KH_*^{\text{na}}(Y;R)=\{0\}$ for all Y,R. Here is an example which shows why.

Take Y to be a point $\{0\}$, and for any Kuranishi space X write $\boldsymbol{\pi}: X \rightarrow Y$ for the trivial projection.

Let $L \rightarrow \mathbb{CP}^1$ be the line bundle $\mathcal{O}(1)$. Define a compact, oriented Kuranishi space X_k for $k \in \mathbb{Z}$ to be $\mathbb{CP}^1$ with global Kuranishi neighbourhood $(\mathbb{CP}^1, L^k, 0, \mathrm{id}_{\mathbb{CP}^1})$, with obstruction bundle $L^k \rightarrow \mathbb{CP}^1$. Then $\mathrm{vdim}\, X_k = 0$, and $\partial X_k = \emptyset$, so $[X_k, \boldsymbol{\pi}]$ defines a class $[[X_k, \boldsymbol{\pi}]] \in KH_0^{\mathrm{na}}(Y; R)$. We shall show that $[[X_k, \boldsymbol{\pi}]] = k[[Y, \mathrm{id}_Y]]$, and that $[[X_k, \boldsymbol{\pi}]]$ is independent of k. Thus $[[Y, \mathrm{id}_Y]] = 0$. But this is the identity in $KH_*^{\mathrm{na}}(Y; R)$, so $KH_*^{\mathrm{na}}(Y; R) = \{0\}$.

When $k \geqslant 0$ we can choose a generic smooth section s of $L^k \to \mathbb{CP}^1$ which has exactly k zeroes $x_1, \ldots, x_k$, each of multiplicity 1. Let t be the coordinate on $[0,1]$. Then ts is a section of $L^k \to [0,1] \times \mathbb{CP}^1$, with $(ts)^{-1}(0) = \{0\} \times \mathbb{CP}^1 \cup [0,1] \times \{x_1, \ldots, x_k\}$, and $\bigl([0,1] \times \mathbb{CP}^1, L^k, ts, \mathrm{id}_{(ts)^{-1}(0)}\bigr)$ is a Kuranishi neighbourhood on $(ts)^{-1}(0)$, making it a Kuranishi space of virtual dimension 1. By taking the boundary of this we find

$$\begin{aligned} \bigl[[X_k, \boldsymbol{\pi}]\bigr] &= \bigl[[\{x_1, \ldots, x_k\}, \boldsymbol{\pi}]\bigr] \\ &= k\bigl[[Y, \mathrm{id}_Y]\bigr] \end{aligned}$$

in $KH_0^{\mathrm{na}}(Y; R)$.

Write $[z_0, z_1]$ for the homogeneous coordinates on $\mathbb{CP}^1$, and define

$$V = \bigl\{(t, [z_0, z_1]) \in \mathbb{R} \times \mathbb{CP}^1 : \\ \min(|z_0|^2, |z_1|^2) \max(|z_0|^2, |z_1|^2)^{-1} \leqslant t \leqslant 2\bigr\}.$$

Then V is a compact oriented 3-manifold with corners, and ∂V is the disjoint union of three pieces, a copy of $\mathbb{CP}^1$ with $t = 2$, the hemisphere $H_+ = \bigl\{[z_0, z_1] \in \mathbb{CP}^1 : |z_0| \leqslant |z_1|\bigr\}$ with $t = |z_0|^2/|z_1|^2$, and the hemisphere $H_- = \bigl\{[z_0, z_1] \in \mathbb{CP}^1 : |z_0| \geqslant |z_1|\bigr\}$ with $t = |z_1|^2/|z_0|^2$.

Define the Kuranishi space W_k for $k \in \mathbb{Z}$ to be V with Kuranishi neighbourhood $(V, \pi^*(L^k), 0, \mathrm{id}_V)$, where $\pi : V \to \mathbb{CP}^1$ is the projection. Define Kuranishi spaces X_+, X_- to be H_+, H_- with Kuranishi neighbourhoods $(H_\pm, L^0, 0, \mathrm{id}_{H_\pm})$. Now the line bundles $L^k \to H^\pm$ are for $k \in \mathbb{Z}$ are isomorphic to $L^0 \to H^\pm$. Thus there is an isomorphism of oriented Kuranishi spaces $\partial W_k \cong X_k \amalg -X_+ \amalg -X_-$, so in $KH_0^{\mathrm{na}}(Y;R)$ we have

$$\big[[X_k, \boldsymbol{\pi}]\big] = \big[[X_+, \boldsymbol{\pi}] + [X_-, \boldsymbol{\pi}]\big],$$

and $\big[[X_k, \boldsymbol{\pi}]\big]$ is independent of k.

It follows easily that $KH_*^{\mathrm{na}}(Y;R) = \{0\}$ for all Y, R. What went wrong?

The problem is with the Kuranishi space $C = \partial X_+ = -\partial X_-$. This is a compact Kuranishi space with $\operatorname{vdim} C = -1$, topological space $\mathcal{S}^1$, obstruction bundle $\mathbb{R}^2 \times \mathcal{S}^1 \to \mathcal{S}^1$, and obstruction map $s \equiv 0$. It has a *large automorphism group*, including automorphisms which fix $\mathcal{S}^1$ but act on the obstruction bundle $\mathbb{R}^2 \times \mathcal{S}^1$ by a map $\mathcal{S}^1 \to \mathrm{SO}(2)$ of degree $d \in \mathbb{Z}$. By cutting X_k into $X_+ \amalg X_-$ along C, 'twisting' C by such an automorphism, and gluing X_+, X_- together again we can make X_{k+d}.

The problem is caused by chains $[X, \boldsymbol{f}]$ with *infinite automorphism groups*. So, we add extra data $\boldsymbol{G}$ to chains to make $\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})$ finite, even if $\mathrm{Aut}(X, \boldsymbol{f})$ is infinite.

II.3. Gauge-fixing data

Chains in $KC_k(Y;R)$ will be R-linear combinations of isomorphism classes $[X,\boldsymbol{f},\boldsymbol{G}]$, where X is a compact, oriented Kuranishi space with corners, $\operatorname{vdim} X=k$, $\boldsymbol{f}:X\rightarrow Y$ is strongly smooth, and $\boldsymbol{G}$ is a (very nonunique) choice of *gauge fixing data* for $(X,\boldsymbol{f})$.

Here are the most important properties of $\boldsymbol{G}$:

(a) Every $(X,\boldsymbol{f})$ has a (nonunique) choice of gauge-fixing data $\boldsymbol{G}$.

(b) The automorphism group $\operatorname{Aut}(X,\boldsymbol{f},\boldsymbol{G})$ is finite for all $(X,\boldsymbol{f},\boldsymbol{G})$.

(c) If $\boldsymbol{G}$ is gauge-fixing data for $(X,\boldsymbol{f})$, it has a restriction $\boldsymbol{G}|_{\partial X}$, which is gauge-fixing data for $(\partial X,\boldsymbol{f}|_{\partial X})$.

(d) There is a natural, orientation-reversing involution $\boldsymbol{\sigma} : \partial^2 X \to \partial^2 X$. Suppose $\boldsymbol{H}$ is gauge-fixing data for $(\partial X, \boldsymbol{f}|_{\partial X})$. Then there exists gauge-fixing data $\boldsymbol{G}$ for $(X, \boldsymbol{f})$ with $\boldsymbol{G}|_{\partial X} = \boldsymbol{H}$ iff $\boldsymbol{H}|_{\partial^2 X}$ is invariant under $\boldsymbol{\sigma}$.
(e) There are good, functorial notions of products, pushforwards, and pullbacks of (co-)gauge-fixing data, as one needs to make cup and cap products, pushforwards, and pullbacks in (co)homology work at the (co)chain level.

It does not matter exactly what (co-)gauge-fixing data is, as long as it has these properties (a)–(e). My definition of gauge-fixing data $\boldsymbol{G}$ for $(X, \boldsymbol{f})$ involves a finite cover of X by Kuranishi neighbourhoods (V^i, E^i, s^i, ψ^i) for $i \in I$, and finite maps $G^i : E^i \to \mathbb{R}^\infty$ for $i \in I$, satisfying complex conditions. The fact that the G^i are *finite* maps (that is, $(G^i)^{-1}(p)$ is finite for all $p \in \mathbb{R}^\infty$) ensures that $\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})$ is finite.

II.4. Kuranishi homology

Let Y be an orbifold. Consider triples $(X, \boldsymbol{f}, \boldsymbol{G})$, for X a compact oriented Kuranishi space, $\boldsymbol{f} : X \to Y$ strongly smooth, and $\boldsymbol{G}$ gauge-fixing data for $(X, \boldsymbol{f})$. Write $[X, \boldsymbol{f}, \boldsymbol{G}]$ for the isomorphism class of $(X, \boldsymbol{f}, \boldsymbol{G})$. Let R be a $\mathbb{Q}$-algebra.

For each $k \in \mathbb{Z}$, define $KC_k(Y; R)$ to be the R-module of finite R-linear combinations of $[X, \boldsymbol{f}, \boldsymbol{G}]$ with $\operatorname{vdim} X = k$, with the relations:

(i) Write $-X$ for X with the opposite orientation. Then

$$[X, \boldsymbol{f}, \boldsymbol{G}] + [-X, \boldsymbol{f}, \boldsymbol{G}] = 0$$

in $KC_k(Y; R)$, for all $[X, \boldsymbol{f}, \boldsymbol{G}]$.

(ii) Suppose there exists an isomorphism $(\boldsymbol{a},\boldsymbol{b}) : (X,\boldsymbol{f},\boldsymbol{G}) \rightarrow (X,\boldsymbol{f},\boldsymbol{G})$ which reverses the orientation of X. Then $[X,\boldsymbol{f},\boldsymbol{G}] = 0$ in $KC_k(Y;R)$.

(iii) Let $[X,\boldsymbol{f},\boldsymbol{G}]$ be an isomorphism class. Suppose that X may be written as a disjoint union $X = X_+ \amalg X_-$ of compact oriented Kuranishi spaces in a way compatible with $\boldsymbol{G}$. Then in $KC_k(Y;R)$ we have

$$[X,\boldsymbol{f},\boldsymbol{G}] = [X_+,\boldsymbol{f}|_{X_+},\boldsymbol{G}|_{X_+}] + [X_-,\boldsymbol{f}|_{X_-},\boldsymbol{G}|_{X_-}].$$

(iv) Suppose Γ is a finite subgroup of $\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})$. Then $\tilde{X} = X/\Gamma$ is a compact oriented Kuranishi space, with $\boldsymbol{\pi} : X \to \tilde{X}$, and $\boldsymbol{f}, \boldsymbol{G}$ push down to $\boldsymbol{\pi}_*(\boldsymbol{f}), \boldsymbol{\pi}_*(\boldsymbol{G})$. We require that

$$\bigl[X/\Gamma, \boldsymbol{\pi}_*(\boldsymbol{f}), \boldsymbol{\pi}_*(\boldsymbol{G})\bigr] = \frac{1}{|\Gamma|}\bigl[X, \boldsymbol{f}, \boldsymbol{G}\bigr]$$

in $KC_k(Y; R)$.

We must take R to be a $\mathbb{Q}$-*algebra* so that $1/|\Gamma|$ makes sense in R.

Elements of $KC_k(Y; R)$ will be called *Kuranishi chains*.

Define the *boundary operator* ∂ or $\partial_k : KC_k(Y;R) \to KC_{k-1}(Y;R)$ by

$$\partial : \sum_{a\in A} \rho_a [X_a, \boldsymbol{f}_a, \boldsymbol{G}_a] \longmapsto \sum_{a\in A} \rho_a [\partial X_a, \boldsymbol{f}_a|_{\partial X_a}, \boldsymbol{G}_a|_{\partial X_a}],$$

where A is a finite indexing set and $\rho_a \in R$ for $a \in A$. Using a natural orientation-reversing involution $\boldsymbol{\sigma} : \partial^2 X_a \to \partial^2 X_a$ and relation (ii), we find that $\partial^2 = 0$.

— Explain $\boldsymbol{\sigma}$ on the board —

Define the k^{th} *Kuranishi homology group* $KH_k(Y;R)$ to be

$$\frac{\mathrm{Ker}\bigl(\partial_k : KC_k(Y;R) \to KC_{k-1}(Y;R)\bigr)}{\mathrm{Im}\bigl(\partial_{k+1} : KC_{k+1}(Y;R) \to KC_k(Y;R)\bigr)}.$$

Let Y, Z be orbifolds, and $h : Y \to Z$ a smooth map. Define the *pushforward* $h_* : KC_k(Y;R) \to KC_k(Z;R)$ on Kuranishi chains by

$$h_* : [X, \boldsymbol{f}, \boldsymbol{G}] \mapsto [X, h \circ \boldsymbol{f}, h_*(\boldsymbol{G})].$$

Then $h_* \circ \partial = \partial \circ h_*$, so h_* induces $h_* : KH_k(Y;R) \to KH_k(Z;R)$ on Kuranishi homology.

II.5. Isomorphism with $H_*^{\mathrm{si}}(Y;R)$

For $k \geqslant 0$, the *k-simplex* Δ_k is

$$\Delta_k = \{(x_0,\ldots,x_k) \in \mathbb{R}^{k+1} : x_i \geqslant 0, \quad x_0 + \ldots + x_k = 1\}.$$

Let Y be an orbifold, and R a $\mathbb{Q}$-algebra. Write $C_k^{\mathrm{si}}(Y;R)$ for the R-module spanned by smooth maps $\sigma : \Delta_k \to Y$. By identifying $\partial\Delta_k$ with the disjoint union of $k+1$ copies of Δ_{k-1} we define $\partial : C_k^{\mathrm{si}}(Y;R) \to C_{k-1}^{\mathrm{si}}(Y;R)$. *Singular homology* $H_*^{\mathrm{si}}(Y;R)$ is the homology of $(C_*^{\mathrm{si}}(Y;R), \partial)$.

We define a morphism
$\Pi_{\mathrm{si}}^{\mathrm{Kh}} : C_k^{\mathrm{si}}(Y;R) \to KC_k(Y;R)$ by
$\Pi_{\mathrm{si}}^{\mathrm{Kh}} : \sum_{a\in A} \rho_a \sigma_a \mapsto \sum_{a\in A} \rho_a \big[\Delta_k, \sigma_a, \boldsymbol{G}_{\Delta_k}\big]$.
Here $\boldsymbol{G}_{\Delta_k}$ is an explicit choice of gauge-fixing data for (Δ_k, σ_a). Then $\partial \circ \Pi_{\mathrm{si}}^{\mathrm{Kh}} = \Pi_{\mathrm{si}}^{\mathrm{Kh}} \circ \partial$, and so $\Pi_{\mathrm{si}}^{\mathrm{Kh}}$ induces R-module morphisms

$$\Pi_{\mathrm{si}}^{\mathrm{Kh}} : H_k^{\mathrm{si}}(Y;R) \to KH_k(Y;R).$$

Our main result is:

Theorem 1. *For Y an orbifold and R a $\mathbb{Q}$-algebra, $\Pi_{\mathrm{si}}^{\mathrm{Kh}} : H_k^{\mathrm{si}}(Y;R) \to KH_k(Y;R)$ is an isomorphism.*

This shows Kuranishi homology can be used instead of singular homology in applications, e.g. G–W theory, Lagrangian Floer homology.

To prove Theorem 1 we must construct an inverse $(\Pi_{\mathrm{si}}^{\mathrm{Kh}})^{-1} : KH_k^{\mathrm{Kh}}(Y;R) \to H_k^{\mathrm{si}}(Y;R)$. Basically this is a *virtual cycle construction*: $(\Pi_{\mathrm{si}}^{\mathrm{Kh}})^{-1}$ should take $[X, \boldsymbol{f}, \boldsymbol{G}]$ to a virtual chain for X. We use some ideas of Fukaya–Ono. However, ensuring the virtual chains are functorial, and compatible at boundary and corners of X, makes the proof very complex and difficult. A lot of the technical issues in [FOOO] are transferred to the proof of Theorem 1 in my set-up.

We prove Theorem 1 by taking a class α in $KH_k(Y;R)$ and representing it by cycles with better and better properties, until we get a cycle in the image of $\Pi_{\rm si}^{\rm Kh}$.
It is essential in the proof that R is a $\mathbb{Q}$-algebra, and that $\mathrm{Aut}(X,\boldsymbol{f},\boldsymbol{G})$ is always finite as in II.3(b) above. Some steps in the proof involve modifying $[X,\boldsymbol{f},\boldsymbol{G}]$ somehow, and then averaging this over $\mathrm{Aut}(X,\boldsymbol{f},\boldsymbol{G})$ to get something with the original symmetry group. Symmetries are important as they represent choices in the way you can identify chains, and so in how chains can cancel.

II.6. Kuranishi cohomology

In Kuranishi homology we used triples $(X, \boldsymbol{f}, \boldsymbol{G})$, in which $\boldsymbol{f} : X \to Y$ was strongly smooth, X was oriented, and $\boldsymbol{G}$ was gauge-fixing data. For Kuranishi cohomology we use triples $(X, \boldsymbol{f}, \boldsymbol{C})$, in which $\boldsymbol{f} : X \to Y$ is a *strong submersion*, $(X, \boldsymbol{f})$ is *cooriented*, and $\boldsymbol{C}$ is *co-gauge-fixing data*. Here a *coorientation* of a (strong) submersion is a relative orientation.

If $f : X \to Y$ is a submersion of manifolds, then a coorientation is an orientation on the fibres of the vector bundle $\mathrm{Ker}\big(\mathrm{d}f : TX \to f^*(TY)\big)$ over X.

Let Y be an orbifold. Consider triples $(X, \boldsymbol{f}, \boldsymbol{C})$, for X a compact Kuranishi space, $\boldsymbol{f} : X \to Y$ a cooriented strong submersion, and $\boldsymbol{C}$ co-gauge-fixing data for $(X, \boldsymbol{f})$. Write $[X, \boldsymbol{f}, \boldsymbol{C}]$ for the isomorphism class of $(X, \boldsymbol{f}, \boldsymbol{C})$. Let R be a $\mathbb{Q}$-algebra.

For each $k \in \mathbb{Z}$, define $KC^k(Y; R)$ to be the R-module of finite R-linear combinations of $[X, \boldsymbol{f}, \boldsymbol{C}]$ with $\operatorname{vdim} X = \dim Y - k$, with the analogues of relations (i)–(iv) in §II.4.

Define d or $\mathrm{d}_k : KC^k(Y;R) \to KC^{k+1}(Y;R)$ by

$$\mathrm{d} : \sum_{a\in A} \rho_a [X_a, \boldsymbol{f}_a, \boldsymbol{C}_a] \longmapsto \sum_{a\in A} \rho_a [\partial X_a, \boldsymbol{f}_a|_{\partial X_a}, \boldsymbol{C}_a|_{\partial X_a}].$$

Then $\mathrm{d}^2 = 0$.
Define the k^{th} *Kuranishi cohomology group* $KH^k(Y;R)$ to be

$$\frac{\mathrm{Ker}\bigl(\mathrm{d}_k : KC^k(Y;R) \to KC^{k+1}(Y;R)\bigr)}{\mathrm{Im}\bigl(\mathrm{d}_{k-1} : KC^{k-1}(Y;R) \to KC^k(Y;R)\bigr)}.$$

Remarks. In general, homology or bordism involves structures on X, such as orientations, and cohomology or cobordism involves the corresponding *relative structures* for $f : X \to Y$, such as coorientations.

Also note: for manifolds X, Y, if $f : X \to Y$ is a submersion, then $\dim X \geqslant \dim Y$. So if we defined $KC^k(Y; R)$ using $[X, f, C]$ for X a manifold with $\dim X = \dim Y - k$, f a submersion then $KC^k(Y; R) = 0$ for $k > 0$, which would be no use. However, if X is a Kuranishi space and $f : X \to Y$ a strong submersion, can have $\operatorname{vdim} X < \dim Y$. Strong submersions are easy to produce.

Let Y, Z be orbifolds, and $h : Y \to Z$ a smooth map. Define the *pull-back* $h^* : KC^k(Z;R) \to KC^k(Y;R)$ on Kuranishi chains by

$$h^* : [X, \boldsymbol{f}, \boldsymbol{C}] \mapsto [Y \times_{h,Z,\boldsymbol{f}} X, \boldsymbol{\pi}_Y, h^*(\boldsymbol{C})].$$

Here $Y \times_{h,Z,\boldsymbol{f}} X$ is the *fibre product of Kuranishi spaces*, defined as $\boldsymbol{f}$ is a strong submersion. The coorientation for $(X, \boldsymbol{f})$ pulls back to a coorientation for $(Y \times_{h,Z,\boldsymbol{f}} X, \boldsymbol{\pi}_Y)$. Then $h^* \circ \mathrm{d} = \mathrm{d} \circ h^*$, so h^* induces $h^* : KH^k(Z;R) \to KH^k(Y;R)$ on Kuranishi cohomology.

Define a *cup product* $KC^k(Y;R)\times KC^l(Y;R)\rightarrow KC^{k+l}(Y;R)$ by

$$[X,\boldsymbol{f},\boldsymbol{C}]\cup[\tilde X,\tilde{\boldsymbol{f}},\tilde{\boldsymbol{C}}]= \bigl[X\times_{\boldsymbol{f},Y,\tilde{\boldsymbol{f}}}\tilde X,\boldsymbol{\pi}_Y,\boldsymbol{C}\times_Y\tilde{\boldsymbol{C}}\bigr],$$

using fibre products of Kuranishi spaces. This is *associative* and *supercommutative*, at the cochain level, and compatible with d, so it induces $\cup: KH^k(Y;R)\times KH^l(Y;R)\rightarrow KH^{k+l}(Y;R)$.
Define *cap products* the same way.
Note: products of co-gauge-fixing data $\boldsymbol{C}\times_Y\tilde{\boldsymbol{C}}$ are very nontrivial – it was difficult to make them associative and commutative.

II.7. Poincaré duality

Now suppose Y is *oriented*, of dimension n. For $k \in \mathbb{Z}$ define $\Pi^{\mathrm{Kh}}_{\mathrm{Kch}} : KC^k(Y;R) \to KC_{n-k}(Y;R)$ by

$$\Pi^{\mathrm{Kh}}_{\mathrm{Kch}} : [X, \boldsymbol{f}, \boldsymbol{C}] \mapsto [X, \boldsymbol{f}, \boldsymbol{G_C}],$$

where $\boldsymbol{G_C}$ is gauge-fixing data for $(X, \boldsymbol{f})$ defined using $\boldsymbol{C}$ (involves an extra choice), and the combining the coorientation for $(X, \boldsymbol{f})$ and the orientation on Y gives an orientation on X. Then $\partial \circ \Pi^{\mathrm{Kh}}_{\mathrm{Kch}} = \Pi^{\mathrm{Kh}}_{\mathrm{Kch}} \circ \mathrm{d}$, so they induce morphisms $\Pi^{\mathrm{Kh}}_{\mathrm{Kch}} : KH^k(Y;R) \to KH_{n-k}(Y;R)$.

We can construct an *inverse* for $\Pi_{\mathrm{Kch}}^{\mathrm{Kh}}$ on (co)homology. On (co)chains we can't define $\Pi_{\mathrm{Kh}}^{\mathrm{Kch}} : [X, \boldsymbol{f}, \boldsymbol{G}] \mapsto [X, \boldsymbol{f}, \boldsymbol{C}_{\boldsymbol{G}}]$, since $\boldsymbol{f}$ may not be a strong submersion. Instead we define $\Pi_{\mathrm{Kh}}^{\mathrm{Kch}} : KC_{n-k}(Y;R) \to KC^k(Y;R)$ by $\Pi_{\mathrm{Kh}}^{\mathrm{Kch}} : [X, \boldsymbol{f}, \boldsymbol{G}] \mapsto [X^Y, \boldsymbol{f}^Y, \boldsymbol{C}_{\boldsymbol{G}}^Y]$. Here X^Y is X with a *new Kuranishi structure*, and $\boldsymbol{f}^Y : X^Y \to Y$ is a strong submersion, a modified version of $\boldsymbol{f}$. Basically, we add a copy of $\boldsymbol{f}^*(TY)$ to both tangent and obstruction bundles of X.

Then $\mathrm{d}\circ\Pi_{\mathrm{Kh}}^{\mathrm{Kch}}=\Pi_{\mathrm{Kh}}^{\mathrm{Kch}}\circ\partial$, so they induce morphisms on (co)homology $\Pi_{\mathrm{Kh}}^{\mathrm{Kch}}:KH_{n-k}(Y;R)\to KH^k(Y;R)$.

Theorem 2. *For Y an oriented n-orbifold and R a $\mathbb{Q}$-algebra, $\Pi_{\mathrm{Kch}}^{\mathrm{Kh}}:KH^k(Y;R)\to KH_{n-k}(Y;R)$ and $\Pi_{\mathrm{Kh}}^{\mathrm{Kch}}:KH_{n-k}(Y;R)\to KH^k(Y;R)$ are inverse, so they are both isomorphisms.*

This is *Poincaré duality* for Kuranishi (co)homology.

Now for Y an oriented n-orbifold and R a $\mathbb{Q}$-algebra we have $H^{\mathrm{si}}_{n-k}(Y;R) \cong H^k_{\mathrm{cs}}(Y;R)$, where $H^*_{\mathrm{cs}}(Y;R)$ is *compactly-supported cohomology*. Combining this with Theorems 1 and 2 gives $H^k_{\mathrm{cs}}(Y;R) \cong KH^k(Y;R)$. We can remove the assumption Y oriented, giving:

Corollary 3. *For Y an orbifold and R a $\mathbb{Q}$-algebra, we have $H^k_{\mathrm{cs}}(Y;R) \cong KH^k(Y;R)$, with $KH^k(Y;R) = 0$ for $k < 0$.*

Note: we get *compactly-supported* cohomology as chains are $[X, \boldsymbol{f}, \boldsymbol{C}]$ with X compact, so supported in a compact set $\boldsymbol{f}(X)$.

By comparing cup product with the intersection product on $H_*^{\mathrm{si}}(Y;R)$ for Y oriented, we can prove:

Theorem 4. *The isomorphisms* $H_*^{\mathrm{si}}(Y;R) \cong KH_*(Y;R)$, $H^*_{\mathrm{cs}}(Y;R) \cong KH^*(Y;R)$ *identify the cup and cap products on* $H^*_{\mathrm{cs}}(Y;R), H_*^{\mathrm{si}}(Y;R)$ *with those on* $KH^*(Y;R), KH_*(Y;R)$*.*

II.8. Conclusions.

Kuranishi (co)homology is well adapted for use in symplectic geometry, because we can turn moduli spaces of J-holomorphic curves $\bar{\mathcal{M}}$ with evaluation maps $\mathbf{ev}: \bar{\mathcal{M}} \to Y$ directly into (co)chains $[\bar{\mathcal{M}}, \mathbf{ev}, \boldsymbol{G}]$ or $[\bar{\mathcal{M}}, \mathbf{ev}, \boldsymbol{C}]$ just by choosing (co-) gauge-fixing data $\boldsymbol{G}, \boldsymbol{C}$. There is *no need to perturb moduli spaces*. Choosing (co-)gauge-fixing data is a much milder process, there is no problem in making infinitely many compatible choices.

Also, Kuranishi (co)homology is *very well behaved at the cochain level*, with cup products that are everywhere defined, associative, and supercommutative on Kuranishi cochains. This is useful in Lagrangian Floer cohomology, in which the moduli spaces $\bar{\mathcal{M}}$ are Kuranishi spaces with boundary and corners, and the boundary $\partial\bar{\mathcal{M}}$ is written as a disjoint union $\coprod_{\bar{\mathcal{M}}',\bar{\mathcal{M}}''} \pm \bar{\mathcal{M}}' \times_L \bar{\mathcal{M}}''$ of fibre products $\bar{\mathcal{M}}' \times_L \bar{\mathcal{M}}''$ of other curve moduli spaces $\bar{\mathcal{M}}', \bar{\mathcal{M}}''$.

We translate this into an *exact algebraic identity* on Kuranishi cochains, roughly of the form

$$\mathrm{d}[\bar{\mathcal{M}}, \mathrm{ev}, \boldsymbol{C}] = \sum_{\bar{\mathcal{M}}', \bar{\mathcal{M}}''} \pm [\bar{\mathcal{M}}', \mathrm{ev}', \boldsymbol{C}'] \cup_L [\bar{\mathcal{M}}'', \mathrm{ev}'', \boldsymbol{C}''].$$

We can then define a geometric A_∞ algebra directly on the completed Kuranishi cochains $\widehat{KC}^*(Y;\Lambda_{\mathrm{nov}})$, bypassing many steps in [FOOO].