Calibrated Submanifolds and the Exceptional Geometries

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To Mom and Dad, without whom none of this would have been possible.

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Preface

The knowledge of which geometry aims is the knowledge of the eternal.

– Plato

Calibrated geometry, since conception, has been tied to the exceptional geometries occurring in seven and eight dimensions. Harvey and Lawson [17, §IV], in their seminal paper on the subject, dedicate a chapter to the relationship between these two fields. In seven dimensions the relevant holonomy group is G_2 and the calibrated submanifolds are known as *associative 3-folds* and *coassociative 4-folds*, whereas in eight the group is Spin(7) and *Cayley 4-folds* form the calibrated geometry.

The dearth of concrete formulations of calibrated submanifolds in manifolds with exceptional holonomy, even in the simple cases of 7- and 8-dimensional Euclidean space, is markedly evident. The exhibition of examples in mathematics, perhaps particularly in geometry, is crucial to our understanding of the theory involved. Part I of this thesis addresses this need by presenting methods of constructing calibrated submanifolds of $\mathbb{R}^7$ and $\mathbb{R}^8$ related to the exceptional geometries and, moreover, utilising them to produce explicit descriptions of associative, coassociative and Cayley submanifolds. Much of this work has already appeared in the author's papers [38] and [39]: specifically, Chapter 4, apart from §4.2, and §5.4 form the material of the former and the latter respectively, with the addition of further examples.

Having studied what one may tentatively label the 'applied' aspects of the subject, Part 2 tackles, remaining cautious with our verbiage, more 'abstract' problems. It is concerned with *deformations* of two distinguished classes of *noncompact* coassociative 4-folds which are, in a sense, dual to one another: *asymptotically conical* (AC) and 4-folds with *conical singularities* (CS). Informally, AC submanifolds 'look like' cones near infinity, whereas CS submanifolds have a finite number of points at which they locally have the appearance of a cone near its vertex. McLean [45, §4] fully describes the deformation theory of compact coassociative 4-folds, demonstrating that the moduli space of deformations is a smooth manifold of known dimension. This result motivates our study as both AC and CS submanifolds are natural extensions from the compact case. The material in Chapter 7 on AC deformations forms the core of the author's paper [40].

The incentive for the research detailed within this dissertation is not limited to mathematics. In high-energy theoretical physics there are areas of study known as *String Theory*, *M-theory* and *F-theory*. The ultimate goal of String Theory is to combine Quantum Theory and General Relativity. The key concept is that particles are not modelled as points in space but rather as a 1-dimensional object, called a ‘string’. An unusual feature of the theory is that the universe is forced to have a dimension higher than four. The most popular version of String Theory requires the universe to have ten dimensions. However, the dimension of the universe may be eleven (for M-theory), twelve (for F-theory) or possibly even twenty-six.

String theorists hypothesise that, geometrically, the universe comprises a large, observable, 4-dimensional piece and a very small extra piece which has six, seven, eight or more dimensions. For an 11-dimensional universe the additional constituent must be a compact G_2 manifold; that is, a compact 7-dimensional Riemannian manifold with holonomy group contained in G_2. Associative and coassociative submanifolds of a G_2 manifold then have physical significance in M-theory, with a particular interest shown for their *singularities*. A similar picture may hold for a 12-dimensional universe in F-theory, where the relevant 8-dimensional piece might be a Spin(7) manifold, but the physics is as yet unclear.

An area of conjectures in String Theory is called *Mirror Symmetry*. There is a proposed geometric explanation of this result for 10-dimensional String Theory known as the *SYZ conjecture*, which involves the consideration of fibrations of a compact Calabi–Yau 3-fold by 3-dimensional calibrated submanifolds, known as *special Lagrangian 3-folds*, that are allowed to have singularities. It has been conjectured, based on physical arguments, that the analogous situation in seven dimensions is true. The hope is that the work in this document will aid in the solution of this difficult problem.

It may appear that we have spent much of this preface detailing the usefulness of the research discussed in this thesis. Although this is certainly true, it would only be honest to close with the following quotation with which the author agrees wholeheartedly.

> *The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.*
>
> – Henri Poincaré

Chapter 1

Introduction

The research presented here lies within the broader subject of *calibrated geometry*. In this chapter the basic theory underlying this topic is given. Certain types of asymptotic behaviour of submanifolds at infinity are also discussed and a section is dedicated to some elementary properties of number systems and their relationship with group theory. This final section of the chapter provides the reader with background material before the exposition of the *octonions*, or *Cayley numbers*, in Chapter 2.

The focus of the second chapter is on the exceptional Lie groups G_2 and Spin(7) and the formulation of calibrations on $\mathbb{R}^7$ and $\mathbb{R}^8$ which are associated to these groups. This allows the definition of *associative 3-folds* and *coassociative 4-folds* in $\mathbb{R}^7$ and *Cayley 4-folds* in $\mathbb{R}^8$. Moreover, it is demonstrated that there is a generalisation of these calibrations and calibrated submanifolds for particular 7- and 8-dimensional Riemannian manifolds, known as G_2 and Spin(7) manifolds respectively.

Following these two preliminary chapters, the rest of the dissertation is split into two parts each containing three chapters. Part I begins, in Chapter 3, with a review of the theory and constructions of *special Lagrangian m-folds* in $\mathbb{C}^m$ which shall be pertinent in the sequel. Chapter 4 gives construction methods and examples for associative 3-folds in $\mathbb{R}^7$. The final chapter in Part I is a similar presentation for coassociative 4-folds in $\mathbb{R}^7$ and Cayley 4-folds in $\mathbb{R}^8$.

Chapter 6, the first chapter in Part II, reviews various definitions and results from the study of analysis on *asymptotically conical* (AC) manifolds and manifolds with *conical singularities* (CS). This material is predominately based on a paper by Lockhart and McOwen [37]. The final chapters are dedicated to the study of *deformations* of AC coassociative 4-folds in $\mathbb{R}^7$ and CS coassociative 4-folds in a G_2 manifold. In Chapter 7 it is proved that an AC coassociative 4-fold, which converges with generic rate in a specified range to a cone at infinity, has a locally smooth moduli space of deformations of known dimension. In Chapter 8, three different deformation problems for CS coassociative 4-folds are studied. For each case there is a weaker result: the moduli space is locally homeomorphic to the kernel of a smooth map between smooth manifolds. However, if the obstructions in the problem are known to be zero, the moduli space is locally smooth and a lower bound is given on its dimension.

In this thesis, manifolds are taken to be smooth and nonsingular almost everywhere and submanifolds are assumed to be immersed, unless otherwise stated.

1.1 Calibrated Geometry

We define *calibrations* and *calibrated submanifolds* following the approach in [17].

Definition 1.1.1. Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane V* on M is an oriented k-dimensional vector subspace V of T_xM, for some x in M. Given an oriented tangent k-plane V on M, $g|_V$ is a Euclidean metric on V and hence, using $g|_V$ and the orientation on V, there is a natural volume form, vol_V, which is a k-form on V.

A closed k-form η on M is a *calibration* on M if $\eta|_V \leq \mathrm{vol}_V$ for all oriented tangent k-planes V on M, where $\eta|_V = \kappa \cdot \mathrm{vol}_V$ for some $\kappa \in \mathbb{R}$, so $\eta|_V \leq \mathrm{vol}_V$ if $\kappa \leq 1$. An oriented k-dimensional submanifold N of M is a *calibrated submanifold* or *η-submanifold* if $\eta|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

Calibrated submanifolds are *minimal* submanifolds [17, Theorem II.4.2]. Minimal submanifolds of $\mathbb{R}^n$ are related to *harmonic* functions, i.e. functions f satisfying $\Delta f = d^*df = 0$, by the following elementary result [35, Corollary 9].

Theorem 1.1.2. *A submanifold of $\mathbb{R}^n$, with immersion ι, is minimal if and only if ι is harmonic; that is, each component of ι mapping to $\mathbb{R}$ is harmonic.*

If η is a k-form satisfying the restrictions in Definition 1.1.1 but is *not* closed, η-submanifolds are no longer minimal, yet may still be defined. We shall return to this point in Sections 2.3.2 and 2.4.2 and, with greatest interest, in §8.5. However, the minimality of calibrated submanifolds provides the following property, as discussed in [17].

Theorem 1.1.3. *A calibrated submanifold is real analytic wherever it is nonsingular.*

One may think of the calibrated condition as corresponding to a *partial differential equation* which has solutions described by calibrated submanifolds. Given the analytic techniques employed in this thesis and the many appearances which differential equations make during the course of our study, this viewpoint is one we encourage the reader to adopt. Having chosen this perspective, the inherent difficulty we face lies within the fact that the calibrations we consider correspond to equations that are strictly *nonlinear*, for which solutions are hard to find in general.

A particular result from the theory of partial differential equations that we use is the *Cauchy–Kowalevsky Theorem* [48, Theorem B.1], which we now state.

Theorem 1.1.4 (Cauchy–Kowalevsky Theorem). *Let* $\mathbf{u} = (u_1, \ldots, u_N) = \mathbf{u}(\mathbf{x}, t)$ *be a vector-valued function of* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$. *Let* a^i_{jk} *and* b_j *be real analytic functions of* $\mathbf{z} = (\mathbf{x}, \mathbf{u})$ *in a neighbourhood of zero in* $\mathbb{R}^{n+N}$ *for* $i = 1, \ldots, n$ *and* $j, k = 1, \ldots, N$. *The system of differential equations*

$$\frac{\partial u_j}{\partial t} = b_j(\mathbf{z}) + \sum_{i=1}^{n}\sum_{k=1}^{N} a^i_{jk}(\mathbf{z})\frac{\partial u_k}{\partial x_i}, \qquad j = 1, \ldots, N,$$

with initial condition $\mathbf{u}(\mathbf{x}, 0) = 0$ *has a real analytic solution in a neighbourhood of zero in* $\mathbb{R}^{n+1}$. *Moreover, this solution is unique in the class of real analytic functions.*

1.2 Asymptotics

On a number of occasions we study the asymptotic behaviour of submanifolds and thus make a few definitions relating to this area.

Definition 1.2.1. Let M and M_0 be closed submanifolds of $\mathbb{R}^n$. We say that M is *asymptotic with rate* λ *at infinity in* $\mathbb{R}^n$ *to* M_0 if there exist constants $R > 0$ and $\lambda < 1$, a compact subset K of M and a diffeomorphism $\Psi : M_0 \setminus \bar{B}_R \to M \setminus K$ such that

$$|\Psi(\mathbf{x}) - \mathbf{x}| = O(r^{\lambda}) \qquad \text{as } r \to \infty,$$

where r is the radius function on $\mathbb{R}^n$ and $\bar{B}_R$ is the closed ball of radius R.

We continue by considering cones and conical behaviour at infinity, using the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition 1.2.2. A *cone* in $\mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$ which is invariant under dilations and is nonsingular except possibly at 0. A cone C is said to be *two-sided* if $C = -C$.

Definition 1.2.3. Let M_0 be a closed cone in $\mathbb{R}^n$ and let M be a closed nonsingular submanifold of $\mathbb{R}^n$. Then M is *asymptotically conical* (AC) *to* M_0 *with rate* λ if there exist constants $R > 0$ and $\lambda < 1$, a compact subset K of M and a diffeomorphism $\Psi : M_0 \setminus \bar{B}_R \to M \setminus K$ such that

$$|\nabla^j(\Psi(\mathbf{x}) - \iota(\mathbf{x}))| = O(r^{\lambda - j}) \qquad \text{for } j \in \mathbb{N} \text{ as } r \to \infty, \tag{1.1}$$

where $\bar{B}_R$ is the closed ball of radius R in $\mathbb{R}^n$, $\iota : M_0 \to \mathbb{R}^n$ is the inclusion map and r is the radius function on $\mathbb{R}^n$. Here $|\,.\,|$ is calculated using the cone metric on $M_0 \setminus \bar{B}_R$, and ∇ is a combination of the Levi–Civita connection derived from the cone metric and the flat connection on $\mathbb{R}^n$ acting as partial differentiation.

If $\lambda \geq 0$ in the definition above, M is AC with rate λ to any translation of M_0 in $\mathbb{R}^n$, since translations correspond to $O(1)$ displacements. If we could take $\lambda \geq 1$, M would be AC with rate λ to any $\mathrm{SO}(n) \ltimes \mathbb{R}^n$ transformation of M_0, since rotations in $\mathbb{R}^n$ are $O(r)$ displacements, which would be a very weak kind of convergence.

The proposition below shows the relationship of AC behaviour to calibrated geometry.

Proposition 1.2.4. *Let η be a calibration k-form on $\mathbb{R}^n$ and let M be an η-submanifold of $\mathbb{R}^n$ which is AC with rate λ to a closed cone M_0 in $\mathbb{R}^n$. Provided M_0 is k-dimensional, it is calibrated with respect to η.*

Proof. Use the notation of Definition 1.2.3. Write $M_0 = (0, \infty) \times \Sigma$, where $\Sigma = M_0 \cap \mathcal{S}^{n-1}$, let (r, σ) be coordinates on M_0 and let g_Σ be the induced metric on Σ from the round metric on $\mathcal{S}^{n-1}$. Note that there are two different metrics on M_0: the *cylindrical* metric $g_{\mathrm{cyl}} = dr^2 + g_\Sigma$ and the *conical* metric $g_{\mathrm{cone}} = dr^2 + r^2 g_\Sigma$.

Let g_0 be the Euclidean metric on $\mathbb{R}^n$. Since M is calibrated with respect to η,

$$\left| d\Psi|^*_{(r,\,\sigma)}(\eta) \right|_{d\Psi|^*_{(r,\,\sigma)}(g_0)} = 1 \quad \text{and hence} \quad \left| r^{-k} d\Psi|^*_{(r,\,\sigma)}(\eta) \right|_{r^{-2} d\Psi|^*_{(r,\,\sigma)}(g_0)} = 1 \tag{1.2}$$

by the scaling properties of η and g_0 under dilations.

From (1.1),

$$\left| d\Psi|^*_{(r,\,\sigma)}(\eta) - d\iota|^*_{(r,\,\sigma)}(\eta) \right|_{g_{\mathrm{cone}}} = O(r^{\lambda-1}) \qquad \text{as } r \to \infty.$$

Therefore,

$$\left| r^{-k} d\Psi|^*_{(r,\,\sigma)}(\eta) - r^{-k} d\iota|^*_{(r,\,\sigma)}(\eta) \right|_{g_{\mathrm{cyl}}} = O(r^{\lambda-1}) \qquad \text{as } r \to \infty \tag{1.3}$$

by the relationship between g_{cone} and g_{cyl}. Similarly,

$$\left| r^{-2} d\Psi|^*_{(r,\,\sigma)}(g_0) - r^{-2} d\iota|^*_{(r,\,\sigma)}(g_0) \right|_{g_{\mathrm{cyl}}} = O(r^{\lambda-1}) \qquad \text{as } r \to \infty. \tag{1.4}$$

Since $\lambda < 1$, $r^{\lambda-1} \to 0$ as $r \to \infty$. Moreover, g_{cyl} is independent of r, so using (1.2)-(1.4) we deduce

$$\left| r^{-k} d\iota|^*_{(r,\,\sigma)}(\eta) \right|_{r^{-2} d\iota|^*_{(r,\,\sigma)}(g_0)} \to 1 \qquad \text{as } r \to \infty.$$

However, since $\iota(r, \sigma) = r\sigma$, $r^{-k} d\iota|^*_{(r,\,\sigma)}(\eta)$ and $r^{-2} d\iota|^*_{(r,\,\sigma)}(g_0)$ are independent of r. Thus, $d\iota^*(\eta)$ is equal to the volume form with respect to $d\iota^*(g_0)$ on $T_{(1,\,\sigma)} M_0$ for all $\sigma \in \Sigma$. Finally note that, since M_0 is a cone, $T_{(r,\,\sigma)} M_0 = T_{(1,\,\sigma)} M_0$ for all $r > 0$. The result follows. $\square$

Another required result related to asymptotics is a *Maximum Principle* for harmonic functions due to Hopf [35, p. 12].

Theorem 1.2.5 (Maximum Principle). *Let f be a smooth function on a Riemannian manifold M with boundary ∂M. If f is harmonic and assumes a local maximum (or minimum) at a point in $M \setminus \partial M$ it is constant.*

1.3 Number Systems

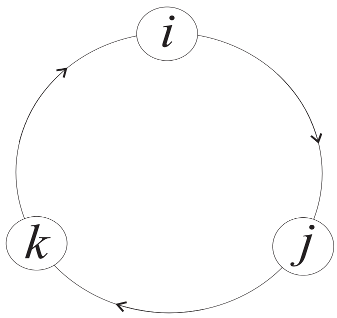
We reiterate that throughout this document we suppose that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

We may define the *orthogonal group* $\mathrm{O}(n)$ as the set of $n\times n$ real matrices preserving the Euclidean metric on $\mathbb{R}^n$. Alternatively, $\mathrm{O}(n)$ preserves the *dot product* on $\mathbb{R}^n$. The *special orthogonal group* $\mathrm{SO}(n)$ is the subgroup of $\mathrm{O}(n)$ of determinant 1 matrices; i.e. it preserves the orientation on $\mathbb{R}^n$. Similarly, on $\mathbb{C}^n$, we have the *unitary* and *special unitary groups*, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ respectively, for which we simply replace real matrices by complex matrices. For the next stage we want to describe the *compact symplectic group* $\mathrm{Sp}(n)$, for which we choose to consider the *quaternions* $\mathbb{H}$.

The quaternions are a 4-dimensional generalisation of complex numbers discovered by Hamilton in 1843. They are spanned by 1 and elements i, j and k satisfying the following multiplication law:

$$\begin{array}{c|ccc} & i & j & k \\ \hline i & -1 & k & -j \\ j & -k & -1 & i \\ k & j & -i & -1. \end{array}$$

This can be summarised in the following elegant diagram.



Then $\mathrm{Sp}(n)$ is the group of $n \times n$ quaternionic matrices preserving the dot product on $\mathbb{H}^n$. The orientation preserving group, or 'determinant 1' group if you will, associated with $\mathrm{Sp}(n)$ is $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$.

We complete our discussion of number systems with the octonions, or Cayley numbers, $\mathbb{O}$. They are an 8-dimensional generalisation of complex numbers discovered in 1843 by Hamilton's college friend John Graves, but first appeared in a publication by Arthur Cayley in 1845. They will be discussed in detail in §2.1. Here it is not as obvious to define the groups $\mathrm{Spin}(7)$ and G_2 associated with $\mathbb{O}$ to complete the sequence of groups described here, so we shall leave this until §2.1.3. However, we hope that the reader will appreciate that, rather than G_2 and $\mathrm{Spin}(7)$ appearing 'out of the blue', they are the final stage of a natural progression often called the Cayley–Dickson process.

Chapter 2

The Exceptional Geometries

We study the exceptional geometries in seven and eight dimensions, described by the Lie groups G_2 and Spin(7), making extensive use of the *octonions* which are discussed in §2.1. In Sections 2.3 and 2.4, we expose the relationship of G_2 and Spin(7) with calibrated geometry, both for Euclidean spaces and, more generally, for certain types of Riemannian manifold.

2.1 The Octonions

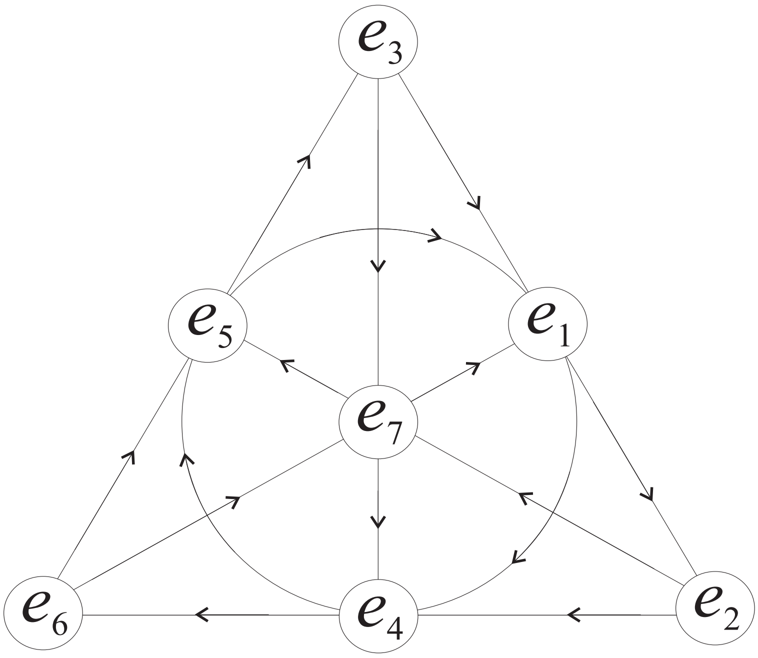
The octonions, or Cayley numbers, $\mathbb{O}$ help us provide an elegant description of the exceptional geometries and are used on many occasions in Chapters 4 and 5.

2.1.1 Cayley multiplication table

Let $\{e_1, \ldots, e_7\}$ be a basis for $\operatorname{Im}\mathbb{O}$. Then a Cayley multiplication table for $\mathbb{O}$ is as shown.

$$\begin{array}{c|cccccccc}
 & 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
1 & 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & e_1 & -1 & e_3 & -e_2 & e_5 & -e_4 & e_7 & -e_6 \\
e_2 & e_2 & -e_3 & -1 & e_1 & e_6 & -e_7 & -e_4 & e_5 \\
e_3 & e_3 & e_2 & -e_1 & -1 & -e_7 & -e_6 & e_5 & e_4 \\
e_4 & e_4 & -e_5 & -e_6 & e_7 & -1 & e_1 & e_2 & -e_3 \\
e_5 & e_5 & e_4 & e_7 & e_6 & -e_1 & -1 & -e_3 & -e_2 \\
e_6 & e_6 & -e_7 & e_4 & -e_5 & -e_2 & e_3 & -1 & e_1 \\
e_7 & e_7 & e_6 & -e_5 & -e_4 & e_3 & e_2 & -e_1 & -1
\end{array} \tag{2.1}$$

This table is not standard but chosen to agree with our later conventions. The information can be encoded in a diagram known as the Fano projective plane, which is referred to, for example, in [49, p. 157].



The multiplication is neither commutative nor associative. However, there are 4-dimensional associative subalgebras of $\mathbb{O}$ which, along with their complements, will be used to describe calibrated 3-planes and 4-planes in $\mathbb{R}^7$.

2.1.2 Cross products

We define cross products and multiple cross products of octonions which help us to describe and interpret the geometry of $\mathbb{R}^7$ and $\mathbb{R}^8$. The material is mainly derived from [17, Appendix IV.B]. We must note however that, because of the conventions adopted, there are some minor modifications to the formulae given in [17]. The differences come from our choice of basis for $\mathbb{O}$, so our formulae are equivalent up to a coordinate transformation and possible reversal of orientation, amounting to a change in sign. We endeavour to make these discrepancies apparent to the reader.

Definition 2.1.1. Let $x, y, z, w \in \mathbb{O}$. The *cross product* of x and y is

$$x \times y = -\frac{1}{2}\,(\bar{x}y - \bar{y}x). \tag{2.2}$$

Note that $x \times y = \mathrm{Im}(\bar{y}x)$, which shows that the cross product is imaginary-valued.

The *triple cross product* of x, y, z is

$$x \times y \times z = -\frac{1}{2}\big(x(\bar{y}z) - z(\bar{y}x)\big) \tag{2.3}$$

and the *fourfold cross product* of x, y, z, w is

$$x \times y \times z \times w = \frac{1}{4}\big(\bar{x}(y \times z \times w) + \bar{y}(z \times x \times w) + \bar{z}(x \times y \times w) + \bar{w}(y \times x \times z)\big). \tag{2.4}$$

The triple cross product of x, y and z can also be defined as the alternation of $-x(\bar{y}z)$. Hence, the triple and fourfold cross products are alternating multilinear forms.

Equations (2.2)-(2.3) are the opposite sign to the equivalent formulae in [17] and (2.4) is unaltered because the sign change is already accounted for by the choice in (2.3). Note that $x \times y \times z \neq x \times (y \times z) \neq (x \times y) \times z$ in general and a similar statement is true for the fourfold cross product.

We make some observations about the real parts of the products [17, Lemma IV.B.9].

Proposition 2.1.2. *If $x, y, z, w \in \mathbb{O}$ and x', y', z' are the imaginary parts of x, y, z respectively,*

$$\mathrm{Re}(x \times y) = 0,$$
$$\mathrm{Re}(x \times y \times z) = g_0(x' \times y', z') \text{ and}$$
$$\mathrm{Re}(x \times y \times z \times w) = g_0(x \times y \times z, w),$$

where g_0 is the Euclidean metric on $\mathbb{O} \cong \mathbb{R}^8$.

Recall the commutator $[x, y] = xy - yx$ of x and y. This leads us to define the associator.

Definition 2.1.3. The *associator* $[x, y, z]$ of $x, y, z \in \mathbb{O}$ is given by:

$$[x, y, z] = (xy)z - x(yz).$$

Whereas the commutator measures the extent to which commutativity fails, the associator gives the degree to which associativity fails in $\mathbb{O}$. In §4.4 we require some properties of the associator which we state as a proposition taken from [17, Proposition IV.B.16].

Proposition 2.1.4. *The associator $[x, y, z]$ of $x, y, z \in \mathbb{O}$ is alternating, imaginary-valued and orthogonal to x, y, z and to $[a, b]$ for any subset $\{a, b\}$ of $\{x, y, z\}$.*

On $\mathrm{Im}\,\mathbb{O}$ we can also define the coassociator.

Definition 2.1.5. The *coassociator* $[x, y, z, w]$ of $x, y, z, w \in \mathrm{Im}\,\mathbb{O}$ is given by:

$$[x, y, z, w] = -\big(g_0(y, zw)x + g_0(z, xw)y + g_0(x, yw)z + g_0(y, xz)w\big),$$

where g_0 is the Euclidean metric on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$.

If we restrict to considering $\mathrm{Im}\,\mathbb{O}$ we get the following neat result [17, Proposition IV.B.14].

Proposition 2.1.6. *For $x, y, z, w \in \operatorname{Im} \mathbb{O}$,*

$$2 \operatorname{Im}(x \times y) = [x, y],$$
$$2 \operatorname{Im}(x \times y \times z) = [x, y, z] \textit{ and}$$
$$2 \operatorname{Im}(x \times y \times z \times w) = [x, y, z, w].$$

We conclude by focusing on the cross product (2.2) restricted to $\operatorname{Im} \mathbb{O}$. Let $x, y \in \operatorname{Im} \mathbb{O}$ and write $x = x_1 e_1 + \ldots + x_7 e_7$, $y = y_1 e_1 + \ldots + y_7 e_7$ and $x \times y = z_1 e_1 + \ldots + z_7 e_7$. Then

$$\begin{aligned}
z_1 &= x_2 y_3 - x_3 y_2 + x_4 y_5 - x_5 y_4 + x_6 y_7 - x_7 y_6, \\
z_2 &= x_3 y_1 - x_1 y_3 + x_4 y_6 - x_6 y_4 + x_7 y_5 - x_5 y_7, \\
z_3 &= x_1 y_2 - x_2 y_1 + x_7 y_4 - x_4 y_7 + x_6 y_5 - x_5 y_6, \\
z_4 &= x_5 y_1 - x_1 y_5 + x_6 y_2 - x_2 y_6 + x_3 y_7 - x_7 y_3, \\
z_5 &= x_1 y_4 - x_4 y_1 + x_2 y_7 - x_7 y_2 + x_3 y_6 - x_6 y_3, \\
z_6 &= x_7 y_1 - x_1 y_7 + x_2 y_4 - x_4 y_2 + x_5 y_3 - x_3 y_5 \text{ and} \\
z_7 &= x_1 y_6 - x_6 y_1 + x_5 y_2 - x_2 y_5 + x_4 y_3 - x_3 y_4.
\end{aligned}$$

We can use this to prove the proposition below.

Proposition 2.1.7. *If $x, y \in \operatorname{Im} \mathbb{O}$ and g_0 is the Euclidean metric on $\operatorname{Im} \mathbb{O} \cong \mathbb{R}^7$,*

$$|x \times y|^2 + (g_0(x, y))^2 = |x|^2 |y|^2.$$

Hence $x \times y = 0$ if and only if x and y are linearly dependent.

Proof. Establishing the formula is a straightforward calculation. We have that $g_0(x, y) = |x||y| \cos \theta$, where θ is the angle between x and y, so $(g_0(x, y))^2 = |x|^2 |y|^2$ if and only if $\theta = 0$, i.e. when x and y are linearly dependent. We deduce the result. $\square$

2.1.3 $\mathbf{G}_2$ and $\mathrm{Spin}(7)$

We now define the exceptional Lie group G_2 in terms of the octonions.

Definition 2.1.8. The subgroup of the automorphisms of $\operatorname{Im} \mathbb{O}$ preserving the cross product (2.2) is G_2. It is a compact, connected, simply connected, simple, 14-dimensional Lie group which preserves the orientation on $\operatorname{Im} \mathbb{O}$.

This definition leads to the next proposition, which is in fact an instance of a general result from the theory of normed algebras, as discussed in [16, Chapter 6].

Proposition 2.1.9. *The Euclidean metric g_0 on $\operatorname{Im}\mathbb{O}$ is preserved by* G_2. *Thus* $\mathrm{G}_2 \subseteq \mathrm{SO}(7)$.

Proof. Let $x, y \in \operatorname{Im}\mathbb{O}$ and define

$$cp_x(u) = x \times u \quad \text{and} \quad cp_y(u) = y \times u \qquad \text{for } u \in \operatorname{Im}\mathbb{O}.$$

Then cp_x and cp_y are linear maps and so have matrices A_x and A_y with respect to the basis $\{e_1, \ldots, e_7\}$ of $\operatorname{Im}\mathbb{O}$. If $x = x_1e_1 + \ldots + x_7e_7$, consultation of the table (2.1) shows that

$$A_x = \begin{pmatrix} 0 & x_3 & -x_2 & x_5 & -x_4 & x_7 & -x_6 \\ -x_3 & 0 & x_1 & x_6 & -x_7 & -x_4 & x_5 \\ x_2 & -x_1 & 0 & -x_7 & -x_6 & x_5 & x_4 \\ -x_5 & -x_6 & x_7 & 0 & x_1 & x_2 & -x_3 \\ x_4 & x_7 & x_6 & -x_1 & 0 & -x_3 & -x_2 \\ -x_7 & x_4 & -x_5 & -x_2 & x_3 & 0 & x_1 \\ x_6 & -x_5 & -x_4 & x_3 & x_2 & -x_1 & 0 \end{pmatrix} \tag{2.5}$$

and we have a similar expression for A_y.

A straightforward calculation using formula (2.5) then gives:

$$-6g_0(x, y) = \operatorname{Tr}(A_x A_y).$$

We deduce the result from the definition of G_2. $\square$

We can also define Spin(7) using $\mathbb{O}$.

Definition 2.1.10. The subgroup of the automorphisms of $\mathbb{O}$ preserving the triple cross product (2.3) is Spin(7). It is a compact, connected, simply connected, simple, 21-dimensional Lie group, which preserves the Euclidean metric and the orientation on $\mathbb{O}$. Thus $\mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$. It is isomorphic to the double cover of SO(7).

The fact that Spin(7) preserves the metric can be proved in a similar way to Proposition 2.1.9.

2.2 Holonomy Groups

For this section, let (M, g) be an n-dimensional Riemannian manifold and let ∇ denote the Levi–Civita connection of g. We begin with the definition of parallel transport.

Definition 2.2.1. Let $\gamma : [0, 1] \to M$ be a piecewise-smooth path from x to y in M. For each $v \in T_xM$ there exists a unique section s of $\gamma^*(TM)$, which is smooth whenever γ is smooth, such that $s(0) = v$ and $\nabla_{\dot\gamma(t)} s(t) = 0$ for all $t \in [0,1]$. Define $P_\gamma(v) = s(1)$. Then $P_\gamma : T_xM \to T_yM$ is a linear map called the *parallel transport map*. If $\gamma(0) = \gamma(1) = x$, γ is a *loop based at* x and the parallel transport map P_γ is an invertible linear map from T_xM to itself.

We may now define the holonomy groups of g at $x \in M$.

Definition 2.2.2. The *holonomy group of* g *at* x is

$$\mathrm{Hol}_x(g) = \{P_\gamma \,:\, \gamma \text{ is a loop based at } x\}.$$

The *restricted holonomy group of* g *at* x is

$$\mathrm{Hol}^0_x(g) = \{P_\gamma \,:\, \gamma \text{ is a null-homotopic loop based at } x\}.$$

They are subgroups of $\mathrm{GL}(T_xM)$, the group of invertible linear transformations of T_xM.

If M is connected, [23, Propositions 2.2.3 & 2.5.2] show that the holonomy groups are independent of the point x at which they are based and are subgroups of $\mathrm{O}(n)$.

Proposition 2.2.3. *If* M *is connected,* $\mathrm{Hol}_x(g)$ *and* $\mathrm{Hol}^0_x(g)$ *can be considered as subgroups of* $\mathrm{O}(n)$ *defined up to conjugation in* $\mathrm{O}(n)$ *and are then independent of* x*. We thus use the notation* $\mathrm{Hol}(g)$ *and* $\mathrm{Hol}^0(g)$ *for the subgroups of* $\mathrm{O}(n)$*.*

This allows us to make our main definition.

Definition 2.2.4. Let M be connected. Define the *holonomy group of* g and *restricted holonomy group of* g to be $\mathrm{Hol}(g)$ and $\mathrm{Hol}^0(g)$ respectively. The holonomy group $\mathrm{Hol}(g)$ is a subgroup of $\mathrm{O}(n)$ and, by [23, Theorem 3.2.8], $\mathrm{Hol}^0(g)$ is a closed, hence compact, connected Lie subgroup of $\mathrm{SO}(n)$, each defined up to conjugation in $\mathrm{O}(n)$. Note that if M is simply connected, $\mathrm{Hol}(g) = \mathrm{Hol}^0(g)$ since all loops in M are null-homotopic.

In 1955, Berger made an important advance in the classification of holonomy groups of Riemannian metrics with the following theorem [4, Theorem 3] which also appears in [49, p. 1].

Theorem 2.2.5. *If* (M, g) *is an oriented, simply connected, connected,* n*-dimensional Riemannian manifold which is neither locally a product nor symmetric,* $\mathrm{Hol}(g)$ *must equal one of*

$$\mathrm{SO}(n),\quad \mathrm{U}(\tfrac{n}{2}),\quad \mathrm{SU}(\tfrac{n}{2}),\quad \mathrm{Sp}(\tfrac{n}{4})\,\mathrm{Sp}(1),\quad \mathrm{Sp}(\tfrac{n}{4}),\quad \mathrm{G}_2 \ (\textit{for } n=7) \ \textit{or} \ \mathrm{Spin}(7) \ (\textit{for } n=8).$$

The theorem also gives the particular subgroups of $\mathrm{O}(n)$ which are isomorphic to each of the Lie groups in the list. In Berger's Theorem, $\mathrm{Spin}(9)$ was included as a possible holonomy group but Alekseevskii [2] first showed that any Riemannian metric with holonomy $\mathrm{Spin}(9)$ is symmetric. The group $\mathrm{SO}(n)$ is the holonomy group of a generic metric.

Suppose M is of dimension $n = 2m$. If $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$ then g is a *Kähler* metric and if $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ then g is called a *Calabi–Yau* metric. Suppose further that $n = 4k$. If $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(k)$, g is called a *hyperkähler* metric and if $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(k)\,\mathrm{Sp}(1)$, g is a *quaternionic Kähler* metric.

Examples of all but the quaternionic Kähler metrics are known to exist in both the compact and complete cases. Calabi–Yau and hyperkähler metrics are Kähler and Ricci flat, even though generic Kähler metrics, which have holonomy $\mathrm{U}(m)$, are not. Quaternionic Kähler metrics are neither Kähler nor Ricci flat but they are Einstein.

The groups G_2 and $\mathrm{Spin}(7)$ in Theorem 2.2.5 are thus the exceptions in the list of possible holonomy groups. They are therefore known as the *exceptional holonomy groups*. Metrics with exceptional holonomy are Ricci flat. The local existence of metrics with holonomy G_2 and $\mathrm{Spin}(7)$ was proved in 1985 by Bryant [6]. Complete examples were given by Bryant and Salamon in [9] and compact examples were constructed by Joyce in [20], [21] and [22].

2.3 The G_2 Calibrations

2.3.1 G_2 geometry on $\mathbb{R}^7$

We start by defining calibrations on $\mathbb{R}^7$ as in [23, Chapter 10].

Definition 2.3.1. Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$ and write $d\mathbf{x}_{ij\ldots k}$ for the form $dx_i \wedge dx_j \wedge \ldots \wedge dx_k$. Define a 3-form φ_0 by:

$$\varphi_0 = d\mathbf{x}_{123} + d\mathbf{x}_{145} + d\mathbf{x}_{167} + d\mathbf{x}_{246} - d\mathbf{x}_{257} - d\mathbf{x}_{347} - d\mathbf{x}_{356}. \tag{2.6}$$

By [17, Theorem IV.1.4], φ_0 is a calibration on $\mathbb{R}^7$ and submanifolds calibrated with respect to φ_0 are called *associative 3-folds*.

The 4-form $*\varphi_0$, where φ_0 and $*\varphi_0$ are related by the Hodge star, is given by:

$$*\varphi_0 = d\mathbf{x}_{4567} + d\mathbf{x}_{2367} + d\mathbf{x}_{2345} + d\mathbf{x}_{1357} - d\mathbf{x}_{1346} - d\mathbf{x}_{1256} - d\mathbf{x}_{1247}. \tag{2.7}$$

By [17, Theorem IV.1.16], $*\varphi_0$ is a calibration on $\mathbb{R}^7$ and $*\varphi_0$-submanifolds are called *coassociative 4-folds*.

We make a few observations, identifying the standard basis on $\mathbb{R}^7$ with $(e_1, \ldots, e_7)$ on $\mathrm{Im}\,\mathbb{O}$.

Proposition 2.3.2. *If $x, y, z, w \in \mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$ and g_0 is the Euclidean metric on $\mathrm{Im}\,\mathbb{O}$,*

$$\varphi_0(x, y, z) = g_0(x \times y, z) \quad \textit{and} \quad *\,\varphi_0(x, y, z, w) = \tfrac{1}{2}\, g_0([x, y, z], w), \tag{2.8}$$

where the cross product is defined by (2.2) *and the associator is given in Definition 2.1.3.*

The proof of this proposition is immediate from (2.1) and Definitions 2.1.1, 2.1.3 and 2.3.1. We are now able to make further definitions.

Definition 2.3.3. Let g_0 be the Euclidean metric on $\mathbb{R}^7$. Define a *cross product* for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^7$ using index notation by

$$(\mathbf{x} \times \mathbf{y})^d = (\varphi_0)_{abc}\mathbf{x}^a\mathbf{y}^b(g_0)^{cd}. \tag{2.9}$$

This coincides with the definition of the octonionic cross product in (2.2) by (2.8).

We also define the *associator* on $\mathbb{R}^7$ by

$$[\mathbf{x}, \mathbf{y}, \mathbf{z}]^e = 2(*\varphi_0)_{abcd}\mathbf{x}^a\mathbf{y}^b\mathbf{z}^c(g_0)^{de} \tag{2.10}$$

for vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^7$. This agrees with the identification of $\mathbb{R}^7$ with $\operatorname{Im}\mathbb{O}$ by (2.8).

Finally, if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^7$, we define the *triple cross product* of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ by

$$(\mathbf{x} \times \mathbf{y} \times \mathbf{z})^e = (*\varphi_0)_{abcd}\mathbf{x}^a\mathbf{y}^b\mathbf{z}^c(g_0)^{de}. \tag{2.11}$$

This agrees with the triple cross product (2.3) if $\varphi_0(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$ by Propositions 2.1.2 and 2.1.6, the formula for φ_0 given in (2.8) and equation (2.10) for the associator.

We can use the associator to characterise associative 3-planes [17, Corollary IV.1.7].

Proposition 2.3.4. *A 3-plane in $\mathbb{R}^7$ with basis $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, appropriately oriented, is associative if and only if $[\mathbf{x}, \mathbf{y}, \mathbf{z}] = 0$.*

This has a useful corollary which follows from Proposition 2.1.7 and a straightforward calculation using (2.9) and (2.10).

Corollary 2.3.5. *If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent vectors in $\mathbb{R}^7$, $(\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y})$ is an oriented basis for a 3-plane V in $\mathbb{R}^7$. Moreover, V, with this orientation, is associative.*

We may also characterise coassociative 4-planes in $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ as ones on which the coassociator, given in Definition 2.1.5, vanishes [17, Proposition IV.1.25]. However, we have a far more useful description of coassociative 4-folds which follows from [17, Proposition IV.4.5 & Theorem IV.4.6].

Proposition 2.3.6. *A 4-dimensional submanifold M of $\mathbb{R}^7$, with an appropriate orientation, is coassociative if and only if $\varphi_0|_M \equiv 0$.*

We now give an alternative characterisation of G_2.

Proposition 2.3.7. *The stabilizer of φ_0 in $\mathrm{GL}(7, \mathbb{R})$ is G_2. Moreover, G_2 preserves $*\varphi_0$.*

Proof. Since G_2 preserves the cross product on $\operatorname{Im}\mathbb{O}$ by Definition 2.1.8 and g_0 by Proposition 2.1.9, it preserves φ_0 by (2.8). Note that G_2 preserves the Hodge star on $\mathbb{R}^7$ and hence preserves $*\varphi_0$. $\square$

The form φ_0 is often referred to as the G_2 3-form on $\mathbb{R}^7$, which is justified by Proposition 2.3.7. It, along with $*\varphi_0$, which we may call the G_2 4-form, is the central component of our study of calibrated geometry on $\mathbb{R}^7$.

2.3.2 G_2 structures on 7-manifolds

We start with two definitions following [8, p. 7] and [23, p. 243].

Definition 2.3.8. Let M be an oriented 7-manifold. For each $x \in M$ there exists an orientation preserving isomorphism ι_x from T_xM to $\mathbb{R}^7$ and thus $\iota_x^*(\varphi_0) \in \Lambda^3 T_x^*M$. Since $\dim \mathrm{G}_2 = 14$, $\dim \mathrm{GL}_+(T_xM) = 49$ and $\dim \Lambda^3 T_x^*M = 35$ for all $x \in M$, the $\mathrm{GL}_+(T_xM)$ orbit of $\iota_x^*(\varphi_0)$ in $\Lambda^3 T_x^*M$, denoted $\Lambda^3_+ T_x^*M$, is open. A 3-form φ on M is *definite*, or *positive*, if $\varphi|_{T_xM} \in \Lambda^3_+ T_x^*M$ for all $x \in M$. Denote the bundle of definite 3-forms $\Lambda^3_+ T^*M$. It is a bundle with fibre $\mathrm{GL}_+(7,\mathbb{R})/\,\mathrm{G}_2$ which is *not* a vector subbundle of $\Lambda^3 T^*M$.

Essentially, a definite 3-form is identified with the G_2 3-form on $\mathbb{R}^7$ at each point in M. Therefore, to each definite 3-form φ we can uniquely associate a 4-form $*\varphi$ and a metric g on M such that the triple $(\varphi, *\varphi, g)$ corresponds to $(\varphi_0, *\varphi_0, g_0)$ at each point. This leads us to our next definition.

Definition 2.3.9. Let M be an oriented 7-manifold, let φ be a definite 3-form on M and let g be the metric associated to φ. We call (φ, g) a G_2 *structure* on M. If $d\varphi = 0$, (φ, g) is a *closed* G_2 structure, and if $d^*\varphi = 0$ then (φ, g) is a *coclosed* G_2 structure. A closed and coclosed G_2 structure is called *torsion-free*.

Our choice of notation here agrees with [8]. Fernàndez [14] calls closed G_2 structures *associative* and coclosed G_2 structures *coassociative*. However, the author feels that this is potentially confusing, which justifies our choice of notation. Moreover, a G_2 structure (φ, g) defines a unique principal G_2 subbundle of the frame bundle and so there is a 1-1 correspondence between pairs (φ, g) and G_2 structures in the sense of bundles.

Our definition of torsion-free G_2 structure is not standard, but agrees with other definitions by the following result [49, Lemma 11.5].

Proposition 2.3.10. *Let (φ, g) be a G_2 structure and let ∇ be the Levi–Civita connection of g. The following are equivalent:*

$$d\varphi = d^*\varphi = 0; \qquad \nabla\varphi = 0; \quad \textit{and} \quad \mathrm{Hol}(g) \subseteq \mathrm{G}_2 \textit{ with } \varphi \textit{ as the associated 3-form.}$$

We now complete our definitions.

Definition 2.3.11. Let M be an oriented 7-manifold endowed with a G_2 structure (φ, g), denoted (M, φ, g). We say that (M, φ, g) is a *φ-closed*, or *φ-coclosed*, 7-manifold if (φ, g) is a closed, respectively coclosed, G_2 structure. If (φ, g) is torsion-free, we call (M, φ, g) a G_2 *manifold*.

If M has a G_2 structure (φ, g) then, since φ_0 and $*\varphi_0$ are calibrations on $\mathbb{R}^7$, φ and $*\varphi$ are calibrations on M if we relax the condition that a calibration is closed. Therefore, although it is most natural to consider calibrated submanifolds of G_2 manifolds, they can be defined for G_2 structures which are not torsion-free.

Definition 2.3.12. An oriented 3-dimensional submanifold N of (M, φ, g) is *associative* if it is calibrated with respect to φ. An oriented 4-dimensional submanifold N of (M, φ, g) is *coassociative* if it is calibrated with respect to $*\varphi$.

Note that, by Proposition 2.3.6, we have an alternative characterisation of coassociative 4-folds.

Proposition 2.3.13. *A 4-dimensional submanifold N of (M, φ, g), with an appropriate orientation, is coassociative if and only if $\varphi|_N \equiv 0$.*

McLean [45] studies the *deformation theory* of *compact* coassociative 4-folds in a G_2 manifold. We state a key result, required in Chapters 7 and 8, which follows from [45, Proposition 4.2].

Proposition 2.3.14. *Let N be a coassociative 4-fold in (M, φ, g). There is an isomorphism between the normal bundle $\nu(N)$ of N in M and $\Lambda^2_+ T^*N$ given by $v \mapsto (v \cdot \varphi)|_{TN}$.*

We finish this section with McLean's main result on compact coassociative 4-folds [45, Theorem 4.5].

Theorem 2.3.15. *Let N be a compact coassociative 4-fold in a φ-closed 7-manifold (M, φ, g). The moduli space of compact coassociative deformations of N in M is a smooth manifold of dimension $b^2_+(N)$.*

The result is actually for G_2 manifolds but analysis of the proof, as noted in [15], shows that closed G_2 structures suffice. McLean [45, §5] also considers deformations of compact associative 3-folds N in a G_2 manifold. As shown in [1, §4], the proof works for any G_2 structure and, for generic choices of (φ, g), one expects that N will admit no deformations.

2.4 The $\mathrm{Spin}(7)$ Calibration

2.4.1 $\mathrm{Spin}(7)$ geometry on $\mathbb{R}^8$

We define a 4-form on $\mathbb{R}^8$ as in [23, Chapter 10]

Definition 2.4.1. Let $(x_1, \ldots, x_8)$ be coordinates on $\mathbb{R}^8$ and write $d\mathbf{x}_{ij\ldots k}$ for the form $dx_i \wedge dx_j \wedge \ldots \wedge dx_k$. Define a 4-form Φ_0 by:

$$\begin{aligned}\Phi_0 = {} & d\mathbf{x}_{1234} + d\mathbf{x}_{1256} + d\mathbf{x}_{1278} + d\mathbf{x}_{1357} - d\mathbf{x}_{1368} - d\mathbf{x}_{1458} - d\mathbf{x}_{1467} \\ & + d\mathbf{x}_{5678} + d\mathbf{x}_{3478} + d\mathbf{x}_{3456} + d\mathbf{x}_{2468} - d\mathbf{x}_{2457} - d\mathbf{x}_{2367} - d\mathbf{x}_{2358}.\end{aligned} \tag{2.12}$$

By [17, Theorem IV.1.24], Φ_0 is a calibration on $\mathbb{R}^8$ and submanifolds calibrated with respect to Φ_0 are called *Cayley 4-folds*.

We first relate Φ_0 to octonionic multiplication, where we take $\mathbb{R}^8 \cong \mathbb{O}$ by identifying the standard basis on $\mathbb{R}^8$ with $(1, e_1, \ldots, e_7)$.

Proposition 2.4.2. *If $x, y, z, w \in \mathbb{O} \cong \mathbb{R}^8$ and g_0 is the Euclidean metric on $\mathbb{O}$,*

$$\Phi_0(x, y, z, w) = g_0(x \times y \times z, w), \tag{2.13}$$

where the triple cross product is defined by (2.3).

The proof of this result is immediate from inspection of (2.1), (2.3) and (2.12). We are thus led to make the next definition.

Definition 2.4.3. Let g_0 be the Euclidean metric on $\mathbb{R}^8$. Define the *triple cross product* of vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^8$ using index notation by

$$(\mathbf{x} \times \mathbf{y} \times \mathbf{z})^e = (\Phi_0)_{abcd}\mathbf{x}^a\mathbf{y}^b\mathbf{z}^c(g_0)^{de}. \tag{2.14}$$

This agrees with the identification of $\mathbb{R}^8$ with $\mathbb{O}$ by Proposition 2.4.2. Note, from (2.13), that $\mathbf{x} \times \mathbf{y} \times \mathbf{z}$ is orthogonal to $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

We may characterise Cayley 4-planes using the fourfold cross product on $\mathbb{O}$, defined by (2.4), as in [17, Corollary IV.1.29].

Proposition 2.4.4. *A 4-plane in $\mathbb{O} \cong \mathbb{R}^8$ with basis (x, y, z, w), appropriately oriented, is Cayley if and only if* $\mathrm{Im}(x \times y \times z \times w) = 0$.

This has a useful corollary, which follows from a straightforward calculation using (2.4) and (2.14).

Corollary 2.4.5. *If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly independent vectors in $\mathbb{R}^8$, $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x} \times \mathbf{y} \times \mathbf{z})$ is an oriented basis for a 4-plane V in $\mathbb{R}^8$. Moreover, V, with this orientation, is Cayley.*

In order to use Proposition 2.4.4 effectively, as required at various stages in the proof of Theorem 5.4.3, we desire efficient methods for computing the fourfold cross product. We give details of these methods now.

For our purposes we need only consider fourfold cross products of the form

$$f_{jk} = e_0 \times e_1 \times e_j \times e_k,$$

where we take $e_0 = 1$. It is clear by Definition 2.1.1 that f_{jk} is antisymmetric and that $f_{jk} = 0$ for $0 \leq j, k \leq 1$ since the fourfold cross product is alternating.

We only want to consider the case when $\operatorname{Im} f_{jk} \neq 0$. By Proposition 2.4.4 this occurs if and only if $\{e_0, e_1, e_j, e_k\}$ does not lie in a Cayley 4-plane. Hence $\operatorname{Im} f_{jk} = 0$ if $\{j, k\}$ is $\{2, 3\}$, $\{4, 5\}$ or $\{6, 7\}$.

We next make the following observation. By the invariance of the fourfold cross product under $\mathrm{Spin}(7)$, if $\{e_j, e_k, e_l, e_m\}$ is an ordered basis for a Cayley 4-plane, then either $f_{jk} = f_{lm}$ or $f_{jk} = -f_{lm}$, depending on whether $\{e_j, e_k, e_l, e_m\}$ is a positively oriented basis or not.

Therefore, the only fourfold cross products we require are given below.

Proposition 2.4.6. *Let $f_{jk} = 1 \times e_1 \times e_j \times e_k \in \mathbb{O}$. The only f_{jk} with $\operatorname{Im} f_{jk} \neq 0$ are:*

$$\begin{array}{lll} f_{47} = e_2 = f_{56}; & f_{46} = e_3 = f_{75}; & f_{63} = e_4 = f_{72}; \\ f_{37} = e_5 = f_{62}; & f_{25} = e_6 = f_{34}; \text{ and} & f_{24} = e_7 = f_{53}. \end{array}$$

We observe that associative 3-folds and coassociative 4-folds in $\mathbb{R}^7$ can be thought of as special cases of Cayley 4-folds in $\mathbb{R}^8$ in the following sense.

Proposition 2.4.7. *If we consider $\mathbb{R}^8 \cong \mathbb{R} \oplus \mathbb{R}^7$, with x_1 as the coordinate on $\mathbb{R}$ and coordinates on $\mathbb{R}^7$ labelled as $(x_2, \ldots, x_8)$, then*

$$\Phi_0 = dx_1 \wedge \varphi_0 + *\varphi_0. \tag{2.15}$$

Hence, L is an associative 3-fold in $\mathbb{R}^7$ if and only if $\mathbb{R} \times L \subseteq \mathbb{R} \oplus \mathbb{R}^7$ is a Cayley 4-fold in $\mathbb{R}^8$ and N is a coassociative 4-fold in $\mathbb{R}^7$ if and only if $\{0\} \times N \subseteq \mathbb{R} \oplus \mathbb{R}^7$ is a Cayley 4-fold in $\mathbb{R}^8$.

The proof follows from (2.6), (2.7) and (2.12) and Definitions 2.3.1 and 2.4.1. The equation (2.15) is also given in [23, Proposition 13.1.3].

We now give an alternative characterisation of $\mathrm{Spin}(7)$.

Proposition 2.4.8. *The stabilizer of Φ_0 in $\mathrm{GL}(8, \mathbb{R})$ is $\mathrm{Spin}(7)$.*

Proof. From Definition 2.1.10, $\mathrm{Spin}(7)$ preserves the triple cross product on $\mathbb{O}$ and the metric g_0. Thus, by Proposition 2.4.2, $\mathrm{Spin}(7)$ preserves Φ_0. $\square$

This result justifies referring to Φ_0 as the $\mathrm{Spin}(7)$ 4-form.

2.4.2 Spin(7) structures on 8-manifolds

For completeness we discuss the analogy of G_2 structures for 8-manifolds, following [23, p. 255].

Definition 2.4.9. Let M be an oriented 8-manifold. For each $x \in M$ there exists an orientation preserving isomorphism ι_x from T_xM to $\mathbb{R}^8$ and thus $\iota_x^*(\Phi_0) \in \Lambda^4 T_x^* M$. Since $\dim \mathrm{Spin}(7) = 21$, $\dim \mathrm{GL}_+(T_xM) = 64$ and $\dim \Lambda^4 T_x^* M = 70$ for all $x \in M$, the $\mathrm{GL}_+(T_xM)$ orbit of $\iota_x^*(\Phi_0)$ in $\Lambda^4 T_x^* M$, denoted $\Lambda_a^4 T_x^* M$, has codimension 27. A 4-form Φ on M is *admissible* if $\Phi|_{T_xM} \in \Lambda_a^4 T_x^* M$ for all $x \in M$. Denote the bundle of admissible 4-forms $\Lambda_a^4 T^* M$. It is a bundle with fibre $\mathrm{GL}_+(8,\mathbb{R})/\mathrm{Spin}(7)$ which is *not* a vector subbundle of $\Lambda^4 T^* M$.

An admissible 4-form Φ defines an isomorphism between T_xM and $\mathbb{R}^8$ for all $x \in M$ such that $\Phi|_{T_xM}$ is identified with Φ_0. Thus, Φ uniquely defines a metric g on M such that (Φ, g) corresponds to (Φ_0, g_0) for all $x \in M$.

Definition 2.4.10. Let M be an oriented 8-manifold, let Φ be an admissible 4-form on M and let g be the metric associated to Φ. We call (Φ, g) a Spin(7) *structure* on M. If $d\Phi = 0$, we say that the Spin(7) structure is *torsion-free*.

As for G_2 structures, there is a 1-1 correspondence between pairs (Φ, g) and Spin(7) structures as in the theory of bundles. Again, our definition of torsion-free is not standard, but is equivalent to other definitions by the following result [49, Lemma 12.4].

Proposition 2.4.11. *Let (Φ, g) be a* Spin(7) *structure and let ∇ be the Levi–Civita connection of g. The following are equivalent:*

$$d\Phi = 0; \qquad \nabla\Phi = 0; \quad \textit{and} \quad \mathrm{Hol}(g) \subseteq \mathrm{Spin}(7) \textit{ with } \Phi \textit{ as the associated 4-form.}$$

We now define Spin(7) manifolds.

Definition 2.4.12. Let M be an oriented 8-manifold with a Spin(7) structure (Φ, g), which we denote (M, Φ, g). We call (M, Φ, g) a Spin(7) *manifold* if (Φ, g) is torsion-free.

As for G_2 structures, if (Φ, g) is a Spin(7) structure on M then Φ satisfies the calibration condition since Φ_0 does and is a genuine calibration if it is closed. It is therefore most natural to describe Cayley 4-folds in Spin(7) manifolds, but they can be defined for any Spin(7) structure by relaxing the closed restriction.

Definition 2.4.13. An oriented 4-dimensional submanifold N of (M, Φ, g) is *Cayley* if it is calibrated with respect to Φ.

McLean [45, §6] studies the deformation theory of Cayley 4-folds N in a Spin(7) manifold (M, Φ, g). His work involves a twisted Dirac operator $\not{D}$. For generic choices of (Φ, g) one expects N to admit a moduli space of deformations with dimension equal to the index of $\not{D}$, provided this index is non-negative.

Part I

Constructions in Euclidean Space

...the source of all great mathematics is the special case, the concrete example. It is frequent in mathematics that every instance of a concept of seemingly great generality is in essence the same as a small and concrete special case.

– Paul R. Halmos

Chapter 3

Special Lagrangian m-folds in $\mathbb{C}^m$

Of all the work on calibrated submanifolds, the greatest attention has been paid to the special Lagrangian category, particularly the area of special Lagrangian 3-folds. This chapter presents a selection of definitions and results relating to special Lagrangian geometry, with the papers [24], [25] and [26] as the main sources. We shall show that special Lagrangian 3-folds and 4-folds can be considered as particular examples of, respectively, associative 3-folds and Cayley 4-folds. This observation will aid us in Chapters 4 and 5.

3.1 Basic Theory

We begin with the definition of special Lagrangian m-folds in $\mathbb{C}^m$.

Definition 3.1.1. Let $(z_1, \ldots, z_m)$ be complex coordinates on $\mathbb{C}^m$. Define a metric g_m, a real 2-form ω_m and a complex m-form Ω_m on $\mathbb{C}^m$ by:

$$\begin{aligned} g_m &= |dz_1|^2 + \ldots + |dz_m|^2; \\ \omega_m &= \frac{i}{2}\left(dz_1 \wedge d\bar{z}_1 + \ldots + dz_m \wedge d\bar{z}_m\right); \text{ and} \\ \Omega_m &= dz_1 \wedge \ldots \wedge dz_m. \end{aligned}$$

A real oriented m-dimensional submanifold L of $\mathbb{C}^m$ is a *special Lagrangian* (SL) m-fold in $\mathbb{C}^m$ with *phase* $e^{i\theta}$ if L is calibrated with respect to the real m-form $\cos\theta \operatorname{Re}\Omega_m + \sin\theta \operatorname{Im}\Omega_m$. If the phase of L is unspecified it is taken to be one so that L is a $\operatorname{Re}\Omega_m$-submanifold of $\mathbb{C}^m$.

One may define SL m-folds more generally in *Calabi–Yau m-folds*, or even in *almost Calabi–Yau m-folds*, but this shall not be required. Harvey and Lawson [17, Corollary III.1.11] give the following alternative characterisation of SL m-folds in $\mathbb{C}^m$.

Proposition 3.1.2. *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. There is an orientation on L making it into an SL m-fold in $\mathbb{C}^m$ with phase $e^{i\theta}$ if and only if $\omega_m|_L \equiv 0$ and $(\sin\theta \operatorname{Re}\Omega_m - \cos\theta \operatorname{Im}\Omega_m)|_L \equiv 0$.*

The relationship between the G_2 and Spin(7) forms and the special Lagrangian calibrations is given in the propositions below, the proofs of which are immediate from Definitions 2.3.1, 2.4.1 and 3.1.1. These results are also given in [23, Propositions 11.1.2 & 13.1.4]

Proposition 3.1.3. *Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$. Consider $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ with x_1 as the coordinate on $\mathbb{R}$ and (z_1, z_2, z_3) as coordinates on $\mathbb{C}^3$, where $z_1 = x_2 + ix_3$, $z_2 = x_4 + ix_5$ and $z_3 = x_6 + ix_7$. Then*

$$\varphi_0 = dx_1 \wedge \omega_3 + \operatorname{Re}\Omega_3 \quad \text{and} \quad *\varphi_0 = \frac{1}{2}\,\omega_3 \wedge \omega_3 - dx_1 \wedge \operatorname{Im}\Omega_3. \tag{3.1}$$

Thus, L is an SL 3-fold in $\mathbb{C}^3$ with phase 1 if and only if $\{0\} \times L$ is an associative 3-fold in $\mathbb{R}^7$. Moreover, L is an SL 3-fold in $\mathbb{C}^3$ with phase $-i$ if and only if $\mathbb{R} \times L$ is a coassociative 4-fold in $\mathbb{R}^7$ with $\omega_3|_L \equiv 0$.

Proposition 3.1.4. *Let $(x_1, \ldots, x_8)$ be coordinates on $\mathbb{R}^8$. Consider $\mathbb{R}^8 \cong \mathbb{C}^4$ with (z_1, z_2, z_3, z_4) as coordinates on $\mathbb{C}^4$, where $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, $z_3 = x_5 + ix_6$ and $z_4 = x_7 + ix_8$. Then*

$$\Phi_0 = \frac{1}{2}\,\omega_4 \wedge \omega_4 + \operatorname{Re}\Omega_4. \tag{3.2}$$

Hence, L is an SL 4-fold in $\mathbb{C}^4$ if and only if L is a Cayley 4-fold in $\mathbb{R}^8$ with $\omega_4|_L \equiv 0$.

In $\mathbb{C}^3$ we may define a cross product which will be useful in describing the construction in §3.3.

Definition 3.1.5. Define a cross product $\times : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$, using index notation, by

$$(\mathbf{x} \times \mathbf{y})^d = (\operatorname{Re}\Omega_3)_{abc}\mathbf{x}^a\mathbf{y}^b(g_3)^{cd} \tag{3.3}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^3$, regarding $\mathbb{C}^3$ as a real vector space.

We can relate this product to the cross product on $\mathbb{R}^7$ given in Definition 2.3.3.

Proposition 3.1.6. *Consider $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ as in Proposition 3.1.3. Let $\times_{\mathbb{R}^7}$ and $\times_{\mathbb{C}^3}$ denote the cross products on $\mathbb{R}^7$ and $\mathbb{C}^3$ given by (2.9) and (3.3) respectively. For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^3 \subseteq \mathbb{R}^7$,*

$$\mathbf{x} \times_{\mathbb{R}^7} \mathbf{y} = (\omega_3(\mathbf{x}, \mathbf{y}),\, \mathbf{x} \times_{\mathbb{C}^3} \mathbf{y}). \tag{3.4}$$

Thus, the cross products on $\mathbb{R}^7$ and $\mathbb{C}^3$ are equivalent for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^3$ such that $\omega_3(\mathbf{x}, \mathbf{y}) = 0$.

The proof is immediate from the definitions of the cross products and (3.1). Proposition 3.1.6 has the following corollary.

Corollary 3.1.7. *Use the notation of Proposition 3.1.6. If* $\mathbf{x}$ *and* $\mathbf{y}$ *are linearly independent vectors in* $\mathbb{C}^3$ *such that* $\omega_3(\mathbf{x}, \mathbf{y}) = 0$*,* $(\mathbf{x}, \mathbf{y}, \mathbf{x} \times_{\mathbb{C}^3} \mathbf{y})$ *is an oriented basis for a 3-plane* V *in* $\mathbb{C}^3$*. Moreover,* V*, with this orientation, is an SL 3-plane.*

Proof. From Corollary 2.3.5, $(\mathbf{x}, \mathbf{y}, \mathbf{x} \times_{\mathbb{R}^7} \mathbf{y})$ is an oriented basis for a 3-plane U in $\mathbb{R}^7$ which is associative. However, since $\mathbf{x} \times_{\mathbb{R}^7} \mathbf{y} = (0, \mathbf{x} \times_{\mathbb{C}^3} \mathbf{y})$ by (3.4), $U = \{0\} \times V$ where V is an oriented 3-plane in $\mathbb{C}^3$ with basis $(\mathbf{x}, \mathbf{y}, \mathbf{x} \times_{\mathbb{C}^3} \mathbf{y})$. Then V is an SL 3-plane in $\mathbb{C}^3$ by Proposition 3.1.3. □

On $\mathbb{C}^4$ we have a triple cross product.

Definition 3.1.8. The *triple cross product* of vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^4$ is

$$(\mathbf{x} \times \mathbf{y} \times \mathbf{z})^e = (\operatorname{Re} \Omega_4)_{abcd} \mathbf{x}^a \mathbf{y}^b \mathbf{z}^c (g_4)^{de}, \tag{3.5}$$

using index notation for tensors on $\mathbb{C}^4$. By (3.2), this triple cross product agrees with (2.14) when $\omega_4(\mathbf{x}, \mathbf{y}) = \omega_4(\mathbf{y}, \mathbf{z}) = \omega_4(\mathbf{z}, \mathbf{x}) = 0$.

We may then prove the following.

Proposition 3.1.9. *If* $\mathbf{x}$*,* $\mathbf{y}$ *and* $\mathbf{z}$ *are linearly independent vectors in* $\mathbb{C}^4$ *such that* $\omega_4(\mathbf{x}, \mathbf{y}) = \omega_4(\mathbf{y}, \mathbf{z}) = \omega_4(\mathbf{z}, \mathbf{x}) = 0$*,* $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x} \times \mathbf{y} \times \mathbf{z})$ *is an oriented basis for a real 4-plane* V *in* $\mathbb{C}^4$*. Moreover,* V*, with this orientation, is SL.*

Proof. By Definition 3.1.8, $\mathbf{x} \times \mathbf{y} \times \mathbf{z}$ agrees with equation (2.14) for the product on $\mathbb{R}^8 \cong \mathbb{C}^4$. By Corollary 2.4.5, $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x} \times \mathbf{y} \times \mathbf{z})$ is an oriented basis for a Cayley 4-plane V in $\mathbb{R}^8$. Calculation in coordinates shows that, for $\mathbf{w} \in \mathbb{C}^4$ such that $\omega_4(\mathbf{w}, \mathbf{x}) = \omega_4(\mathbf{w}, \mathbf{y}) = \omega_4(\mathbf{w}, \mathbf{z}) = 0$,

$$\omega_4(\mathbf{x} \times \mathbf{y} \times \mathbf{z}, \mathbf{w}) = \operatorname{Im}(\epsilon_{abcd} \mathbf{x}^a \mathbf{y}^b \mathbf{z}^c \mathbf{w}^d),$$

where ϵ_{abcd} is the permutation symbol. Hence $\omega_4|_V \equiv 0$, so V is SL by Proposition 3.1.4. □

3.2 Evolution Equations

In this section we review the work of Joyce in [24] and [25] on the construction of special Lagrangian m-folds in $\mathbb{C}^m$ by *evolution equations.*

Joyce, in [24], derives an evolution equation for SL m-folds, the proof of which requires the following theorem [17, Theorem III.5.5].

Theorem 3.2.1. *Let* P *be a real analytic* $(m-1)$*-dimensional submanifold of* $\mathbb{C}^m$ *with* $\omega_m|_P \equiv 0$*. There locally exists an SL* m*-fold* L *in* $\mathbb{C}^m$ *containing* P*. Moreover,* L *is locally unique.*

The use of the *Cartan–Kähler Theorem* in the proof of Theorem 3.2.1 necessitates the requirement that P be real analytic. We now give a key result from [24], taken from [24, Theorem 3.3].

Theorem 3.2.2. *Let P be a compact, orientable, $(m-1)$-dimensional, real analytic manifold, χ a real analytic nowhere vanishing section of $\Lambda^{m-1}TP$ and $\psi : P \to \mathbb{C}^m$ a real analytic embedding (immersion) such that $\psi^*(\omega_m) \equiv 0$ on P. There exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{C}^m$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^b = (\psi_t)_*(\chi)^{a_1 \dots a_{m-1}} (\operatorname{Re}\Omega_m)_{a_1 \dots a_{m-1} a_m} (g_m)^{a_m b},$$

using index notation for tensors on $\mathbb{C}^m$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{C}^m$ by $\Psi(t,p) = \psi_t(p)$. Then $M = \operatorname{Image}\Psi$ is a nonsingular embedded (immersed) SL m-fold in $\mathbb{C}^m$.

In [25, §3] Joyce introduces the idea of affine evolution data with which he is able to derive an evolution equation, and therefore reduces the infinite-dimensional problem of Theorem 3.2.2 to a finite-dimensional one.

Definition 3.2.3. Let $2 \leq m \leq n$ be integers. A set of *affine evolution data* is a pair (P, χ), where P is an $(m-1)$-dimensional submanifold of $\mathbb{R}^n$ and $\chi : \mathbb{R}^n \to \Lambda^{m-1}\mathbb{R}^n$ is an affine map, such that $\chi(p)$ is a nonzero element of $\Lambda^{m-1}TP$ in $\Lambda^{m-1}\mathbb{R}^n$ for each nonsingular $p \in P$. We suppose also that P is not contained in any proper affine subspace $\mathbb{R}^k$ of $\mathbb{R}^n$.

Let $\operatorname{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ be the affine space of affine maps $\psi : \mathbb{R}^n \to \mathbb{C}^m$ and define $\mathcal{C}_P^{\omega_m}$ to be the set of $\psi \in \operatorname{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ satisfying:

(i) $\psi^*(\omega_m)|_P \equiv 0$;

(ii) $\psi|_{T_pP} : T_pP \to \mathbb{C}^m$ is injective for all p in a dense open subset of P.

Then (i) is a quadratic condition on ψ and (ii) is an open condition on ψ, so $\mathcal{C}_P^{\omega_m}$ is a nonempty open set in the intersection of a finite number of quadrics in $\operatorname{Aff}(\mathbb{R}^n, \mathbb{C}^m)$.

The conditions upon χ in Definition 3.2.3 are strong. The result is that there are few known examples of affine evolution data. The evolution equation derived in [25, Theorem 3.5] is given below.

Theorem 3.2.4. *Let (P, χ) be a set of affine evolution data and let $\psi \in \mathcal{C}_P^{\omega_m}$, where $\mathcal{C}_P^{\omega_m}$ is defined in Definition 3.2.3. There exist $\epsilon > 0$ and a unique real analytic family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ in $\mathcal{C}_P^{\omega_m}$ with $\psi_0 = \psi$, satisfying*

$$\left(\frac{d\psi_t}{dt}(x)\right)^b = (\psi_t)_*(\chi(x))^{a_1 \dots a_{m-1}} (\operatorname{Re}\Omega_m)_{a_1 \dots a_{m-1} a_m} (g_m)^{a_m b}$$

for all $x \in \mathbb{R}^n$, using index notation for tensors in $\mathbb{C}^m$. Moreover, $M = \{\psi_t(p) : t \in (-\epsilon, \epsilon),\, p \in P\}$ is an SL m-fold in $\mathbb{C}^m$ wherever it is nonsingular.

3.3 An Explicit Construction in $\mathbb{C}^3$

We conclude this chapter by discussing the material in [26], which is particularly pertinent to §4.4, where Joyce, for the majority of the paper, focuses on constructing SL 3-folds in $\mathbb{C}^3$ using the set of affine evolution data given below [26, p. 352].

Example 3.3.1. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^5$ be the embedding of $\mathbb{R}^2$ in $\mathbb{R}^5$ given by

$$\phi(y_1, y_2) = \left(\frac{1}{2}\,(y_1^2 + y_2^2),\ \frac{1}{2}\,(y_1^2 - y_2^2),\ y_1 y_2,\ y_1,\ y_2\right). \tag{3.6}$$

Then $P = \operatorname{Image}\phi$ can be written as

$$P = \left\{(x_1, \ldots, x_5) \in \mathbb{R}^5 : x_1 = \frac{1}{2}\,(x_4^2 + x_5^2),\ x_2 = \frac{1}{2}\,(x_4^2 - x_5^2),\ x_3 = x_4 x_5\right\},$$

which is diffeomorphic to $\mathbb{R}^2$. Writing $\mathbf{e}_j = \frac{\partial}{\partial x_j}$, we calculate from (3.6):

$$\phi_*\left(\frac{\partial}{\partial y_1}\right) = y_1\mathbf{e}_1 + y_1\mathbf{e}_2 + y_2\mathbf{e}_3 + \mathbf{e}_4; \quad \phi_*\left(\frac{\partial}{\partial y_2}\right) = y_2\mathbf{e}_1 - y_2\mathbf{e}_2 + y_1\mathbf{e}_3 + \mathbf{e}_5;$$

and thus

$$\begin{aligned}\phi_*\left(\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}\right) &= \left(y_1^2 + y_2^2\right)\mathbf{e}_2 \wedge \mathbf{e}_3 + \left(y_1^2 - y_2^2\right)\mathbf{e}_1 \wedge \mathbf{e}_3 - 2y_1 y_2\,\mathbf{e}_1 \wedge \mathbf{e}_2 \\ &\quad + y_1\left(\mathbf{e}_1 \wedge \mathbf{e}_5 + \mathbf{e}_2 \wedge \mathbf{e}_5 - \mathbf{e}_3 \wedge \mathbf{e}_4\right) + y_2\left(-\mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_4 + \mathbf{e}_3 \wedge \mathbf{e}_5\right) + \mathbf{e}_4 \wedge \mathbf{e}_5.\end{aligned}$$

Hence, if we define an affine map $\chi : \mathbb{R}^5 \to \Lambda^2\mathbb{R}^5$ by

$$\begin{aligned}\chi(x_1, \ldots, x_5) &= 2x_1\,\mathbf{e}_2 \wedge \mathbf{e}_3 + 2x_2\,\mathbf{e}_1 \wedge \mathbf{e}_3 - 2x_3\,\mathbf{e}_1 \wedge \mathbf{e}_2 + x_4\left(\mathbf{e}_1 \wedge \mathbf{e}_5 + \mathbf{e}_2 \wedge \mathbf{e}_5 - \mathbf{e}_3 \wedge \mathbf{e}_4\right) \\ &\quad + x_5\left(-\mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_4 + \mathbf{e}_3 \wedge \mathbf{e}_5\right) + \mathbf{e}_4 \wedge \mathbf{e}_5,\end{aligned} \tag{3.7}$$

then $\chi = \phi_*\left(\frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}\right)$ on P. Therefore (P, χ) is a set of affine evolution data with $m = 3$ and $n = 5$, in the notation of Definition 3.2.3.

The main result [26, Theorem 5.1] is stated below.

Theorem 3.3.2. *Suppose that $\mathbf{z}_1, \ldots, \mathbf{z}_6 : \mathbb{R} \to \mathbb{C}^3$ are differentiable functions satisfying*

$$\omega_3(\mathbf{z}_2, \mathbf{z}_3) = \omega_3(\mathbf{z}_1, \mathbf{z}_3) = \omega_3(\mathbf{z}_1, \mathbf{z}_2) = 0, \tag{3.8}$$

$$\omega_3(\mathbf{z}_1, \mathbf{z}_5) + \omega_3(\mathbf{z}_2, \mathbf{z}_5) - \omega_3(\mathbf{z}_3, \mathbf{z}_4) = 0, \tag{3.9}$$

$$-\,\omega_3(\mathbf{z}_1, \mathbf{z}_4) + \omega_3(\mathbf{z}_2, \mathbf{z}_4) + \omega_3(\mathbf{z}_3, \mathbf{z}_5) = 0 \ \textit{and} \tag{3.10}$$

$$\omega_3(\mathbf{z}_4, \mathbf{z}_5) = 0 \tag{3.11}$$

at $t = 0$, and the equations:

$$\frac{d\mathbf{z}_1}{dt} = 2\mathbf{z}_2 \times \mathbf{z}_3; \qquad \frac{d\mathbf{z}_2}{dt} = 2\mathbf{z}_1 \times \mathbf{z}_3; \qquad \frac{d\mathbf{z}_3}{dt} = -2\mathbf{z}_1 \times \mathbf{z}_2; \tag{3.12}$$

$$\frac{d\mathbf{z}_4}{dt} = \mathbf{z}_1 \times \mathbf{z}_5 + \mathbf{z}_2 \times \mathbf{z}_5 - \mathbf{z}_3 \times \mathbf{z}_4; \qquad \frac{d\mathbf{z}_5}{dt} = -\mathbf{z}_1 \times \mathbf{z}_4 + \mathbf{z}_2 \times \mathbf{z}_4 + \mathbf{z}_3 \times \mathbf{z}_5; \text{ and} \tag{3.13}$$

$$\frac{d\mathbf{z}_6}{dt} = \mathbf{z}_4 \times \mathbf{z}_5 \tag{3.14}$$

for all $t \in \mathbb{R}$, where $\times$ is defined by (3.3)*. The subset M of $\mathbb{C}^3$ given by*

$$M = \left\{ \frac{1}{2}(y_1^2 + y_2^2)\,\mathbf{z}_1(t) + \frac{1}{2}(y_1^2 - y_2^2)\,\mathbf{z}_2(t) + y_1 y_2\,\mathbf{z}_3(t) + y_1\,\mathbf{z}_4(t) + y_2\,\mathbf{z}_5(t) + \mathbf{z}_6(t) : y_1, y_2, t \in \mathbb{R} \right\}$$

is a special Lagrangian 3-fold in $\mathbb{C}^3$ wherever it is nonsingular.

Joyce [26] solves (3.12)-(3.14) subject to the conditions (3.8)-(3.11), dividing the solutions into sets based on the dimension of $\langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)\rangle_{\mathbb{R}}$ for generic $t \in \mathbb{R}$. Our concern shall lie with the case $\dim \langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)\rangle_{\mathbb{R}} = 3$, which forms the bulk of the results of [26]. The solutions here involve the *Jacobi elliptic functions*: $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ for $k \in [0,1]$. A description of these functions may be found in [10, Chapter VII].

The embedding given in Example 3.3.1 was constructed by considering the action of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$. Hence, Joyce [26, Proposition 9.1] shows that solutions of (3.12), satisfying the condition (3.8), are equivalent under the natural actions of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(3)$ to a solution $\mathbf{z}_1 = (z_1, 0, 0)$, $\mathbf{z}_2 = (0, z_2, 0)$, $\mathbf{z}_3 = (0, 0, z_3)$, for differentiable functions $z_1, z_2, z_3 : \mathbb{R} \to \mathbb{C}$. We therefore assume $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$ are of this form. Equations (3.12) become

$$\frac{dz_1}{dt} = 2\,\overline{z_2 z_3}, \qquad \frac{dz_2}{dt} = -2\,\overline{z_3 z_1} \quad \text{and} \quad \frac{dz_3}{dt} = -2\,\overline{z_1 z_2}. \tag{3.15}$$

The next result is taken from [26, Proposition 9.2].

Proposition 3.3.3. *Given any initial data, $z_1(0)$, $z_2(0)$ and $z_3(0)$, solutions to* (3.15) *exist for all $t \in \mathbb{R}$. Wherever the $z_j(t)$ are nonzero they may be written as:*

$$2z_1 = e^{i\theta_1}\sqrt{\alpha_1^2 + v}\,; \qquad 2z_2 = e^{i\theta_2}\sqrt{\alpha_2^2 - v}\,; \quad \text{and} \quad 2z_3 = e^{i\theta_3}\sqrt{\alpha_3^2 - v},$$

where $\alpha_j \in \mathbb{R}$ for all j and $v, \theta_1, \theta_2, \theta_3 : \mathbb{R} \to \mathbb{R}$ are differentiable functions. Let $\theta = \theta_1 + \theta_2 + \theta_3$ and let $Q(v) = (\alpha_1^2 + v)(\alpha_2^2 - v)(\alpha_3^2 - v)$. There exists $A \in \mathbb{R}$ such that $Q(v)^{\frac{1}{2}} \sin\theta = A$.

We conclude with the statement of a theorem [26, Theorem 9.3] that shall be of great use in §4.4.

Theorem 3.3.4. *Use the notation of Proposition 3.3.3. Let $\alpha_j > 0$ for all j with $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$. Suppose that v has a minimum at $t = 0$, $\theta_2(0) = \theta_3(0) = 0$, $A \geq 0$ and $\alpha_2 \leq \alpha_3$. Exactly one of the following four cases holds.*

(i) $A = 0$, $\alpha_2 = \alpha_3$ *and* z_1, z_2, z_3 *are given by:*

$$2z_1(t) = \sqrt{3}\alpha_1 \tanh\left(\sqrt{3}\alpha_1 t\right) \quad \textit{and} \quad 2z_2(t) = 2z_3(t) = \sqrt{3}\alpha_1 \operatorname{sech}\left(\sqrt{3}\alpha_1 t\right).$$

(ii) $A = 0$, $\alpha_2 < \alpha_3$ *and* z_1, z_2, z_3 *are given by:*

$$\begin{aligned} 2z_1(t) &= \sqrt{\alpha_1^2 + \alpha_2^2}\ \operatorname{sn}(\sigma t, \tau); \\ 2z_2(t) &= \sqrt{\alpha_1^2 + \alpha_2^2}\ \operatorname{cn}(\sigma t, \tau); \textit{ and} \\ 2z_3(t) &= \sqrt{\alpha_1^2 + \alpha_3^2}\ \operatorname{dn}(\sigma t, \tau), \end{aligned}$$

where $\sigma = \sqrt{\alpha_1^2 + \alpha_3^2}$ *and* $\tau = \sqrt{\dfrac{\alpha_1^2 + \alpha_2^2}{\alpha_1^2 + \alpha_3^2}}$.

(iii) $0 < A < \alpha_1\alpha_2\alpha_3$. *Let the roots of* $Q(v) - A^2$ *be* $\gamma_1, \gamma_2, \gamma_3$, *ordered such that* $\gamma_1 \leq 0 \leq \gamma_2 \leq \gamma_3$. *Then* $v, \theta_1, \theta_2, \theta_3$ *are given by:*

$$\begin{aligned} v(t) &= \gamma_1 + (\gamma_2 - \gamma_1)\operatorname{sn}^2(\sigma t, \tau); \\ \theta_1(t) &= \theta_1(0) - A\int_0^t \frac{ds}{\alpha_1^2 + \gamma_1 + (\gamma_2 - \gamma_1)\operatorname{sn}^2(\sigma s, \tau)}; \\ \theta_2(t) &= A\int_0^t \frac{ds}{\alpha_2^2 - \gamma_1 - (\gamma_2 - \gamma_1)\operatorname{sn}^2(\sigma s, \tau)}; \textit{ and} \\ \theta_3(t) &= A\int_0^t \frac{ds}{\alpha_3^2 - \gamma_1 - (\gamma_2 - \gamma_1)\operatorname{sn}^2(\sigma s, \tau)}, \end{aligned}$$

where $\sigma = \sqrt{\gamma_3 - \gamma_1}$ *and* $\tau = \sqrt{\dfrac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_1}}$.

(iv) $A = \alpha_1\alpha_2\alpha_3$. *Define* $a_1, a_2, a_3 \in \mathbb{R}$ *by:*

$$a_1 = -\frac{\alpha_2\alpha_3}{\alpha_1}; \qquad a_2 = \frac{\alpha_3\alpha_1}{\alpha_2}; \quad \textit{and} \quad a_3 = \frac{\alpha_1\alpha_2}{\alpha_3}.$$

Then $a_1 + a_2 + a_3 = 0$, *since* $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$, *and* z_1, z_2, z_3 *are given by:*

$$2z_1(t) = i\alpha_1 e^{ia_1 t}; \qquad 2z_2(t) = \alpha_2 e^{ia_2 t}; \quad \textit{and} \quad 2z_3(t) = \alpha_3 e^{ia_3 t}.$$

Chapter 4

Associative 3-folds in $\mathbb{R}^7$

In this chapter we construct examples of associative 3-folds in $\mathbb{R}^7$ by three separate methods. The first method, described in §4.2, produces 3-folds with *symmetries* using *evolution equations.* In §4.4 an *affine* evolution equation gives a 14-dimensional family of associative 3-folds. Finally, in §4.5, we consider *1-ruled* 3-folds. The material exhibited is a generalisation of the work of Joyce in [24], [25], [26] and [27].

4.1 The First Evolution Equation

To derive our evolution equation we shall require two results related to *real analyticity.* The first is an immediate corollary of Theorem 1.1.3.

Theorem 4.1.1. *An associative 3-fold in $\mathbb{R}^7$ is real analytic wherever it is nonsingular.*

The proof of the next result [17, Theorem IV.4.1] relies on the *Cartan–Kähler Theorem*, which is only applicable in the real analytic category.

Theorem 4.1.2. *Let P be a 2-dimensional real analytic submanifold of $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$. There locally exists a real analytic associative 3-fold N in $\mathbb{R}^7$ which contains P. Moreover, N is locally unique.*

We now formulate an evolution equation for associative 3-folds, given a 2-dimensional real analytic submanifold of $\mathbb{R}^7$, following Theorem 3.2.2.

Theorem 4.1.3. *Let P be a compact, orientable, 2-dimensional, real analytic manifold, χ a real analytic nowhere vanishing section of $\Lambda^2 TP$ and $\psi : P \to \mathbb{R}^7$ a real analytic embedding (immersion). There exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{R}^7$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^d = (\psi_t)_*(\chi)^{ab}(\varphi_0)_{abc}(g_0)^{cd}, \tag{4.1}$$

where $(g_0)^{cd}$ is the inverse of the Euclidean metric on $\mathbb{R}^7$, using index notation for tensors on $\mathbb{R}^7$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^7$ by $\Psi(t, p) = \psi_t(p)$. Then $M = \operatorname{Image}\Psi$ is a nonsingular embedded (immersed) associative 3-fold in $\mathbb{R}^7$.

Note that, if $(\psi_t)_*(\chi) = \sum_{j<k} \lambda_{jk}(t)\, \mathbf{e}_j \wedge \mathbf{e}_k$, where $\mathbf{e}_j = \frac{\partial}{\partial x_j}$, the right-hand side of (4.1) is equal to $\sum_{j<k} \lambda_{jk}(t)\, \mathbf{e}_j \times \mathbf{e}_k$ by (2.9).

Proof. The theorem is proved in an entirely similar way to Theorem 3.2.2, so we only give a sketch of the key ideas. Since P is compact and P, χ, ψ are real analytic, the Cauchy–Kowalevsky Theorem (Theorem 1.1.4) gives a family of maps ψ_t and hence M as stated. Theorem 4.1.2 implies there locally exists a locally unique associative 3-fold N containing $\psi(P)$. Showing that N and M agree near $\psi(P)$ allows us to deduce that M is associative. $\square$

4.2 Constructions Using Symmetries

Now that we have a means of constructing associative 3-folds we shall consider the situation where the associative 3-fold has a large symmetry group. The study of symmetric geometry usually leads to more straightforward calculations than the general case, so this motivates our first method of construction. We know from Proposition 2.3.7 that we must consider subgroups of $\mathrm{G}_2 \ltimes \mathbb{R}^7$ as symmetry groups for our associative 3-folds. Suppose that G is a Lie subgroup of $\mathrm{G}_2 \ltimes \mathbb{R}^7$ which has a two-dimensional orbit $\mathcal{O}$. Theorem 4.1.3 allows us to evolve each point in $\mathcal{O}$ orthogonally to the action of G and hence, hopefully, construct an associative 3-fold with symmetry group G.

It may seem natural to first consider $\mathrm{U}(1)^2$-invariant associative 3-folds, but the construction will in fact only yield $\mathrm{U}(1)^2$-invariant special Lagrangian 3-folds in $\mathbb{C}^3$, which have already been studied [24]. However this does motivate our first construction.

4.2.1 A subgroup of $\mathbb{R} \times \mathrm{U}(1)^2$

We may decompose $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ and so the action of $\mathbb{R} \times \mathrm{U}(1)^2$ on $\mathbb{R}^7$ may be written as:

$$(x_1, z_1, z_2, z_3) \longmapsto (x_1 + c,\, e^{i\phi_1} z_1,\, e^{i\phi_2} z_2,\, e^{-i(\phi_1+\phi_2)} z_3) \qquad \text{for } c, \phi_1, \phi_2 \in \mathbb{R}. \tag{4.2}$$

However, we want a two-dimensional orbit, so we choose a two-dimensional subgroup of $\mathbb{R} \times \mathrm{U}(1)^2$.

Definition 4.2.1. Let λ, μ, ν be real numbers which are not all zero. Define G to be the subgroup of $\mathbb{R} \times \mathrm{U}(1)^2$ which acts as in (4.2) with the following imposed:

$$\lambda c + \mu\phi_1 + \nu\phi_2 = 0. \tag{4.3}$$

If $\mu = \nu = 0$ then G is $\mathrm{U}(1)^2$. Suppose $\mu\nu \neq 0$. If there exist coprime integers p and q such that $\mu p + \nu q = 0$ then G is $\mathbb{R} \times \mathrm{U}(1)$ and otherwise it is an $\mathbb{R}^2$ subgroup.

Define smooth maps $\psi_t : \mathrm{G} \to \mathbb{R}^7$ by:

$$\psi_t(c, e^{i\phi_1}, e^{i\phi_2}) = (x_1(t) + c,\, e^{i\phi_1} z_1(t),\, e^{i\phi_2} z_2(t),\, e^{-i(\phi_1+\phi_2)} z_3(t)), \tag{4.4}$$

where $x_1(t)$, $z_1(t) = x_2(t) + ix_3(t)$, $z_2(t) = x_4(t) + ix_5(t)$ and $z_3(t) = x_6(t) + ix_7(t)$ are smooth functions of t.

Using (4.4) we calculate:

$$\begin{aligned}\mathbf{v}_1 = (\psi_t)_*\left(\frac{\partial}{\partial\phi_1}\right) &= i\left(z_1\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1}\right) - i\left(z_3\frac{\partial}{\partial z_3} - \bar{z}_3\frac{\partial}{\partial \bar{z}_3}\right)\\ &= x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} - x_6\frac{\partial}{\partial x_7} + x_7\frac{\partial}{\partial x_6}\,; \end{aligned}\tag{4.5}$$

$$\begin{aligned}\mathbf{v}_2 = (\psi_t)_*\left(\frac{\partial}{\partial\phi_2}\right) &= i\left(z_2\frac{\partial}{\partial z_2} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2}\right) - i\left(z_3\frac{\partial}{\partial z_3} - \bar{z}_3\frac{\partial}{\partial \bar{z}_3}\right)\\ &= x_4\frac{\partial}{\partial x_5} - x_5\frac{\partial}{\partial x_4} - x_6\frac{\partial}{\partial x_7} + x_7\frac{\partial}{\partial x_6}\,; \text{ and} \end{aligned}\tag{4.6}$$

$$\mathbf{v}_3 = (\psi_t)_*\left(\frac{\partial}{\partial c}\right) = \frac{\partial}{\partial x_1}\,, \tag{4.7}$$

where $z_1 = x_2 + ix_3$, $z_2 = x_4 + ix_5$ and $z_3 = x_6 + ix_7$. These three vectors are tangential to the action of $\mathbb{R} \times \mathrm{U}(1)^2$; the vector $\mathbf{v}_1$ corresponds to a rotation of z_1 by i and of z_3 by $-i$, and similarly for $\mathbf{v}_2$. The condition (4.3) corresponds to the choice of a nowhere vanishing section χ of $\Lambda^2 T\mathrm{G}$:

$$\chi = \lambda\frac{\partial}{\partial\phi_1}\wedge\frac{\partial}{\partial\phi_2} + \mu\frac{\partial}{\partial\phi_2}\wedge\frac{\partial}{\partial c} + \nu\frac{\partial}{\partial c}\wedge\frac{\partial}{\partial\phi_1}\,.$$

If we let $\mathbf{u} = \mathbf{v}_1 \wedge \mathbf{v}_2$, $\mathbf{v} = \mathbf{v}_2 \wedge \mathbf{v}_3$ and $\mathbf{w} = \mathbf{v}_3 \wedge \mathbf{v}_1$ then

$$(\psi_t)_*(\chi) = \lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w}.$$

Therefore, using (4.5)-(4.7) and equation (2.6) for φ_0 we find that, writing $\mathbf{e}_j = \frac{\partial}{\partial x_j}$,

$$\begin{aligned}(\psi_t)_*(\chi)^{ab}(\varphi_0)_{abc}(g_0)^{cd} = {}& \big(\lambda(x_5x_7 - x_4x_6) - \nu x_2\big)\mathbf{e}_2 + \big(\lambda(x_5x_6 + x_4x_7) - \nu x_3\big)\mathbf{e}_3\\ &+ \big(\lambda(x_3x_7 - x_2x_6) + \mu x_4\big)\mathbf{e}_4 + \big(\lambda(x_3x_6 + x_2x_7) + \mu x_5\big)\mathbf{e}_5\\ &+ \big(\lambda(x_3x_5 - x_2x_4) + (\nu-\mu)x_6\big)\mathbf{e}_6 + \big(\lambda(x_3x_4 + x_2x_5) + (\nu-\mu)x_7\big)\mathbf{e}_7. \end{aligned}\tag{4.8}$$

We also have that

$$\frac{d\psi_t}{dt} = \sum_{j=1}^{7}\frac{dx_j(t)}{dt}\,\mathbf{e}_j. \tag{4.9}$$

Equating both sides of (4.1) using (4.8) and (4.9) and applying Theorem 4.1.3 provides the following theorem.

Theorem 4.2.2. *Let $x_1(t)$ be a real-valued function and let $z_1(t)$, $z_2(t)$, $z_3(t)$ be complex-valued functions such that*

$$\frac{dx_1}{dt} = 0\,, \tag{4.10}$$

$$\frac{dz_1}{dt} = -\nu z_1 - \lambda\overline{z_2 z_3}\,, \tag{4.11}$$

$$\frac{dz_2}{dt} = \mu z_2 - \lambda\overline{z_3 z_1} \text{ and} \tag{4.12}$$

$$\frac{dz_3}{dt} = (\nu-\mu)z_3 - \lambda\overline{z_1 z_2}\,, \tag{4.13}$$

using the notation from Definition 4.2.1. There exists $\epsilon > 0$ such that these equations have a solution for $t \in (-\epsilon, \epsilon)$ and the subset M of $\mathbb{R}\oplus\mathbb{C}^3 \cong \mathbb{R}^7$ defined by

$$M = \left\{\left(x_1(t)+c,\, e^{i\phi_1}z_1(t),\, e^{i\phi_2}z_2(t),\, e^{-i(\phi_1+\phi_2)}z_3(t)\right) : t \in (-\epsilon,\epsilon),\, (c, e^{i\phi_1}, e^{i\phi_2}) \in \mathrm{G}\right\}$$

is an associative 3-fold in $\mathbb{R}^7$. Moreover, M does not lie in $\{x\}\times\mathbb{C}^3$ for any $x \in \mathbb{R}$, as long as not both μ and ν are zero, and (4.11)-(4.13) *imply that* $\mathrm{Im}(z_1z_2z_3) = A$*, where A is a real constant.*

Proof. We only need to prove the last sentence in the statement above. We deduce immediately from (4.10) that x_1 is constant in the direction perpendicular to the group action, though it *is* changing along the group action (as long as not both μ and ν are zero), which means that M does not lie in $\{x\}\times\mathbb{C}^3$ for any real constant x in this case. We also note from (4.11)-(4.13) that

$$\frac{d}{dt}(z_1z_2z_3) = -\lambda(|z_2|^2|z_3|^2 + |z_3|^2|z_1|^2 + |z_1|^2|z_2|^2),$$

which is real, therefore $\mathrm{Im}(z_1z_2z_3)$ is a real constant. □

The case $\lambda = 0$ is not geometrically interesting since it implies that G contains all possible translations in the first coordinate. We can solve (4.11)-(4.13) in this case and show that

$$M = \mathbb{R}\times\left\{\left(A_1e^{i\phi_1-\nu t},\, A_2e^{i\phi_2+\mu t},\, A_3e^{-i(\phi_1+\phi_2)+(\nu-\mu)t}\right) : t\in\mathbb{R},\, \mu\phi_1+\nu\phi_2 = 0\right\},$$

where A_1, A_2, A_3 are complex constants such that $\mathrm{Im}(A_1A_2A_3) = A$. The expression in brackets above defines a holomorphic curve in $\mathbb{C}^3$.

We now restrict to $\lambda \neq 0$ and hence we can normalise so that $\lambda = 1$. We may also suppose that μ and ν are not both zero, since $\mu = \nu = 0$ forces $c = 0$ in G and so there is no translation action in G, which means that M will be an embedded $\mathrm{U}(1)^2$-invariant SL 3-fold as studied in [17, §III.3.A]:

$$M = \left\{(x_1,z_1,z_2,z_3)\in\mathbb{R}^7 : x_1 = x,\, \mathrm{Im}(z_1z_2z_3) = A,\, |z_1|^2-|z_3|^2 = B,\, |z_2|^2-|z_3|^2 = C\right\}$$

for some $x, A, B, C \in \mathbb{R}$.

Writing $z_j(t) = r_j(t)e^{i\theta_j(t)}$ for $j = 1, 2, 3$ and $\theta = \theta_1 + \theta_2 + \theta_3$, equations (4.11)-(4.13) with the condition $\mathrm{Im}(z_1 z_2 z_3) = A$ become:

$$\frac{dr_1}{dt} = -\nu r_1 - r_2 r_3 \cos\theta; \tag{4.14}$$

$$\frac{dr_2}{dt} = \mu r_2 - r_3 r_1 \cos\theta; \tag{4.15}$$

$$\frac{dr_3}{dt} = (\nu - \mu) r_3 - r_1 r_2 \cos\theta; \tag{4.16}$$

$$r_j^2 \frac{d\theta_j}{dt} = A \qquad \text{for } j = 1, 2, 3; \text{ and} \tag{4.17}$$

$$r_1 r_2 r_3 \sin\theta = A. \tag{4.18}$$

Asymptotic Behaviour

Suppose $A > 0$. This forces $\sin\theta > 0$ and $r_j > 0$ for $j = 1, 2, 3$ by equation (4.18). Since we can use the $\mathrm{U}(1)^2$ symmetry to rotate z_2 and z_3 onto the real axis at $t = 0$, we may take $\theta(0) = \theta_1(0) \in (0, \pi)$. Moreover, $\sin\theta(t) > 0$ for all t and $\theta(t)$ is continuous by assumption, so $\theta(t) \in (0, \pi)$ for all t. Then (4.17) implies that the functions θ_j for $j = 1, 2, 3$ are strictly increasing and hence they are positive for all $t \geq 0$. Furthermore, θ is strictly increasing and bounded above, so each θ_j is bounded above. Let $T > 0$ and $T' > 0$ be maximal such that solutions exist for all $t \in (-T', T)$, where we allow T and T' to take the value infinity.

Suppose first that solutions exist for all $t \in \mathbb{R}$. Suppose further, for a contradiction, that $r_j^2 \leq B$ for some $B > 0$ and some j. Then $\frac{d\theta_j}{dt} \geq \frac{A}{B}$ for all t, which implies that $\theta_j \to \infty$ as $t \to \infty$, contradicting the boundedness of θ_j. Therefore, r_j is unbounded for all j as $t \to \pm\infty$.

Alternatively, suppose $T > 0$ and $T' > 0$ are finite. If r_j is bounded for all j then it is possible to show that solutions exist for all $t \in \mathbb{R}$, which is a contradiction to our discussion above. Therefore, at least one of r_j becomes unbounded as $t \nearrow T$. Let $a_1 = \nu$, $a_2 = -\mu$ and $a_3 = \mu - \nu$ and let $f_j = e^{2a_j t} r_j^2$ for all j. Then (4.14)-(4.16) become:

$$\frac{df_j}{dt} = -e^{2a_j t}\sqrt{f_1 f_2 f_3 - A^2} \qquad \text{for } j = 1, 2, 3, \tag{4.19}$$

where we have used (4.18). It is then clear that the ratios $|f_j(t) - f_j(0)|/|f_k(t) - f_k(0)|$ are bounded as $t \nearrow T$ for all j and k. Thus, r_j is unbounded as $t \nearrow T$ for all j. Similar arguments show that r_j becomes unbounded as $t \searrow -T'$ for all j.

In both cases, r_j is unbounded for all j as $t \nearrow T$ and $t \searrow -T'$. Suppose, for a contradication, that $r_1 r_2 r_3$ remains bounded. Equations (4.19) and the boundedness of $f_1 f_2 f_3$ imply that r_j is bounded for any j such that $a_j \geq 0$: our required contradiction. Thus, $r_1 r_2 r_3$ is unbounded and equation (4.18) implies that there exist sequences (t_n) and $(-t'_n)$, tending to T and $-T'$ respectively, such that $(\sin\theta(t_n))$ and $(\sin\theta(-t'_n))$ are strictly decreasing sequences tending to 0. However, θ is a strictly increasing function taking values in $(0,\pi)$ and hence $\sin\theta$ must be eventually decreasing as $t\nearrow T$ and $t\searrow -T'$. Therefore, $\sin\theta\to 0$ as $t\nearrow T$ and $t\searrow -T'$. We conclude that $\theta\to\pi$ as $t\nearrow T$ and $\theta\to 0$ as $t\searrow -T'$. Moreover, $r_1r_2r_3\to\infty$ as $t\nearrow T$ and $t\searrow -T'$.

The upshot of this analysis is that we can now write down the asymptotic differential equations. Since $\cos\theta\to -1$, the approximate equations as $t\nearrow T$ are thus:

$$\frac{dr_1}{dt}=-\nu r_1+r_2r_3;\qquad \frac{dr_2}{dt}=\mu r_2+r_3r_1;\quad\text{and}\quad \frac{dr_3}{dt}=(\nu-\mu)r_3+r_1r_2.$$

Equations (4.19) imply that r_1, r_2 and r_3 must eventually grow at the same rate as $t\nearrow T$, so we approximate further by neglecting the linear terms, which will be dominated by the quadratic parts as $t\nearrow T$, to give:

$$r_j\frac{dr_j}{dt}=r_1r_2r_3\qquad\text{for } j=1,2,3.$$

These equations can be solved to show that, for t sufficiently near T, which must be finite,

$$r_j\sim\frac{1}{T-t}\qquad\text{for } j=1,2,3.$$

Hence the functions r_1,r_2 and r_3 reach infinity in finite positive time. Similarly, since $\cos\theta\to 1$ as $t\searrow -T'$ we deduce that T' must be finite and that, for t sufficiently near $-T'$,

$$r_j\sim\frac{1}{T'+t}\qquad\text{for } j=1,2,3.$$

Hence the functions r_1,r_2 and r_3 reach infinity in finite negative time.

We conclude that, at infinity, these associative 3-folds behave like the standard $\mathrm{U}(1)^2$-invariant SL 3-fold as studied by Harvey and Lawson [17, §III.3.A]:

$$\{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|=|z_2|=|z_3|,\ \mathrm{Im}(z_1z_2z_3)=A,\ \mathrm{Re}(z_1z_2z_3)>0\}.$$

This is the result expected since the translation component of the group action becomes negligible in comparison to the $\mathrm{U}(1)^2$ action as r_1, r_2 and r_3 become large.

4.2.2 U(1)-invariant cones

In this subsection we consider associative 3-folds which are invariant both under an action of U(1) on the $\mathbb{C}^3$ component of $\mathbb{R}^7\cong\mathbb{R}\oplus\mathbb{C}^3$, and under dilations.

Definition 4.2.3. Let $\mathbb{R}^+$ denote the group of positive real numbers under multiplication. The group action of $\mathbb{R}^+\times\mathrm{U}(1)$ on $\mathbb{R}^7\cong\mathbb{R}\oplus\mathbb{C}^3$ is given by, for some fixed $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$:

$$(x_1,z_1,z_2,z_3)\longmapsto(rx_1,\,re^{is\alpha_1}z_1,\,re^{is\alpha_2}z_2,\,re^{is\alpha_3}z_3)\qquad r>0,\ s\in\mathbb{R}.$$

To ensure we have a U(1) action in G_2, we choose α_1, α_2, α_3 to be coprime integers satisfying $\alpha_1+\alpha_2+\alpha_3=0$.

Define smooth maps $\psi_t:\mathbb{R}^+\times \mathrm{U}(1)\to\mathbb{R}^7$ by:

$$\psi_t(r,e^{is})=(rx_1(t),\, re^{is\alpha_1}z_1(t),\, re^{is\alpha_2}z_2(t),\, re^{is\alpha_3}z_3(t)), \tag{4.20}$$

where $x_1(t)$, $z_1(t)=x_2(t)+ix_3(t)$, $z_2(t)=x_4(t)+ix_5(t)$ and $z_3(t)=x_6(t)+ix_7(t)$ are smooth functions of t.

Using (4.20) we calculate the tangent vectors to the group action given in Definition 4.2.3:

$$\mathbf{u}=(\psi_t)_*\left(\frac{\partial}{\partial r}\right)=\sum_{j=1}^{7}x_j\frac{\partial}{\partial x_j}\ \text{and}$$

$$\mathbf{v}=(\psi_t)_*\left(\frac{\partial}{\partial s}\right)=\alpha_1\left(x_2\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_2}\right)+\alpha_2\left(x_4\frac{\partial}{\partial x_5}-x_5\frac{\partial}{\partial x_4}\right)+\alpha_3\left(x_6\frac{\partial}{\partial x_7}-x_7\frac{\partial}{\partial x_6}\right).$$

If we take $\chi=\frac{\partial}{\partial r}\wedge\frac{\partial}{\partial s}$ then $(\psi_t)_*(\chi)=\mathbf{u}\wedge\mathbf{v}$. We deduce that, writing $\mathbf{e}_j=\frac{\partial}{\partial x_j}$,

$$\begin{aligned}\mathbf{u}\times\mathbf{v}&=\big(\alpha_1(x_2^2+x_3^2)+\alpha_2(x_4^2+x_5^2)+\alpha_3(x_6^2+x_7^2)\big)\mathbf{e}_1\\&\quad+\big(-\alpha_1x_1x_2+(\alpha_2-\alpha_3)(x_4x_7+x_5x_6)\big)\mathbf{e}_2+\big(-\alpha_1x_1x_3+(\alpha_2-\alpha_3)(x_4x_6-x_5x_7)\big)\mathbf{e}_3\\&\quad+\big(-\alpha_2x_1x_4+(\alpha_3-\alpha_1)(x_2x_7+x_3x_6)\big)\mathbf{e}_4+\big(-\alpha_2x_1x_5+(\alpha_3-\alpha_1)(x_2x_6-x_3x_7)\big)\mathbf{e}_5\\&\quad+\big(-\alpha_3x_1x_6+(\alpha_1-\alpha_2)(x_2x_5+x_3x_4)\big)\mathbf{e}_6+\big(-\alpha_3x_1x_7+(\alpha_1-\alpha_2)(x_2x_4-x_3x_5)\big)\mathbf{e}_7.\end{aligned}$$

We also have that

$$\frac{d\psi_t}{dt}=\sum_{j=1}^{7}\frac{dx_j(t)}{dt}\mathbf{e}_j.$$

Equating both sides of (4.1) using the above formulae as in §4.2.1 we obtain the following theorem.

Theorem 4.2.4. *Use the notation of Definition 4.2.3. Let $\beta_1=\alpha_2-\alpha_3$, $\beta_2=\alpha_3-\alpha_1$ and $\beta_3=\alpha_1-\alpha_2$. Let $x_1(t)$ be a real-valued function and let $z_1(t)$, $z_2(t)$, $z_3(t)$ be complex-valued functions such that*

$$\frac{dx_1}{dt}=\alpha_1|z_1|^2+\alpha_2|z_2|^2+\alpha_3|z_3|^2, \tag{4.21}$$

$$\frac{dz_1}{dt}=-\alpha_1x_1z_1+i\beta_1\overline{z_2z_3}, \tag{4.22}$$

$$\frac{dz_2}{dt}=-\alpha_2x_1z_2+i\beta_2\overline{z_3z_1}\ \textit{and} \tag{4.23}$$

$$\frac{dz_3}{dt}=-\alpha_3x_1z_3+i\beta_3\overline{z_1z_2}. \tag{4.24}$$

These equations have a solution for all $t\in\mathbb{R}$ and the subset M of $\mathbb{R}\oplus\mathbb{C}^3\cong\mathbb{R}^7$ defined by

$$M=\big\{\big(rx_1(t),\, re^{is\alpha_1}z_1(t),\, re^{is\alpha_2}z_2(t),\, re^{is\alpha_3}z_3(t)\big)\,:\, r\in\mathbb{R}^+,\ s,t\in\mathbb{R}\big\}$$

is an associative 3-fold in $\mathbb{R}^7$. *Moreover,* (4.21)-(4.24) *imply that* $x_1^2+|z_1|^2+|z_2|^2+|z_3|^2$ *can be chosen to be* 1 *and that* $\mathrm{Re}(z_1z_2z_3)=A$*, where* A *is a real constant.*

Proof. Noting that $\beta_1+\beta_2+\beta_3=0$, we immediately see that $x_1^2+|z_1|^2+|z_2|^2+|z_3|^3$ is a constant which we can take to be one. This is hardly surprising since the associative 3-fold was constructed so as to be a cone. We also see from (4.22)-(4.24) that

$$\frac{d}{dt}(z_1z_2z_3)=i(\beta_1|z_2|^2|z_3|^2+\beta_2|z_3|^2|z_1|^2+\beta_3|z_1|^2|z_2|^2),$$

which is purely imaginary, and therefore $\mathrm{Re}(z_1z_2z_3)=A$ is a real constant.

The proof that (4.21)-(4.24) have solutions for all $t\in\mathbb{R}$ is deferred until later in the subsection. □

Writing $z_j(t)=r_j(t)e^{i\theta_j(t)}$ for $j=1,2,3$ and $\theta=\theta_1+\theta_2+\theta_3$, (4.21)-(4.24) become:

$$\frac{dx_1}{dt}=\alpha_1r_1^2+\alpha_2r_2^2+\alpha_3r_3^2; \tag{4.25}$$

$$\frac{dr_1}{dt}=-\alpha_1x_1r_1+\beta_1r_2r_3\sin\theta; \tag{4.26}$$

$$\frac{dr_2}{dt}=-\alpha_2x_1r_2+\beta_2r_3r_1\sin\theta; \tag{4.27}$$

$$\frac{dr_3}{dt}=-\alpha_3x_1r_3+\beta_3r_1r_2\sin\theta;\ \text{and} \tag{4.28}$$

$$r_j^2\frac{d\theta_j}{dt}=\beta_jA \qquad \text{for } j=1,2,3, \tag{4.29}$$

with the conditions

$$x_1^2+r_1^2+r_2^2+r_3^2=1\ \text{ and} \tag{4.30}$$

$$r_1r_2r_3\cos\theta=A. \tag{4.31}$$

Unlike in §4.2.1, we are restricted in our choices of the real parameter A. The problem of maximising A^2, by (4.30) and (4.31), is equivalent to the problem of maximising $r_1^2r_2^2r_3^2$ subject to $r_1^2+r_2^2+r_3^2=1$. By direct calculation the solution is $r_1^2=r_2^2=r_3^2=\frac{1}{3}$. Therefore $A\in\left[-\frac{1}{3\sqrt{3}},\frac{1}{3\sqrt{3}}\right]$. We can restrict to $A\geq 0$ since changing the sign of A corresponds to reversing the sign of $\cos\theta$, so the addition of π to θ.

The case where $A=\frac{1}{3\sqrt{3}}$ is immediately soluble since this forces $r_1=r_2=r_3=\frac{1}{\sqrt{3}}$, which implies $x_1=0$ by (4.30) and $\cos\theta=1$ by (4.31), so we can take $\theta=0$. Equations (4.29) become

$$\frac{1}{3}\frac{d\theta_j}{dt}=\frac{1}{3\sqrt{3}}\beta_j \qquad \text{for } j=1,2,3,$$

which can easily be solved, along with the condition $\theta=0$, to give:

$$\theta_j(t)=\frac{\beta_j}{\sqrt{3}}t+\gamma_j \qquad \text{for } j=1,2,3,$$

where γ_1, γ_2, γ_3 are real constants which sum to zero. Then

$$M = \left\{\left(0,\, re^{i\phi_1},\, re^{i\phi_2},\, re^{i\phi_3}\right) : r > 0,\, \phi_1, \phi_2, \phi_3 \in \mathbb{R},\, \phi_1 + \phi_2 + \phi_3 = 0\right\},$$

which is a $\mathrm{U}(1)^2$-invariant special Lagrangian cone, as studied in [17, §III.3.A], embedded in $\mathbb{R}^7$ and is therefore in itself not an interesting object of study here. Any associative 3-fold constructed with $x_1 = 0$ will be at least a U(1)-invariant special Lagrangian cone and so we shall not consider this situation further. However, we know that M must be the limiting case of the family of associative 3-folds parameterised by A as it tends to $\frac{1}{3\sqrt{3}}$.

We may also solve the equations in the following special case.

Theorem 4.2.5. *Use the notation of Theorem 4.2.4. Suppose that $\alpha_2 = \alpha_3$. Then x_1, z_1, z_2 and z_3 may be chosen to satisfy $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ and $\operatorname{Im} z_1 = 0$. Moreover, they satisfy:*

$$\operatorname{Re}(z_1z_2z_3) = A; \quad |z_1|(x_1^2 + |z_1|^2 - 1) = B; \quad \operatorname{Re}(z_1(z_2^2 - z_3^2)) = C; \quad \textit{and} \quad \operatorname{Im}(z_1(z_2^2 + z_3^2)) = D$$

for some real constants A, B, C and D.

Proof. Since $\beta_1 = 0$, (4.29) implies that the argument of z_1 is constant. Using U(1) we can take it to be zero so that z_1 is real. Moreover, $\beta_1 = 0$ and (4.30) imply that x_1 and z_1 evolve amongst themselves and hence, using (4.21) and (4.22), we deduce that the real function $f = |z_1|(x_1^2+|z_1|^2-1)$ is constant. Note that SU(2) acts on the (z_2, z_3)-plane. We are thus led to calculate

$$\frac{d}{dt}\left(z_1(az_2 + bz_3)(-\bar{b}z_2 + \bar{a}z_3)\right) = -4i\beta_2|z_1|^2 \operatorname{Re}(a\bar{b}z_2z_3) + i\beta_2|z_1|^2(|a|^2 - |b|^2)(|z_3|^2 - |z_2|^2)$$

for constants $a, b \in \mathbb{C}$, which is purely imaginary. Equating real parts for $(a, b) = (1, -1)$ and $(a, b) = (i, 1)$ leads to the final two conserved quantities in the statement of the theorem. $\square$

Note that in the theorem above we have six conditions on seven variables, which thus determine the associative cone constructed by Theorem 4.2.4 for $\alpha_2 = \alpha_3$. Moreover, we may construct a function $\pi : \mathbb{R} \oplus \mathbb{C}^3 \to \mathbb{R}^6$ by mapping (x_1, z_1, z_2, z_3) to the six real constant functions given in Theorem 4.2.5, which are defined by the initial values $(x_1(0), z_1(0), z_2(0), z_3(0))$.

Sard's Theorem [33, p. 173] states that, if $f : X \to Y$ is a smooth map between finite-dimensional manifolds, the set of $y \in Y$ with some $x \in f^{-1}(y)$ such that $df|_x : T_xX \to T_yY$ is *not* surjective is of measure zero in Y. Therefore, $f^{-1}(y)$ is a submanifold of X of dimension $\dim X - \dim Y$ for almost all $y \in Y$. Applying Sard's Theorem, generically the fibres of π will be 1-dimensional submanifolds of $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$. Moreover, we know that these fibres are compact by the conditions in Theorem 4.2.5. Hence, the variables form loops in $\mathbb{R}^7$ for generic initial values; i.e. the solutions are *periodic* in t. We deduce the following result.

Theorem 4.2.6. *Use the notation of Theorem 4.2.4 and suppose that $\alpha_2 = \alpha_3$. For generic values of the functions x_1, z_1, z_2 and z_3 at $t = 0$, the associative 3-folds constructed by Theorem 4.2.4 are closed* $\mathrm{U}(1)$*-invariant cones over T^2 in $\mathbb{R}^7$.*

Domain of Definition

Theorem 4.1.3 gives us that a solution to (4.21)-(4.24) in Theorem 4.2.4 exists for sufficiently small values of t, but we want to be able to say more about the domain of the solutions.

Rewriting (4.25)-(4.29) using $y = r_1 r_2 r_3 \sin\theta$ and $s_j = r_j^2$ for $j = 1, 2, 3$:

$$\frac{dx_1}{dt} = \sum_{j=1}^{3} \alpha_j s_j; \quad \frac{ds_j}{dt} = -2\alpha_j x_1 s_j + 2\beta_j y \text{ for } j = 1, 2, 3; \text{ and } \frac{dy}{dt} = \beta_1 s_2 s_3 + \beta_2 s_3 s_1 + \beta_3 s_1 s_2. \tag{4.32}$$

Using (4.30) and (4.31), the solutions to (4.32) lie on the manifold

$$N_A = \{(x_1, s_1, s_2, s_3, y) \, : \, x_1^2 + s_1 + s_2 + s_3 = 1, \; A^2 + y^2 = s_1 s_2 s_3, \; s_1, s_2, s_3 \geq 0\}.$$

If we write $\mathbf{x} = (x_1, s_1, s_2, s_3, y)$, (4.32) can be cast in the form

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

for some appropriate function $f : \mathbb{R}^5 \to \mathbb{R}^5$. It is clear that f is continuous on N_A, and since the solution lies in N_A it follows that f is bounded on N_A, as N_A is a closed and bounded and hence compact subset of $\mathbb{R}^5$. Moreover, f will be bounded on a sufficiently small open neighbourhood U of N_A. We may then multiply f by a smooth bump function which takes the value one on N_A and is zero outside of U. We have thus constructed a function which equals f on N_A and is continuous and bounded on $\mathbb{R}^5$. We are then able to apply a result from the theory of ordinary differential equations taken from [11, p. 53] after a couple of definitions also taken from [11].

Definition 4.2.7. Given a solution $x(t)$ to an ordinary differential equation defined on an interval (a, b), an *extension* is another solution $\tilde{x}(t)$ defined on $(c, d) \supseteq (a, b)$ which equals $x(t)$ on (a, b).

Definition 4.2.8. A solution $x(t)$ to an ordinary differential equation defined on an interval (a, b) is *maximal* if for any extension $\tilde{x}(t)$ which is defined on (c, d) then $(a, b) = (c, d)$, hence $x = \tilde{x}$.

Theorem 4.2.9. *Suppose $\tilde{f}$ is continuous on $(t, \mathbf{x})$-space and that $\tilde{f}$ is bounded. Any maximal solution $\mathbf{x}(t)$ to the equation*

$$\frac{d\mathbf{x}}{dt} = \tilde{f}(t, \mathbf{x})$$

has the t-axis as its domain.

Therefore, since we can find maximal solutions to equations (4.32), they must be defined for all $t \in \mathbb{R}$, and hence there exist solutions to (4.21)-(4.24) defined for all $t \in \mathbb{R}$. This provides the missing detail in the proof of Theorem 4.2.4.

4.3 The Second Evolution Equation

In general it is difficult to use Theorem 4.1.3 as stated to construct associative 3-folds since it is an *infinite-dimensional* evolution problem. We follow the material in [25, §3] to reduce the theorem to a *finite-dimensional* problem.

Definition 4.3.1. Let (P, χ) be a set of affine evolution data, in the sense of Definition 3.2.3, for $m = 3$ and some $n \geq 3$. Let $\mathrm{Aff}\,(\mathbb{R}^n, \mathbb{R}^7)$ be the affine space of affine maps $\psi : \mathbb{R}^n \to \mathbb{R}^7$. Define $\mathcal{C}_P$ as the set of $\psi \in \mathrm{Aff}\,(\mathbb{R}^n, \mathbb{R}^7)$ such that $\psi|_{T_pP} : T_pP \to \mathbb{R}^7$ is injective for all p in a dense open subset of P. Let V be an associative 3-plane in $\mathbb{R}^7$. Generic linear maps $\psi : \mathbb{R}^n \to V$ will satisfy the condition to be members of $\mathcal{C}_P$. Hence $\mathcal{C}_P$ is non-empty.

We formulate our second evolution equation following Theorem 3.2.4.

Theorem 4.3.2. *Let (P, χ) be a set of affine evolution data and n, $\mathrm{Aff}\,(\mathbb{R}^n, \mathbb{R}^7)$ and $\mathcal{C}_P$ be as in Definition 4.3.1. Suppose $\psi \in \mathcal{C}_P$. There exist $\epsilon > 0$ and a unique real analytic family $\{\psi_t : t \in (-\epsilon, \epsilon)\} \subseteq \mathcal{C}_P$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}(x)\right)^d = (\psi_t)_*(\chi(x))^{ab}(\varphi_0)_{abc}(g_0)^{cd} \tag{4.33}$$

for all $x \in \mathbb{R}^n$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^7$ by $\Psi(t, p) = \psi_t(p)$. Then $M = \mathrm{Image}\,\Psi$ is an associative 3-fold wherever it is nonsingular.

Proof. As for Theorem 4.1.3, the proof is almost identical to the special Lagrangian case (Theorem 3.2.4), so we omit the details. Since (4.33) is of the form $\frac{d\psi_t}{dt} = Q(\psi_t)$ where Q is a quadratic, the *Cauchy–Picard Theorem* [11, p. 14] of ordinary differential equations gives existence and uniqueness of the family of maps stated. The proof then mirrors that of Theorem 3.2.2, noting that we can drop the compactness condition for P since it was only needed previously to establish the existence of the maps ψ_t. Finally, $\psi_t \in \mathcal{C}_P$ for t sufficiently small since $\psi_0 \in \mathcal{C}_P$ and the condition to lie in $\mathcal{C}_P$ is an open one. □

Before we construct associative 3-folds using this result, it is worth noting that using quadrics to provide affine evolution data as in [25] would not be a worthwhile enterprise. Suppose $Q \subseteq \mathbb{R}^3$ is a quadric and that $L : \mathbb{R}^3 \to \mathbb{R}^7$ is a linear map. Then we can transform $\mathbb{R}^7$ using G_2 such that, if we write $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$, then $L(\mathbb{R}^3) \subseteq \mathbb{C}^3$ is a Lagrangian plane. Therefore, evolving Q using (4.33) will only produce SL 3-folds, which have already been studied in [25].

4.4 An Explicit Affine Evolution Construction

Let us now return to the affine evolution data described in §3.3 and use Theorem 4.3.2 to construct associative 3-folds. Let (P, χ) be as in Example 3.3.1 and define affine maps $\psi_t : \mathbb{R}^5 \to \mathbb{R}^7$ by

$$\psi_t(x_1, \ldots, x_5) = \mathbf{w}_1(t)x_1 + \ldots + \mathbf{w}_5(t)x_5 + \mathbf{w}_6(t),$$

where $\mathbf{w}_j : \mathbb{R} \to \mathbb{R}^7$ are smooth functions for all j. Using the notation of Example 3.3.1, $(\psi_t)_*(\mathbf{e}_j) = \mathbf{w}_j$ for $j = 1, \ldots, 5$. Hence, by equations (3.7) for χ, (2.9) for the cross product on $\mathbb{R}^7$ and (4.33),

$$\begin{aligned}\frac{d\psi_t}{dt}(x_1, \ldots, x_5) = {} & 2x_1\mathbf{w}_2 \times \mathbf{w}_3 + 2x_2\mathbf{w}_1 \times \mathbf{w}_3 - 2x_3\mathbf{w}_1 \times \mathbf{w}_2 + x_4(\mathbf{w}_1 \times \mathbf{w}_5 + \mathbf{w}_2 \times \mathbf{w}_5 - \mathbf{w}_3 \times \mathbf{w}_4) \\ & + x_5(-\mathbf{w}_1 \times \mathbf{w}_4 + \mathbf{w}_2 \times \mathbf{w}_4 + \mathbf{w}_3 \times \mathbf{w}_5) + \mathbf{w}_4 \times \mathbf{w}_5\end{aligned}$$

for all $(x_1, \ldots, x_5) \in \mathbb{R}^5$. Using the formulae above and Theorem 4.3.2 gives the following.

Theorem 4.4.1. *Let $\mathbf{w}_1, \ldots, \mathbf{w}_6 : \mathbb{R} \to \mathbb{R}^7$ be differentiable functions satisfying*

$$\frac{d\mathbf{w}_1}{dt} = 2\mathbf{w}_2 \times \mathbf{w}_3, \qquad \frac{d\mathbf{w}_2}{dt} = 2\mathbf{w}_1 \times \mathbf{w}_3, \qquad \frac{d\mathbf{w}_3}{dt} = -2\mathbf{w}_1 \times \mathbf{w}_2, \tag{4.34}$$

$$\frac{d\mathbf{w}_4}{dt} = \mathbf{w}_1 \times \mathbf{w}_5 + \mathbf{w}_2 \times \mathbf{w}_5 - \mathbf{w}_3 \times \mathbf{w}_4, \qquad \frac{d\mathbf{w}_5}{dt} = -\mathbf{w}_1 \times \mathbf{w}_4 + \mathbf{w}_2 \times \mathbf{w}_4 + \mathbf{w}_3 \times \mathbf{w}_5 \ \textit{and} \tag{4.35}$$

$$\frac{d\mathbf{w}_6}{dt} = \mathbf{w}_4 \times \mathbf{w}_5. \tag{4.36}$$

The subset M of $\mathbb{R}^7$ defined by

$$M = \left\{ \frac{1}{2}\,(y_1^2 + y_2^2)\mathbf{w}_1(t) + \frac{1}{2}\,(y_1^2 - y_2^2)\mathbf{w}_2(t) + y_1y_2\mathbf{w}_3(t) + y_1\mathbf{w}_4(t) + y_2\mathbf{w}_5(t) + \mathbf{w}_6(t) \ : \ y_1, y_2, t \in \mathbb{R} \right\}$$

is an associative 3-fold in $\mathbb{R}^7$ wherever it is nonsingular.

Theorem 4.3.2 only gives us that the associative 3-fold M is defined for t in some small open neighbourhood of zero, but work later in this section shows that M is indeed defined for all t as stated in the above theorem.

The equations we have just obtained fall naturally into three parts: (4.34) shows that $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ evolve amongst themselves; (4.35) are *linear* equations for $\mathbf{w}_4$ and $\mathbf{w}_5$ once $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ are known; and (4.36) defines $\mathbf{w}_6$ once the functions $\mathbf{w}_4$ and $\mathbf{w}_5$ are known. Moreover, these equations are very similar to (3.12)-(3.14), given in Theorem 3.3.2, the only difference being that here our functions and cross products are defined on $\mathbb{R}^7$ rather than $\mathbb{C}^3$. If we could show that any solutions $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$

are equivalent to functions $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$, lying in $\mathbb{C}^3$, satisfying (3.8) and (3.12), then we would be able to use results from [26] to hopefully construct associative 3-folds which are not SL 3-folds. It is to this end that we now proceed.

Suppose that $\mathbf{w}_1(t), \mathbf{w}_2(t), \mathbf{w}_3(t)$ are solutions to (4.34). Let $w_j = \mathbf{w}_j(0)$ for all j and let $v = [w_1, w_2, w_3]$, as defined by (2.10).

If $v = 0$, then, by Proposition 2.3.4, $\langle w_1, w_2, w_3 \rangle_{\mathbb{R}}$ lies in an associative 3-plane which we can map to $\mathbb{R}^3 \subseteq \mathbb{C}^3 \subseteq \mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$, since G_2 acts transitively on associative 3-planes [17, Theorem IV.1.8]. Let z_1, z_2, z_3 be the images of w_1, w_2, w_3 under this transformation and let ω_3 be the standard symplectic form on $\mathbb{C}^3$, given in Definition 3.1.1. Then z_1, z_2, z_3 lie in $\mathbb{R}^3 \subseteq \mathbb{C}^3$ and so $\omega_3(z_j, z_k) = 0$ for $j \neq k$.

If $v \neq 0$, then v is orthogonal to w_j for all j by Proposition 2.1.4, so we can split $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ where $\mathbb{R} = \langle v \rangle$ and $\mathbb{C}^3 = \langle v \rangle^{\perp}$. Hence, w_j lies in $\mathbb{C}^3$ for all j with respect to this splitting. By Proposition 2.1.4, v is orthogonal to $[w_j, w_k] = w_j w_k - w_k w_j = 2w_j \times w_k$ and therefore, from (2.8),

$$(\varphi_0)_{abc} v^a {w_j}^b {w_k}^c = 0.$$

Using (3.1), we calculate $(\varphi_0)_{abc} v^a = |v| (\omega_3)_{bc}$ and hence, since $|v| \neq 0$, $\omega_3(w_j, w_k) = 0$.

We have shown that, using a G_2 transformation, we can map the solutions $\mathbf{w}_1(t), \mathbf{w}_2(t), \mathbf{w}_3(t)$ to solutions $\mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)$ such that $\mathbf{z}_j(0) \in \mathbb{C}^3 \subseteq \mathbb{R}^7$ and $\omega_3(\mathbf{z}_j(0), \mathbf{z}_k(0)) = 0$. Our remarks above about (4.34) and the relationship (3.4) between the cross products on $\mathbb{C}^3$ and $\mathbb{R}^7$ show that $\mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)$ must remain in $\mathbb{C}^3$ and satisfy (3.12) along with condition (3.8). Hence, any solution of (4.34) is equivalent up to a G_2 transformation to a solution to the corresponding equations in Theorem 3.3.2.

We now perform a parameter count in order to calculate the dimension of the family of associative 3-folds constructed by Theorem 4.4.1. The initial data $\mathbf{w}_1(0), \ldots, \mathbf{w}_6(0)$ has 42 real parameters, which implies that $\dim \mathcal{C}_P = 42$, using the notation of Definition 4.3.1, so the family of curves in $\mathcal{C}_P$ has dimension 41, which corresponds to factoring out translation in t. It is shown in [26] that $\mathrm{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ acts on this family of curves and, because of the internal symmetry of the evolution data, any two curves related by this group action give the same 3-fold. Therefore we have to reduce the dimension of distinct associative 3-folds up to this group action by 6 to 35. We can also identify any two associative 3-folds which are isomorphic under automorphisms of $\mathbb{R}^7$, i.e. up to the action of $\mathrm{G}_2 \ltimes \mathbb{R}^7$, so we reduce the dimension by 21 to 14.

In conclusion, the family of associative 3-folds constructed in this section has dimension 14, whereas the dimension of the family of SL 3-folds constructed in Theorem 3.3.2 has dimension 9 by a similar parameter count. Thus, not only do we know that we have constructed new geometric objects, but also how many more interesting parameters we expect to find.

4.4.1 Singularities

We study the singularities of the 3-folds constructed by Theorem 4.4.1 by introducing the function $F : \mathbb{R}^3 \to \mathbb{R}^7$ defined by:

$$F(y_1, y_2, t) = \tfrac{1}{2}\,(y_1^2 + y_2^2)\mathbf{w}_1(t) + \tfrac{1}{2}\,(y_1^2 - y_2^2)\mathbf{w}_2(t) + y_1 y_2 \mathbf{w}_3(t) + y_1 \mathbf{w}_4(t) + y_2 \mathbf{w}_5(t) + \mathbf{w}_6(t). \quad (4.37)$$

Clearly, F is smooth and, if $dF|_{(y_1,y_2,t)} : \mathbb{R}^3 \to \mathbb{R}^7$ is injective for all $(y_1, y_2, t) \in \mathbb{R}$, then F is an immersion and $M = \mathrm{Image}\, F$ is nonsingular. Therefore the possible singularities of M correspond to points where dF is not injective. Since we have from (4.34)-(4.36) that

$$\frac{\partial F}{\partial y_1} \times \frac{\partial F}{\partial y_2} = \frac{\partial F}{\partial t}\,,$$

$\frac{\partial F}{\partial t}$ is perpendicular to the other two partial derivatives by (2.8) and it is zero if and only if the y_1 and y_2 partial derivatives are linearly dependent by Proposition 2.1.7. We deduce that F is an immersion if and only if $\frac{\partial F}{\partial y_1}$ and $\frac{\partial F}{\partial y_2}$ are linearly independent, since dF is injective if and only if the three partial derivatives of F are linearly independent. The condition for F to be an immersion at $(0, 0, 0)$ is therefore that $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ are linearly independent.

We perform a parameter count for the family of singular associative 3-folds constructed by Theorem 4.4.1. The set of initial data $\mathbf{w}_1(0), \ldots, \mathbf{w}_6(0)$, with $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ linearly dependent, has dimension $28 + 8 = 36$, since the set of linearly dependent pairs in $\mathbb{R}^7$ has dimension 8. We saw in the earlier parameter count above that the set of initial data without any restrictions had dimension 42. Hence, the condition that F is not an immersion at $(0, 0, 0)$ is of real codimension 6, but this is clearly true for any point in $\mathbb{R}^3$ and therefore it is expected that the family of singular associative 3-folds will be of codimension $6 - 3 = 3$ in the family of all associative 3-folds constructed by Theorem 4.4.1. Therefore the family of distinct singular associative 3-folds up to automorphisms of $\mathbb{R}^7$ should have dimension $14 - 3 = 11$. Thus generic associative 3-folds constructed by Theorem 4.4.1 will be nonsingular. Moreover, the dimension of the family of singular associative 3-folds is greater than the dimension of the family of singular SL 3-folds constructed from the same evolution data, which has dimension 8 by a similar count.

We now model $M = \mathrm{Image}\ F$ near a singular point, which we take to be the origin without loss of generality. Therefore, we expand $\mathbf{w}_1(t), \ldots, \mathbf{w}_6(t)$ about $t = 0$ to study the singularity. Since dF is not injective at the origin, $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ are linearly dependent. As mentioned above, Joyce [26,

§5.1] describes how internal symmetry of the evolution data gives rise to an action of $\mathrm{GL}(2,\mathbb{R})\ltimes\mathbb{R}^2$ upon $\mathbf{w}_1(t),\ldots,\mathbf{w}_6(t)$, under which the associative 3-fold constructed is invariant. A rotation of $\mathbb{R}^2$ by an angle θ transforms $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ to

$$\tilde{\mathbf{w}}_4(0)=\cos\theta\,\mathbf{w}_4(0)-\sin\theta\,\mathbf{w}_5(0)\quad\text{and}\quad\tilde{\mathbf{w}}_5(0)=\sin\theta\,\mathbf{w}_4(0)+\cos\theta\,\mathbf{w}_5(0).$$

Since $\mathbf{w}_4(0)$ and $\mathbf{w}_5(0)$ are linearly dependent, θ may be chosen so that $\tilde{\mathbf{w}}_5(0)=0$. We may therefore suppose that $\mathbf{w}_5(0)=0$ and take our initial data to be:

$$\mathbf{w}_1(0)=\mathbf{v}+\mathbf{w};\quad\mathbf{w}_2(0)=\mathbf{v}-\mathbf{w};\quad\mathbf{w}_3(0)=\mathbf{x};\quad\mathbf{w}_4(0)=\mathbf{u};\quad\text{and}\quad\mathbf{w}_5(0)=\mathbf{w}_6(0)=0,$$

for vectors $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{x}\in\mathbb{R}^7$. Expanding our solutions to (4.34)-(4.36) to low order in t:

$$\begin{aligned}
&\mathbf{w}_1(t)=\mathbf{v}+\mathbf{w}+2t(\mathbf{v}-\mathbf{w})\times\mathbf{x}+O(t^2); && \mathbf{w}_2(t)=\mathbf{v}-\mathbf{w}+2t(\mathbf{v}+\mathbf{w})\times\mathbf{x}+O(t^2);\\
&\mathbf{w}_3(t)=\mathbf{x}+4t\mathbf{v}\times\mathbf{w}+O(t^2); && \mathbf{w}_4(t)=\mathbf{u}+t\mathbf{u}\times\mathbf{x}+O(t^2);\\
&\mathbf{w}_5(t)=2t\mathbf{u}\times\mathbf{w}+8t^2\mathbf{x}\times(\mathbf{u}\times\mathbf{w})+O(t^3);\quad\text{and} && \mathbf{w}_6(0)=10t^3\mathbf{u}\times(\mathbf{x}\times(\mathbf{u}\times\mathbf{w}))+O(t^4).
\end{aligned}$$

Calculating $F(y_1,y_2,t)$ near the origin, we see that the dominant terms in the expansion are dependent upon $\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3$, which we have shown to be equivalent under G_2 to solutions as given in Theorem 3.3.2. Following Joyce [26, p. 363-364], we consider $F(\epsilon^2y_1,\epsilon y_2,\epsilon t)$ for small ϵ:

$$\begin{aligned}
F(\epsilon^2y_1,\epsilon y_2,\epsilon t)&=\epsilon^2[(y_1+\frac{1}{4}g_0(\mathbf{u},\mathbf{w})t^2)\mathbf{u}+(y_2^2-\frac{1}{4}|\mathbf{u}|^2t^2)\mathbf{w}+2y_2t\mathbf{u}\times\mathbf{w}]\\
&+\epsilon^3[4y_2^2t\mathbf{x}\times\mathbf{w}+y_1y_2\mathbf{x}+y_1t\mathbf{u}\times\mathbf{x}+8y_2t^2\mathbf{x}\times(\mathbf{u}\times\mathbf{w})+10t^3\mathbf{u}\times(\mathbf{x}\times(\mathbf{u}\times\mathbf{w}))]+O(\epsilon^4).
\end{aligned}\tag{4.38}$$

Here we have assumed that $\omega_3(\mathbf{u},\mathbf{w})=0$ in order to simplify the coefficient of $\mathbf{u}$. The ϵ^2 terms in (4.38) give us the lowest order description of the singularity. If we suppose that $\mathbf{u}$ and $\mathbf{w}$ are linearly independent, which will be true in the generic case, then $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{u}\times\mathbf{w}$ are linearly independent and generate an SL $\mathbb{R}^3$ by Corollary 3.1.7. Hence, near the origin to lowest order, M is the image of the map from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$(y_1,\,y_2,\,t)\mapsto(y_1+\frac{1}{4}g_0(\mathbf{u},\,\mathbf{w})t^2,y_2^2-\frac{1}{4}|\mathbf{u}|^2t^2,\,2y_2t).\tag{4.39}$$

Note that the first coordinate axis is fixed under (4.39) and, moreover, y_2 and t are allowed to take either sign. Therefore, (4.39) is a *double cover* of an SL $\mathbb{R}^3$ which is branched over the first coordinate axis. This is the same behaviour as occurs in the SL case [26, p. 364].

In order to study the singularity further we consider the ϵ^3 terms in (4.38). It is generally not possible to simplify the final cross product in the ϵ^3 terms to give a neat expression using only four vectors. However, if $\{e_1,\ldots,e_7\}$ is the basis for $\operatorname{Im}\mathbb{O}\cong\mathbb{R}^7$ described in §2.1 and we choose $\mathbf{u}=e_1$,

$\mathbf{w} = e_2$ and $\mathbf{x} = e_4$, we can calculate each of the vectors appearing in (4.38) explicitly. Thus, in this case, the next order of the singularity is the image of the following map from $\mathbb{R}^3$ to $\mathbb{R}^7$:

$$(y_1,\, y_2,\, t) \mapsto (y_1,\, y_2^2 - \frac{1}{4}\,t^2,\, 2y_2t,\, \epsilon y_1y_2,\, \epsilon y_1t,\, -4\epsilon y_2^2t - 10\epsilon t^3,\, 8\epsilon y_2t^2).$$

Note that the singularity does not lie within $\mathbb{C}^3 \subseteq \mathbb{R}^7$ and so we have a model for a singularity which is different from the SL case.

4.4.2 Solving the equations

From the work above, any solution $\mathbf{w}_1(t), \mathbf{w}_2(t), \mathbf{w}_3(t)$ in $\mathbb{R}^7$ to (4.34) is equivalent under a G_2 transformation to a solution $\mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)$ in $\mathbb{C}^3$ to (3.12) satisfying (3.8). We can thus use results from [26] to produce some associative 3-folds. However, we must exercise some caution: we require that $\langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t) : t \in \mathbb{R}\rangle_{\mathbb{R}} = \mathbb{C}^3$. If this does not occur, there may be a further G_2 transformation that preserves the subspace spanned by the $\mathbf{z}_j(t)$, but transforms $\mathbb{C}^3$ so that $\mathbf{w}_4$ and $\mathbf{w}_5$ are mapped into $\mathbb{C}^3$, and thus the submanifold constructed will be an SL 3-fold embedded in $\mathbb{R}^7$.

When $\dim \langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)\rangle_{\mathbb{R}} < 3$, for generic $t \in \mathbb{R}$, the $\mathbf{z}_j(t)$ define a subspace of an SL $\mathbb{R}^3$ in $\mathbb{C}^3$, which corresponds to an associative $\mathbb{R}^3$ in $\mathbb{R}^7$ by Proposition 3.1.3. The subgroup of G_2 preserving an associative $\mathbb{R}^3$ is $\mathrm{SO}(4)$ [17, Theorem IV.1.8], and the subgroup of $\mathrm{SU}(3)$, which is the automorphism group of $\mathbb{C}^3$, preserving the standard $\mathbb{R}^3$ is $\mathrm{SO}(3)$. Hence, the family of different ways of identifying $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ such that $\langle \mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)\rangle_{\mathbb{R}}$ is mapped into the standard $\mathbb{R}^3$ in $\mathbb{C}^3$ contains $\mathrm{SO}(4)/\mathrm{SO}(3) \cong \mathcal{S}^3$. We therefore have sufficient freedom left in using the G_2 symmetry, after mapping $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ into $\mathbb{C}^3$, to map $\mathbf{w}_4$ and $\mathbf{w}_5$ into $\mathbb{C}^3$ as well. This means that these cases will only produce SL 3-folds.

It is also true in (i) and (ii) of Theorem 3.3.4 that the solutions $\mathbf{z}_j(t)$ define a subspace of an SL $\mathbb{R}^3$ in $\mathbb{C}^3$ and so these cases will not provide any new associative 3-folds either. Therefore we need only consider (iii) and (iv) in Theorem 3.3.4.

Suppose we are in the situation of Theorem 3.3.4 so that, if we write $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$, $\mathbf{w}_1 = (0, w_1, 0, 0)$, $\mathbf{w}_2 = (0, 0, w_2, 0)$ and $\mathbf{w}_3 = (0, 0, 0, w_3)$ for differentiable functions $w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C}$. Let $\mathbf{w}_4 = (y, p_1, p_2, q_3)$ and $\mathbf{w}_5 = (-x, q_1, -q_2, p_3)$, where all the functions defined here are differentiable. Equations (4.35) become

$$\frac{dx}{dt} = \mathrm{Im}(\bar{w}_1p_1 - \bar{w}_2p_2 - \bar{w}_3p_3), \tag{4.40}$$

$$\frac{dp_1}{dt} = ixw_1 + \overline{w_2p_3} + \overline{w_3p_2}, \quad \frac{dp_2}{dt} = ixw_2 - \overline{w_3p_1} - \overline{w_1p_3}, \quad \frac{dp_3}{dt} = ixw_3 - \overline{w_1p_2} - \overline{w_2p_1}; \tag{4.41}$$

and

$$\frac{dy}{dt} = \mathrm{Im}(\bar{w}_1 q_1 - \bar{w}_2 q_2 - \bar{w}_3 q_3), \tag{4.42}$$

$$\frac{dq_1}{dt} = iyw_1 + \overline{w_2 q_3} + \overline{w_3 q_2}, \quad \frac{dq_2}{dt} = iyw_2 - \overline{w_3 q_1} - \overline{w_1 q_3}, \quad \frac{dq_3}{dt} = iyw_3 - \overline{w_1 q_2} - \overline{w_2 q_1}. \tag{4.43}$$

Note that the equations on (x, p_1, p_2, p_3) are the same as on (y, q_1, q_2, q_3). Moreover, $(x, p_1, p_2, p_3) = (0, w_1, w_2, w_3)$ gives an automatic solution to (4.40)-(4.41) and $(y, q_1, q_2, q_3) = (0, w_1, w_2, w_3)$ solves (4.42)-(4.43).

If we write $\mathbf{w}_6 = (z, r_1, r_2, r_3)$, where $z : \mathbb{R} \to \mathbb{R}$ and $r_1, r_2, r_3 : \mathbb{R} \to \mathbb{C}$ are differentiable functions, (4.36) becomes

$$\frac{dz}{dt} = \mathrm{Im}(\bar{p}_1 q_1 - \bar{p}_2 q_2 - \bar{p}_3 q_3), \tag{4.44}$$

$$\frac{dr_1}{dt} = ixp_1 + iyq_1 + \overline{p_2 p_3} + \overline{q_2 q_3}, \tag{4.45}$$

$$\frac{dr_2}{dt} = ixp_2 - iyq_2 - \overline{p_3 p_1} + \overline{q_3 q_1}, \tag{4.46}$$

$$\frac{dr_3}{dt} = ixq_3 + iyp_3 - \overline{p_1 q_2} - \overline{p_2 q_1}. \tag{4.47}$$

Note that the conditions that x, y, z are constant correspond to (3.10), (3.9) and (3.11) in Theorem 3.3.2 respectively. Calculation using (4.40)-(4.41) gives

$$\frac{d^2 x}{dt^2} = x(|w_1|^2 - |w_2|^2 - |w_3|^2).$$

Suppose that x is a nonzero constant. Then $|w_1|^2 - |w_2|^2 - |w_3|^2 \equiv 0$. Using (2.8), (4.34) and the alternating properties of φ_0:

$$\begin{aligned}
\frac{d}{dt}\left(|w_1|^2 - |w_2|^2 - |w_3|^2\right) &= 2g_0\left(\frac{d\mathbf{w}_1}{dt}, \mathbf{w}_1\right) - 2g_0\left(\frac{d\mathbf{w}_2}{dt}, \mathbf{w}_2\right) - 2g_0\left(\frac{d\mathbf{w}_3}{dt}, \mathbf{w}_3\right) \\
&= 4(g_0(\mathbf{w}_2 \times \mathbf{w}_3, \mathbf{w}_1) - g_0(\mathbf{w}_1 \times \mathbf{w}_3, \mathbf{w}_2) + g_0(\mathbf{w}_1 \times \mathbf{w}_2, \mathbf{w}_3)) \\
&= 4(\varphi_0(\mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_1) - \varphi_0(\mathbf{w}_1, \mathbf{w}_3, \mathbf{w}_2) + \varphi_0(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)) \\
&= 12\varphi_0(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3).
\end{aligned}$$

By (3.1), $\varphi_0(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \mathrm{Re}\,(w_1 w_2 w_3) \equiv 0$, which occurs if and only if (iv) of Theorem 3.3.4 holds. However, in case (iv), $|w_1|^2 - |w_2|^2 - |w_3|^2 = \alpha_1^2 - \alpha_2^2 - \alpha_3^2$, which, together with the condition $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$, forces $\alpha_j = 0$ for all j which is a contradiction. Hence, if x is constant then x has to be zero, and we have a similar result for y. Therefore (3.9)-(3.11) correspond to $x = y = 0$ and z constant. This is unsurprising since having $x = y = 0$ and z constant corresponds to $\mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6$ remaining in $\mathbb{C}^3$ and thus the associative 3-fold M constructed will be SL and hence satisfy $\omega_3|_M \equiv 0$.

Following the discussion earlier in this subsection we consider (iii) and (iv) of Theorem 3.3.4. However, no solutions are known in case (iii), so we focus on case (iv). We therefore let $\alpha_1, \alpha_2, \alpha_3$ be positive real numbers satisfying $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$ and define a_1, a_2, a_3 by:

$$a_1 = -\frac{\alpha_2\alpha_3}{\alpha_1}, \quad a_2 = \frac{\alpha_3\alpha_1}{\alpha_2}, \quad a_3 = \frac{\alpha_1\alpha_2}{\alpha_3}. \tag{4.48}$$

By Theorem 3.3.4, we have that

$$2w_1(t) = i\alpha_1 e^{ia_1 t}, \quad 2w_2(t) = \alpha_2 e^{ia_2 t} \text{ and } 2w_3(t) = \alpha_3 e^{ia_3 t}.$$

Hence, if we let $\beta_1, \beta_1, \beta_3 : \mathbb{R} \to \mathbb{C}$ be differentiable functions such that

$$p_1(t) = ie^{ia_1 t}\beta_1(t), \quad p_2(t) = e^{ia_2 t}\beta_2(t), \quad p_3(t) = e^{ia_3 t}\beta_3(t),$$

we have the following result.

Proposition 4.4.2. *Using the notation above,* (4.40)-(4.41) *can be written as the following matrix equation for the functions* $x, \beta_1, \beta_2, \beta_3$*:*

$$\frac{d}{dt}\begin{pmatrix} x \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \bar\beta_1 \\ \bar\beta_2 \\ \bar\beta_3 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & -\frac{\alpha_1}{2} & \frac{\alpha_2}{2} & \frac{\alpha_3}{2} & \frac{\alpha_1}{2} & -\frac{\alpha_2}{2} & -\frac{\alpha_3}{2} \\ \alpha_1 & -2a_1 & 0 & 0 & 0 & -\alpha_3 & -\alpha_2 \\ \alpha_2 & 0 & -2a_2 & 0 & \alpha_3 & 0 & \alpha_1 \\ \alpha_3 & 0 & 0 & -2a_3 & \alpha_2 & \alpha_1 & 0 \\ -\alpha_1 & 0 & \alpha_3 & \alpha_2 & 2a_1 & 0 & 0 \\ -\alpha_2 & -\alpha_3 & 0 & -\alpha_1 & 0 & 2a_2 & 0 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 & 0 & 0 & 0 & 2a_3 \end{pmatrix}\begin{pmatrix} x \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \bar\beta_1 \\ \bar\beta_2 \\ \bar\beta_3 \end{pmatrix}. \tag{4.49}$$

Proof. Using (4.40),

$$\frac{dx}{dt} = \frac{1}{2}\operatorname{Im}(\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3) = \frac{i}{4}\left(\alpha_1(\bar\beta_1 - \beta_1) + \alpha_2(\beta_2 - \bar\beta_2) + \alpha_3(\beta_3 - \bar\beta_3)\right),$$

which gives the first row in (4.49). Since $a_1 + a_2 + a_3 = 0$, the equation in (4.41) for p_1 shows that

$$i\frac{d\beta_1}{dt} - a_1\beta_1 = \frac{1}{2}(-\alpha_1 x + \alpha_2\bar\beta_3 + \alpha_3\bar\beta_2),$$

which, upon rearrangement, gives the second row in (4.49). The calculation of the rest of the rows follows in a similar fashion. □

In order to solve (4.49), we find the eigenvalues and corresponding eigenvectors of the matrix.

Proposition 4.4.3. *Let T denote the 7×7 real matrix given in Proposition 4.4.2 and let* $\mathbf{a} = (0, \alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$, *where* ${}^{\mathrm{T}}$ *denotes transpose. There exist nonzero vectors* $\mathbf{b}_\pm$, $\mathbf{c}_\pm$, $\mathbf{d}_\pm \in \mathbb{R}^7$ *such that*

$$T\mathbf{a} = 0, \quad T\mathbf{b}_\pm = \pm\lambda\mathbf{b}_\pm, \quad T\mathbf{c}_\pm = \pm\lambda\mathbf{c}_\pm, \quad T\mathbf{d}_\pm = \pm 3\lambda\mathbf{d}_\pm,$$

where $\lambda > 0$ is such that $\lambda^2 = a_2^2 - a_1 a_3$ and

$$\begin{aligned} \mathbf{b}_+ &= (b_1, b_2, 0, b_3, b_4, 0, b_5)^{\mathrm{T}}, & \mathbf{b}_- &= (b_1, b_4, 0, b_5, b_2, 0, b_3)^{\mathrm{T}}, \\ \mathbf{c}_+ &= (c_1, 0, c_2, c_3, 0, c_4, c_5)^{\mathrm{T}}, & \mathbf{c}_- &= (c_1, 0, c_4, c_5, 0, c_2, c_3)^{\mathrm{T}}, \\ \mathbf{d}_+ &= (0, d_1, d_2, d_3, d_4, d_5, d_6)^{\mathrm{T}}, & \mathbf{d}_- &= (0, d_4, d_5, d_6, d_1, d_2, d_3)^{\mathrm{T}}, \end{aligned} \tag{4.50}$$

for nonzero constants b_j, c_j, $d_j \in \mathbb{R}$. In particular, the pairs $\{\mathbf{b}_\pm, \mathbf{c}_\pm\}$ are linearly independent.

Proof. Most of the results in this proposition are found by direct calculation using Maple. The only point to note is that if $\mathbf{w}$ is a μ-eigenvector of T, for some $\mu \in \mathbb{R}$, and we write $\mathbf{w} = (\ x \quad \mathbf{y} \quad \mathbf{z}\ )^{\mathrm{T}}$, where $x \in \mathbb{R}$ and $\mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, then $\tilde{\mathbf{w}} = (\ x \quad \mathbf{z} \quad \mathbf{y}\ )^{\mathrm{T}}$ is a $-\mu$-eigenvector of T. Hence we can cast the eigenvectors of T into the form as given in (4.50). $\square$

From Proposition 4.4.3 we can write down the general solution to (4.49):

$$\begin{pmatrix} x \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \bar{\beta}_1 \\ \bar{\beta}_2 \\ \bar{\beta}_3 \end{pmatrix} = A \begin{pmatrix} 0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + B_+ e^{\frac{i}{2}\lambda t} \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ b_3 \\ b_4 \\ 0 \\ b_5 \end{pmatrix} + B_- e^{-\frac{i}{2}\lambda t} \begin{pmatrix} b_1 \\ b_4 \\ 0 \\ b_5 \\ b_2 \\ 0 \\ b_3 \end{pmatrix}$$

$$+ C_+ e^{\frac{i}{2}\lambda t} \begin{pmatrix} c_1 \\ 0 \\ c_2 \\ c_3 \\ 0 \\ c_4 \\ c_5 \end{pmatrix} + C_- e^{-\frac{i}{2}\lambda t} \begin{pmatrix} c_1 \\ 0 \\ c_4 \\ c_5 \\ 0 \\ c_2 \\ c_3 \end{pmatrix} + D_+ e^{\frac{3i}{2}\lambda t} \begin{pmatrix} 0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} + D_- e^{-\frac{3i}{2}\lambda t} \begin{pmatrix} 0 \\ d_4 \\ d_5 \\ d_6 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix} \tag{4.51}$$

for constants $A, B_\pm, C_\pm, D_\pm \in \mathbb{C}$. However, the last three rows in (4.51) are equal to the complex conjugate of the three rows above them, which implies that $B_- = \bar{B}_+$, $C_- = \bar{C}_+$, and $D_- = \bar{D}_+$. Moreover, if we translate $\mathbb{R}^2$, as given in the evolution data, from (y_1, y_2) to $(y_1 - A, y_2)$, then $\mathbf{w}_j$ is unaltered for $j = 1, 2, 3$ but $\mathbf{w}_4$ is mapped to $\mathbf{w}_4 - A\mathbf{w}_1$. Therefore, we can set $A = 0$.

From the discussion above, we may now write down the general solution to (4.40)-(4.41) and (4.42)-(4.43) and then simply integrate equations (4.44)-(4.47) to give an explicit description of some associative 3-folds constructed using our second evolution equation. This result is given below.

Theorem 4.4.4. *Define functions $x, y, z : \mathbb{R} \to \mathbb{R}$ and $w_j, p_j, q_j, r_j : \mathbb{R} \to \mathbb{C}$ for $j = 1, 2, 3$ by:*

$$2w_1(t) = i\alpha_1 e^{ia_1 t}, \quad 2w_2(t) = \alpha_2 e^{ia_2 t}, \quad 2w_3(t) = \alpha_3 e^{ia_3 t},$$

where $\alpha_1, \alpha_2, \alpha_3$ are positive constants such that $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$ and a_1, a_2, a_3 are given in (4.48);

$$\begin{aligned}
x(t) &= 2\,\mathrm{Re}\left(Bb_1 e^{\frac{i}{2}\lambda t} + Cc_1 e^{\frac{i}{2}\lambda t}\right),\\
p_1(t) &= ie^{ia_1 t}\left(Bb_2 e^{\frac{i}{2}\lambda t} + \bar{B}b_4 e^{-\frac{i}{2}\lambda t} + Dd_1 e^{\frac{3i}{2}\lambda t} + \bar{D}d_4 e^{-\frac{3i}{2}\lambda t}\right),\\
p_2(t) &= e^{ia_2 t}\left(Cc_2 e^{\frac{i}{2}\lambda t} + \bar{C}c_4 e^{-\frac{i}{2}\lambda t} + Dd_2 e^{\frac{3i}{2}\lambda t} + \bar{D}d_5 e^{-\frac{3i}{2}\lambda t}\right),\\
p_3(t) &= e^{ia_3 t}\Big((Bb_3 + Cc_3)e^{\frac{i}{2}\lambda t} + (\bar{B}b_5 + \bar{C}c_5)e^{-\frac{i}{2}\lambda t} + Dd_3 e^{\frac{3i}{2}\lambda t} + \bar{D}d_6 e^{-\frac{3i}{2}\lambda t}\Big);\\
y(t) &= 2\,\mathrm{Re}\left(B'b_1 e^{\frac{i}{2}\lambda t} + C'c_1 e^{\frac{i}{2}\lambda t}\right),\\
q_1(t) &= ie^{ia_1 t}\left(B'b_2 e^{\frac{i}{2}\lambda t} + \bar{B}'b_4 e^{-\frac{i}{2}\lambda t} + D'd_1 e^{\frac{3i}{2}\lambda t} + \bar{D}'d_4 e^{-\frac{3i}{2}\lambda t}\right),\\
q_2(t) &= e^{ia_2 t}\left(C'c_2 e^{\frac{i}{2}\lambda t} + \bar{C}'c_4 e^{-\frac{i}{2}\lambda t} + D'd_2 e^{\frac{3i}{2}\lambda t} + \bar{D}'d_5 e^{-\frac{3i}{2}\lambda t}\right),\\
q_3(t) &= e^{ia_3 t}\Big((B'b_3 + C'c_3)e^{\frac{i}{2}\lambda t} + (\bar{B}'b_5 + \bar{C}'c_5)e^{-\frac{i}{2}\lambda t} + D'd_3 e^{\frac{3i}{2}\lambda t} + \bar{D}'d_6 e^{-\frac{3i}{2}\lambda t}\Big);\ \text{and}\\
\frac{dz}{dt} &= \mathrm{Im}(\bar{p}_1 q_1 - \bar{p}_2 q_2 - \bar{p}_3 q_3),\\
\frac{dr_1}{dt} &= ixp_1 + iyq_1 + \overline{p_2 p_3} + \overline{q_2 q_3},\\
\frac{dr_2}{dt} &= ixp_2 - iyq_2 - \overline{p_3 p_1} + \overline{q_3 q_1},\\
\frac{dr_3}{dt} &= ixq_3 + iyp_3 - \overline{p_1 q_2} - \overline{p_2 q_1},
\end{aligned}$$

where the real constants λ and b_j, c_j, d_j are as defined in Proposition 4.4.3 and $B, B', C, C', D, D' \in \mathbb{C}$ are arbitrary constants.

The subset M of $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$ given by

$$\begin{aligned}
M = \Bigg\{\Bigg(& y_1 y(t) - y_2 x(t) + z(t),\ \frac{1}{2}(y_1^2 + y_2^2)w_1(t) + y_1 p_1(t) + y_2 q_1(t) + r_1(t),\\
& \frac{1}{2}(y_1^2 - y_2^2)w_2(t) + y_1 p_2(t) - y_2 q_2(t) + r_2(t),\ y_1 y_2 w_3(t) + y_1 q_3(t) + y_2 p_3(t) + r_3(t)\Bigg) : y_1, y_2, t \in \mathbb{R}\Bigg\}
\end{aligned} \tag{4.52}$$

is an associative 3-fold in $\mathbb{R}^7$.

We now count parameters for the associative 3-folds constructed by Theorem 4.4.4. There are four real parameters ($\alpha_1, \alpha_2, \alpha_3$ and the constant of integration for $z(t)$) and nine complex parameters (B, B', C, C', D, D' and the three constants of integration for $r_1(t), r_2(t), r_3(t)$), which makes a total of 22 real parameters. The relationship $\alpha_1^{-2} = \alpha_2^{-2} + \alpha_3^{-2}$ then reduces the number of parameters by one to 21. Recall that we have the symmetry groups $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and $\mathrm{G}_2 \ltimes \mathbb{R}^7$ for these associative 3-folds. By the arguments proceeding Theorem 4.4.1 and the proof of [26, Proposition 9.1], we have used the freedom in G_2 transformations and rotations in $\mathrm{GL}(2,\mathbb{R})$ to transform our solutions $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ of (4.34) to solutions of (3.12), satisfying (3.8), of the form $\mathbf{w}_1 = (0, w_1, 0, 0)$, $\mathbf{w}_2 = (0, 0, w_2, 0)$, $\mathbf{w}_3 = (0, 0, 0, w_3)$. We have also used translations in $\mathbb{R}^2$ to set the constant A in (4.51) and the corresponding constant A' in the general solution to (4.42)-(4.43) both to zero. Therefore, the remaining symmetries available are dilations in $\mathrm{GL}(2,\mathbb{R})$ and translations in $\mathbb{R}^7$, which reduce the number of parameters by eight to 13. Translation in time, say $t \mapsto t + t_0$, corresponds to multiplying B, B', C, C' by $e^{\frac{i}{2}\lambda t_0}$ and D, D' by $e^{\frac{3i}{2}\lambda t_0}$, which thus lowers the parameter count by one. We conclude that the dimension of the family of associative 3-folds generated by Theorem 4.4.4 is 12, whereas the dimension of the whole family generated by Theorem 4.4.1 is 14.

4.4.3 Periodicity

Note that the solutions in Theorem 4.4.4 are all linear combinations of terms of the form $e^{i(a_j + m\lambda)t}$ for $j = 1, 2, 3$ and $m = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3$, since $a_j \pm n\lambda \neq 0$ for $n = 0, 1, 2, 3$, which ensures that r_1, r_2, r_3 do not have any linear terms in t. It is therefore reasonable to search for associative 3-folds M as in (4.52) that are *periodic* in t. Define a map $F : \mathbb{R}^3 \to \mathbb{R}^7$ by (4.37), so that $M = \operatorname{Image} F$. Then M is periodic if and only if there exists some constant $T > 0$ such that $F(y_1, y_2, t + T) = F(y_1, y_2, t)$ for all $y_1, y_2, t \in \mathbb{R}$.

The periods of the exponentials in the functions defined in Theorem 4.4.4 are proportional to $(a_j + m\lambda)^{-1}$ for $j = 1, 2, 3$ and the values of m given above. In general F will be periodic if and only if these periods have a common multiple. By the definition of a_j in (4.48), $a_2 = -xa_1$ and $a_3 = (x - 1)a_1$ for some $x \in (0, 1)$. Then $\lambda^2 = a_2^2 - a_1 a_3 = a_1^2(x^2 - x + 1)$ and, if we let $y = \sqrt{x^2 - x + 1}$, we deduce that $\lambda = -ya_1$ since $a_1 < 0$ and $\lambda, y > 0$. The periods thus have a common multiple if and only if x and y are rational. We have therefore reduced the problem to finding rational points on the conic $y^2 = x^2 - x + 1$. This is a standard problem in number theory and is identical to the one solved by Joyce [26, §11.2], so we are able to prove the following result.

Theorem 4.4.5. *Given $s \in (0, \frac{1}{2}) \cap \mathbb{Q}$, Theorem 4.4.4 gives a family of closed associative 3-folds in $\mathbb{R}^7$ whose generic members are nonsingular immersed 3-folds diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$.*

Proof. Let $s \in (0, \frac{1}{2}) \cap \mathbb{Q}$ and write $s = \frac{p}{q}$ where p and q are coprime positive integers. Then, as in [26, p.390], we define a_1, a_2, a_3, λ either by

$$a_1 = p^2 - q^2 \qquad a_2 = q^2 - 2pq, \qquad a_3 = 2pq - p^2, \qquad \lambda = p^2 - pq + q^2;$$

or, if $p+q$ is divisible by 3, by

$$3a_1 = p^2 - q^2, \qquad 3a_2 = q^2 - 2pq, \qquad 3a_3 = 2pq - p^2, \qquad 3\lambda = p^2 - pq + q^2.$$

In both cases, $\mathrm{hcf}\,(a_1, a_2, a_3) = \mathrm{hcf}\,(a_1, a_2, a_3, \lambda) = 1$. Moreover, λ is odd since at least one of p, q is odd. Thus $a_j + m\lambda$ is an integer for integer values of m and half an integer, but not an integer, for non-integer values of m. Hence, by the form of the functions given in Theorem 4.4.4 and equation (4.37) for F, $F(y_1, y_2, t+2\pi) = F(-y_1, -y_2, t)$ for all y_1, y_2, t. We deduce that F has period 4π, using the condition that $\mathrm{hcf}(a_1, a_2, a_3) = 1$.

If we define an action of $\mathbb{Z}$ on $\mathbb{R}^3$ by requiring, for $n \in \mathbb{Z}$, that

$$(y_1, y_2, t) \mapsto \left((-1)^n y_1, (-1)^n y_2, t + 2n\pi\right),$$

we can consider F as a map from the quotient of $\mathbb{R}^3$ by $\mathbb{Z}$ under this action. Since this quotient is diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$ and generically F is an immersion, $M = \mathrm{Image}\, F$ is generically an immersed 3-fold diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$. $\square$

Joyce [26] has considered the asymptotic behaviour of the SL 3-folds constructed by Theorem 3.3.4(iv) at infinity, which is dependent on the quadratic terms in F. However, since solutions $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ in Theorem 4.4.1 are essentially equivalent to solutions $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ in Theorem 3.3.4, the asymptotic behaviour of the 3-folds given by Theorem 4.4.5 must be identical to that found by Joyce [26, p. 391].

Theorem 4.4.6. *Every closed associative 3-fold defined by $s \in (0, \frac{1}{2}) \cap \mathbb{Q}$, as described in Theorem 4.4.5, is asymptotic with rate $1/2$ at infinity in $\mathbb{R}^7$, in the sense of Definition 1.2.1, to a double cover of the SL T^2 cone given by:*

$$\left\{(0, ie^{ia_1 t}x_1, e^{ia_2 t}x_2, e^{ia_3 t}x_3) : x_1, x_2, x_3, t \in \mathbb{R},\, x_1 \geq 0,\, \sum_{i=1}^{3} a_i x_i^2 = 0\right\},$$

where the constants a_1, a_2, a_3 are defined by s as in the proof of Theorem 4.4.5.

The associative 3-folds in Theorem 4.4.5 actually diverge away from the SL cone given above, but Theorem 4.4.6 gives a measure of the rate of divergence.

We show that if an associative 3-fold were to converge to an SL 3-fold at infinity it would in fact be SL, which we know is not the case for generic members of the family given by Theorem 4.4.5.

Theorem 4.4.7. *Suppose M is an associative 3-fold in $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ and L is an SL 3-fold in $\mathbb{C}^3$. If M is asymptotic with rate $\lambda < 0$ at infinity in $\mathbb{R}^7$ to L, it is an SL 3-fold in $\mathbb{C}^3$ embedded in $\mathbb{R}^7$.*

Proof. Since M is associative, it is minimal [17, Theorem II.4.2]. Therefore, the immersion of M in $\mathbb{R}^7$ is harmonic by Theorem 1.1.2. In particular, if we write coordinates on M as $(x_1, \ldots, x_7)$, x_1 is harmonic. We may assume, without loss of generality, that the SL 3-fold L to which M converges lies in $\{0\} \times \mathbb{C}^3 \subseteq \mathbb{R}^7$. Since M is asymptotic to L at infinity with rate λ where $\lambda < 0$, $x_1 \to 0$ as $r \to \infty$. By the Maximum Principle (Theorem 1.2.5), $x_1 \equiv 0$ and M is an SL 3-fold in $\mathbb{C}^3$ by Proposition 3.1.3. □

4.5 1-Ruled Associative 3-folds

In this final section we focus on *1-ruled* 3-folds and give methods for constructing associative examples. This is a generalisation of the work in Joyce's paper [27] on 1-ruled SL 3-folds in $\mathbb{C}^3$ and it is from this source that we take the definitions below.

Definition 4.5.1. Let M be a 3-dimensional submanifold of $\mathbb{R}^7$. A *1-ruling* of M is a pair (Σ, π), where Σ is a 2-dimensional manifold and $\pi : M \to \Sigma$ is a smooth map, such that for all $\sigma \in \Sigma$ there exist $\mathbf{v}_\sigma \in \mathcal{S}^6$ and $\mathbf{w}_\sigma \in \mathbb{R}^7$ such that $\pi^{-1}(\sigma) = \{r\mathbf{v}_\sigma + \mathbf{w}_\sigma : r \in \mathbb{R}\}$. We call the triple (M, Σ, π) a *1-ruled submanifold* of $\mathbb{R}^7$.

An *r-orientation* for a 1-ruling (Σ, π) of M is a choice of orientation for the affine straight line $\pi^{-1}(\sigma)$ in $\mathbb{R}^7$, for each $\sigma \in \Sigma$, which varies smoothly with σ. A 1-ruled submanifold with an r-orientation for the 1-ruling is called an *r-oriented 1-ruled submanifold*.

Let (M, Σ, π) be an r-oriented 1-ruled submanifold. For each $\sigma \in \Sigma$, let $\phi(\sigma)$ be the unique unit vector in $\mathbb{R}^7$ parallel to $\pi^{-1}(\sigma)$ and in the positive direction with respect to the orientation on $\pi^{-1}(\sigma)$, given by the r-orientation. Then $\phi : \Sigma \to \mathcal{S}^6$ is a smooth map. Define $\psi : \Sigma \to \mathbb{R}^7$ such that, for all $\sigma \in \Sigma$, $\psi(\sigma)$ is the unique vector in $\pi^{-1}(\sigma)$ orthogonal to $\phi(\sigma)$. Then ψ is a smooth map and we may write:

$$M = \{r\phi(\sigma) + \psi(\sigma) : \sigma \in \Sigma,\, r \in \mathbb{R}\}. \tag{4.53}$$

Define the *asymptotic cone* M_0 of a 1-ruled submanifold M by:

$$M_0 = \{\mathbf{v} \in \mathbb{R}^7 : \mathbf{v} \text{ is parallel to } \pi^{-1}(\sigma) \text{ for some } \sigma \in \Sigma\}.$$

If M is also r-oriented,

$$M_0 = \{r\phi(\sigma) : \sigma \in \Sigma,\, r \in \mathbb{R}\} \tag{4.54}$$

and is usually a 3-dimensional two-sided cone; that is, whenever ϕ is an immersion.

Note that we can consider any r-oriented 1-ruled submanifold as being defined by two maps ϕ and ψ as given in Definition 4.5.1. Hence, r-oriented 1-ruled associative 3-folds may be constructed using partial differential equations for ϕ and ψ.

Ionel et al. [19] give a method for constructing associative 3-folds in $\mathbb{R}^7$ which are necessarily 1-ruled. Explicit examples obtained from this construction are given in [19, §4].

Suppose we have a 3-dimensional two-sided cone M_0 in $\mathbb{R}^7$. The *link* of M_0, $M_0 \cap \mathcal{S}^6$, is a nonsingular 2-dimensional submanifold of $\mathcal{S}^6$ closed under the action of $-1 : \mathcal{S}^6 \to \mathcal{S}^6$. Let Σ be the quotient of the link by the ± 1 maps on $\mathcal{S}^6$. Clearly, Σ is a nonsingular 2-dimensional manifold. Define $\tilde{M}_0 \subseteq \Sigma \times \mathbb{R}^7$ by:

$$\tilde{M}_0 = \{(\{\pm\sigma\}, r\sigma) : \sigma \in M_0 \cap \mathcal{S}^6,\, r \in \mathbb{R}\}.$$

Then $\tilde{M}_0$ is a nonsingular 3-fold. Define $\pi : \tilde{M}_0 \to \Sigma$ by $\pi(\{\pm\sigma\}, r\sigma) = \{\pm\sigma\}$ and $\iota : \tilde{M}_0 \to \mathbb{R}^7$ by $\iota(\{\pm\sigma\}, r\sigma) = r\sigma$. Note that $\iota(\tilde{M}_0) = M_0$ and that ι is an immersion except on $\iota^{-1}(0) \cong \Sigma$, so we may consider $\tilde{M}_0$ as a singular immersed submanifold of $\mathbb{R}^7$. Hence $(\tilde{M}_0, \Sigma, \pi)$ is a 1-ruled submanifold of $\mathbb{R}^7$. Therefore, we can regard M_0 as a 1-ruled submanifold and dispense with $\tilde{M}_0$. Suppose further that M_0 is an r-oriented two-sided cone. We can thus write M_0 in the form (4.53) for maps ϕ, ψ, as given in Definition 4.5.1, and see that ψ must be identically zero. It is also clear that any 1-ruled submanifold defined by ϕ, ψ with $\psi \equiv 0$ is an r-oriented two-sided cone.

We now justify the terminology of asymptotic cone as given in Definition 4.5.1. Suppose that M is an r-oriented 1-ruled submanifold and let M_0 be its asymptotic cone. Writing M in the form (4.53) and M_0 in the form (4.54) for maps ϕ, ψ, define a diffeomorphism $\Psi : M_0 \setminus \bar{B}_1 \to M \setminus K$, where K is some compact subset of M and $\bar{B}_1$ is the closed unit ball in $\mathbb{R}^7$, by $\Psi(r\phi(\sigma)) = r\phi(\sigma) + \psi(\sigma)$ for all $\sigma \in \Sigma$ and $|r| > 1$. If Σ is compact, so that ψ is bounded, Ψ satisfies (1.1) as given in Definition 1.2.3 for $\lambda = 0$, which shows that M is asymptotically conical to M_0 with rate 0.

4.5.1 The associative condition

Let Σ be a 2-dimensional, connected, real analytic manifold, let $\phi : \Sigma \to \mathcal{S}^6$ be a real analytic immersion and let $\psi : \Sigma \to \mathbb{R}^7$ be a real analytic map. Define M by (4.53), so that M is the image of the map $\iota : \mathbb{R} \times \Sigma \to \mathbb{R}^7$ given by $\iota(r, \sigma) = r\phi(\sigma) + \psi(\sigma)$. Clearly, $\mathbb{R} \times \Sigma$ is an r-oriented 1-ruled submanifold with 1-ruling (Σ, π), where π is given by $\pi(r, \sigma) = \sigma$. Since ϕ is an immersion, ι is an immersion almost everywhere in $\mathbb{R} \times \Sigma$ and thus M is an r-oriented 1-ruled submanifold.

We now suppose that M is associative in order to discover the conditions that this imposes upon ϕ and ψ. Note that the asymptotic cone M_0 of M, given by (4.54), is the image of $\mathbb{R} \times \Sigma$ under the map ι_0, defined by $\iota_0(r, \sigma) = r\phi(\sigma)$. Since ϕ is an immersion, ι_0 is an immersion except at $r = 0$, so M_0 is a 3-dimensional cone which is nonsingular except at 0.

Let $p \in M$. There exist $r \in \mathbb{R}$, $\sigma \in \Sigma$ such that $p = r\phi(\sigma) + \psi(\sigma)$. Choose local coordinates (s, t) near σ in Σ. Then $T_pM = \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle_{\mathbb{R}}$, where $\mathbf{x} = \phi(\sigma)$, $\mathbf{y} = r\frac{\partial\phi}{\partial s}(\sigma) + \frac{\partial\psi}{\partial s}(\sigma)$ and $\mathbf{z} = r\frac{\partial\phi}{\partial t}(\sigma) + \frac{\partial\psi}{\partial t}(\sigma)$.

Since M is associative, T_pM is an associative 3-plane, which by Proposition 2.3.4 occurs if and only if $[\mathbf{x},\mathbf{y},\mathbf{z}]=0$. This condition forces a quadratic in r to vanish for all $r\in\mathbb{R}$, and thus the coefficient of each power of r must be zero. The following set of equations must therefore hold in Σ:

$$\left[\phi,\frac{\partial\phi}{\partial s},\frac{\partial\phi}{\partial t}\right]=0; \tag{4.55}$$

$$\left[\phi,\frac{\partial\phi}{\partial s},\frac{\partial\psi}{\partial t}\right]+\left[\phi,\frac{\partial\psi}{\partial s},\frac{\partial\phi}{\partial t}\right]=0;\text{ and} \tag{4.56}$$

$$\left[\phi,\frac{\partial\psi}{\partial s},\frac{\partial\psi}{\partial t}\right]=0. \tag{4.57}$$

Note firstly that, if we do not suppose M to be associative but that (4.55)-(4.57) hold locally in Σ, then, following the argument above, each tangent space to M is associative and hence M is associative. Moreover, (4.55) is equivalent to having that tangent spaces to points of the form $r\phi(\sigma)$, for $r\in\mathbb{R}$, $\sigma\in\Sigma$, are associative, which is precisely the condition for the asymptotic cone M_0 to be associative. We deduce the following result.

Proposition 4.5.2. *The asymptotic cone of an r-oriented 1-ruled associative 3-fold in $\mathbb{R}^7$ is associative provided it is 3-dimensional.*

Note that $\phi(\Sigma)$ is the link of an associative cone if and only if it is a holomorphic curve in $\mathcal{S}^6$ [43, Theorem 2.2]. Bryant [5, §4] shows that any compact Riemann surface can be realised as such a curve.

Since M_0 is associative, φ_0 is a non-vanishing 3-form on M_0 that defines the orientation on M_0. This forces Σ to be oriented, for if (s,t) are some local coordinates on Σ, we can define them to be oriented by imposing the condition that

$$\varphi_0\left(\phi,\frac{\partial\phi}{\partial s},\frac{\partial\phi}{\partial t}\right)>0.$$

In addition, if g is the natural metric on $\mathcal{S}^6$, the pullback $\phi^*(g)$ is a metric on Σ making it a *Riemannian* 2-fold, since $\phi:\Sigma\to\mathcal{S}^6$ is an immersion. Therefore we can consider Σ as an oriented Riemannian 2-fold and hence it has a natural *complex structure*, which we denote as J. Locally in Σ we can choose a holomorphic coordinate $u=s+it$, and so the corresponding real coordinates (s,t) satisfy the condition $J\left(\frac{\partial}{\partial s}\right)=\frac{\partial}{\partial t}$. Following Joyce [27, p.241], we say that local real coordinates (s,t) on Σ that have this property are *oriented conformal coordinates*.

We now use oriented conformal coordinates in the proof of the next result, which gives neater equations for maps ϕ,ψ defining an r-oriented 1-ruled associative 3-fold.

Theorem 4.5.3. *Let Σ be a connected real analytic 2-fold, let $\phi:\Sigma\to\mathcal{S}^6$ be a real analytic immersion and let $\psi:\Sigma\to\mathbb{R}^7$ be a real analytic map. If M is defined by* (4.53)*, it is associative if* *and only if*

$$\frac{\partial \phi}{\partial t} = \phi \times \frac{\partial \phi}{\partial s} \tag{4.58}$$

and ψ satisfies

(i) $\dfrac{\partial \psi}{\partial t} = \phi \times \dfrac{\partial \psi}{\partial s} + f\phi$ *for some real analytic function $f : \Sigma \to \mathbb{R}$,*

or

(ii) $\dfrac{\partial \psi}{\partial s}(\sigma), \dfrac{\partial \psi}{\partial t}(\sigma) \in \left\langle \phi(\sigma), \dfrac{\partial \phi}{\partial s}(\sigma), \dfrac{\partial \phi}{\partial t}(\sigma) \right\rangle_{\mathbb{R}}$ *for all $\sigma \in \Sigma$,*

where $\times$ is defined by (2.9) *and (s,t) are oriented conformal coordinates on Σ.*

Proof. Above we noted that (4.55)-(4.57) were equivalent to the condition that M is associative, so we show that (4.58) is equivalent to (4.55) and that (i) and (ii) are equivalent to (4.56) and (4.57).

Let $\sigma \in \Sigma$, $C = |\frac{\partial \phi}{\partial s}(\sigma)| > 0$. Since ϕ maps to the unit sphere in $\mathbb{R}^7$, $\phi(\sigma)$ is orthogonal to $\frac{\partial \phi}{\partial s}(\sigma)$ and $\frac{\partial \phi}{\partial t}(\sigma)$. As (s,t) are oriented conformal coordinates, we also see that $\frac{\partial \phi}{\partial s}(\sigma)$ and $\frac{\partial \phi}{\partial t}(\sigma)$ are orthogonal and that $|\frac{\partial \phi}{\partial t}(\sigma)| = C$. We conclude that the triple $(\phi(\sigma), C^{-1}\frac{\partial \phi}{\partial s}(\sigma), C^{-1}\frac{\partial \phi}{\partial t}(\sigma))$ is an oriented orthonormal triad in $\mathbb{R}^7$, and it is the basis for an associative 3-plane in $\mathbb{R}^7$ if and only if (4.55) holds at σ. Since G_2 acts transitively on the set of associative 3-planes [17, Theorem IV.1.8], if (4.55) holds at σ we can transform coordinates on $\mathbb{R}^7$ using G_2 so that

$$\phi(\sigma) = e_1, \qquad \frac{\partial \phi}{\partial s}(\sigma) = Ce_2 \quad \text{and} \quad \frac{\partial \phi}{\partial t}(\sigma) = Ce_3,$$

where $\{e_1, \ldots, e_7\}$ is a basis for $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$. We note here that (4.58) holds at σ since the cross product is invariant under G_2 by Definition 2.1.8. If (4.58) holds at σ, the 3-plane generated by $\{\phi(\sigma), \frac{\partial \phi}{\partial s}(\sigma), \frac{\partial \phi}{\partial t}(\sigma)\}$ is associative by Corollary 2.3.5.

Under the change of coordinates of $\mathbb{R}^7$ above, we can write $\frac{\partial \psi}{\partial s}(\sigma) = a_1e_1 + \ldots + a_7e_7$ and $\frac{\partial \psi}{\partial t}(\sigma) = b_1e_1 + \ldots + b_7e_7$ for real constants a_j, b_j for $j = 1, \ldots, 7$. Calculations show that (4.56) holds at σ if and only if

$$b_4 = -a_5, \quad b_5 = a_4, \quad b_6 = -a_7, \quad b_7 = a_6, \tag{4.59}$$

and (4.57) holds at σ if and only if

$$-a_4b_7 - a_5b_6 + a_6b_5 + b_4a_7 = 0, \qquad -a_4b_6 + a_5b_7 + a_6b_4 - a_7b_5 = 0, \tag{4.60}$$

$$a_2b_7 + a_3b_6 - a_6b_3 - a_7b_2 = 0, \qquad a_2b_6 - a_3b_7 - a_6b_2 + a_7b_3 = 0, \tag{4.61}$$

$$-a_2b_5 - a_3b_4 + a_4b_3 + a_5b_2 = 0 \text{ and } \qquad -a_2b_4 + a_3b_5 + a_4b_2 - a_5b_3 = 0. \tag{4.62}$$

Substituting condition (4.59) into the above equations, (4.60) is satisfied immediately and (4.61)-(4.62) become:

$$a_6(a_2 - b_3) - a_7(a_3 + b_2) = 0; \qquad -a_6(a_3 + b_2) - a_7(a_2 - b_3) = 0;$$

$$-a_4(a_2 - b_3) + a_5(a_3 + b_2) = 0; \text{ and } \qquad a_4(a_3 + b_2) + a_5(a_2 - b_3) = 0.$$

These equations can then be written in matrix form:

$$\begin{pmatrix} -a_6 & a_7 \\ a_7 & a_6 \end{pmatrix} \begin{pmatrix} a_2 - b_3 \\ a_3 + b_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} -a_4 & a_5 \\ a_5 & a_4 \end{pmatrix} \begin{pmatrix} a_2 - b_3 \\ a_3 + b_2 \end{pmatrix} = 0. \tag{4.63}$$

We see that equations (4.63) hold if and only if the vector appearing in both equations is zero or the determinants of the matrices are zero. We thus have two conditions which we shall show correspond to (i) and (ii):

$$a_2 = b_3, \qquad -a_3 = b_2; \text{ and} \tag{4.64}$$

$$a_4 = a_5 = 0 = a_6 = a_7. \tag{4.65}$$

Using $\phi(\sigma) = e_1$, (4.64) holds if and only if

$$\frac{\partial \psi}{\partial t}(\sigma) = b_1 e_1 - a_3 e_2 + a_2 e_3 - a_5 e_4 + a_4 e_5 - a_7 e_6 + a_6 e_7 = \phi(\sigma) \times \frac{\partial \psi}{\partial s}(\sigma) + f(\sigma)\phi(\sigma),$$

where $f(\sigma) = b_1$. Therefore, (4.64) corresponds to condition (i) holding at σ by virtue of the invariance of the cross product under G_2. The fact that f is real analytic is immediate from the hypotheses that ϕ and ψ are real analytic and that ϕ is nonzero, since ϕ maps to $\mathcal{S}^6$.

Similarly, (4.65) holds if and only if

$$\frac{\partial \psi}{\partial s}(\sigma) = a_1 e_1 + a_2 e_2 + a_3 e_3 \quad \text{and} \quad \frac{\partial \psi}{\partial t}(\sigma) = b_1 e_1 + b_2 e_2 + b_3 e_3,$$

which is equivalent to condition (ii) holding at σ, since $\langle e_1, e_2, e_3\rangle_{\mathbb{R}} = \langle \phi(\sigma), \frac{\partial \phi}{\partial s}(\sigma), \frac{\partial \phi}{\partial t}(\sigma)\rangle_{\mathbb{R}}$.

In conclusion, at each point $\sigma \in \Sigma$, condition (i) or (ii) holds. Let $\Sigma_1 = \{\sigma \in \Sigma : \text{(i) holds at } \sigma\}$ and let $\Sigma_2 = \{\sigma \in \Sigma : \text{(ii) holds at } \sigma\}$. Note that (i) and (ii) are closed conditions on the real analytic maps ϕ, ψ. Therefore, Σ_1 and Σ_2 are closed real analytic subsets of Σ. Since Σ is real analytic and connected, Σ_j must either coincide with Σ or else be of zero measure in Σ for $j = 1, 2$. However, not both Σ_1 and Σ_2 can be of zero measure in Σ since $\Sigma_1 \cup \Sigma_2 = \Sigma$. Hence, $\Sigma_1 = \Sigma$ or $\Sigma_2 = \Sigma$, which completes the proof. □

It is worth making some remarks about Theorem 4.5.3. Note that (i) and (ii) are *linear* conditions on ψ and, by the remarks made above, (4.58) is the condition which makes the asymptotic cone M_0 associative. So, if we start with an associative two-sided cone M_0 defined by a map ϕ, then ϕ and a function ψ satisfying (i) or (ii) will define an r-oriented 1-ruled associative 3-fold M with asymptotic cone M_0. We also note that conditions (i) and (ii) are unchanged if ϕ is fixed and satisfies (4.58), but ψ is replaced by $\psi + \tilde{f}\phi$ where $\tilde{f}$ is a real analytic function. We can thus always locally transform ψ such that f in condition (i) is zero, if we relax the condition that ψ is orthogonal to ϕ.

4.5.2 The partial differential equations

Our first result follows [27, Proposition 5.2]. We make the definition that a function is real analytic on a compact interval I in $\mathbb{R}$ if it extends to a real analytic function on an open set containing I.

Theorem 4.5.4. *Let I be a compact interval in $\mathbb{R}$, let s be a coordinate on I and let $\phi_0 : I \to \mathcal{S}^6$ and $\psi_0 : I \to \mathbb{R}^7$ be real analytic maps. There exist $\epsilon > 0$ and unique real analytic maps $\phi : I \times (-\epsilon, \epsilon) \to \mathcal{S}^6$ and $\psi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^7$ satisfying $\phi(s, 0) = \phi_0(s)$, $\psi(s, 0) = \psi_0(s)$ for all $s \in I$, and*

$$\frac{\partial \phi}{\partial t} = \phi \times \frac{\partial \phi}{\partial s} \quad \text{and} \quad \frac{\partial \psi}{\partial t} = \phi \times \frac{\partial \psi}{\partial s}\,, \tag{4.66}$$

where t is a coordinate on $(-\epsilon, \epsilon)$ and $\times$ is defined in (2.9)*. If M is given by*

$$M = \{r\phi(s,t) + \psi(s,t) : r \in \mathbb{R},\, s \in I,\, t \in (-\epsilon, \epsilon)\},$$

it is an r-oriented 1-ruled associative 3-fold in $\mathbb{R}^7$.

Proof. Since I is compact and ϕ_0, ψ_0 are real analytic, we may use the Cauchy–Kowalevsky Theorem (Theorem 1.1.4) to give us functions $\phi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^7$ and $\psi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^7$ satisfying the initial conditions and (4.66). It is clear that $\frac{\partial}{\partial t}\, g(\phi, \phi) = 2g(\phi, \frac{\partial \phi}{\partial t}) = 0$, since $\frac{\partial \phi}{\partial t}$ is defined by a cross product involving ϕ and hence is orthogonal to ϕ by (2.8). We deduce that $|\phi|$ is independent of t and is therefore one, so that ϕ maps to $\mathcal{S}^6$. We conclude that M is an r-oriented 1-ruled associative 3-fold using (i) of Theorem 4.5.3. □

Theorem 4.5.4 shows that (4.66) can be considered as evolution equations for maps ϕ and ψ satisfying (i) of Theorem 4.5.3. We now show that condition (ii) of Theorem 4.5.3 does not produce any interesting 1-ruled associative 3-folds. We say that two 1-rulings (Σ, π) and $(\tilde{\Sigma}, \tilde{\pi})$ are *distinct* if the families of affine straight lines $\mathcal{F}_\Sigma = \{\pi^{-1}(\sigma) : \sigma \in \Sigma\}$ and $\mathcal{F}_{\tilde{\Sigma}} = \{\tilde{\pi}^{-1}(\tilde{\sigma}) : \tilde{\sigma} \in \tilde{\Sigma}\}$ are different.

Proposition 4.5.5. *Any r-oriented 1-ruled associative 3-fold (M, Σ, π) satisfying condition* (ii) *but not* (i) *of Theorem 4.5.3 is locally isomorphic to an affine associative 3-plane in $\mathbb{R}^7$.*

Proof. By Theorem 4.1.1, M is real analytic wherever it is nonsingular and so we can take (Σ, π) to be locally real analytic. Let $I = [0, 1]$, let $\gamma : I \to \Sigma$ be a real analytic curve in Σ and let ϕ, ψ be the functions defining M. Then we can use Theorem 4.5.4 with initial conditions $\phi_0 = \phi(\gamma(s))$ and

$\psi_0 = \psi(\gamma(s))$ to give us functions $\tilde{\phi}$ and $\tilde{\psi}$ which define an r-oriented 1-ruled associative 3-fold $\tilde{M}$ satisfying (i) of Theorem 4.5.3. However, M and $\tilde{M}$ coincide in the real analytic 2-fold $\pi^{-1}(\gamma(I))$, and hence, by Theorem 4.1.2, they must be locally equal. We conclude that M locally admits a 1-ruling $(\tilde{\Sigma}, \tilde{\pi})$ satisfying (i) of Theorem 4.5.3, which must therefore be distinct from (Σ, π).

The families of affine straight lines $\mathcal{F}_\Sigma$ and $\mathcal{F}_{\tilde{\Sigma}}$, using the notation above, coincide in the family of affine straight lines defined by points on γ, denoted $\mathcal{F}_\gamma$. Using local real analyticity of the families, either $\mathcal{F}_\Sigma$ is equal to $\mathcal{F}_{\tilde{\Sigma}}$ locally or they only meet in $\mathcal{F}_\gamma$ locally. The former possibility is excluded because the 1-rulings (Σ, π) and $(\tilde{\Sigma}, \tilde{\pi})$ are distinct and thus the latter is true.

Let γ_1 and γ_2 be distinct real analytic curves near γ in Σ defining 1-rulings (Σ_1, π_1) and (Σ_2, π_2), respectively, as above. Then $\mathcal{F}_\Sigma \cap \mathcal{F}_{\Sigma_j}$ is locally equal to $\mathcal{F}_{\gamma_j}$ for $j = 1, 2$. Hence, (Σ_1, π_1) and (Σ_2, π_2) are not distinct (that is, $\mathcal{F}_{\Sigma_1} = \mathcal{F}_{\Sigma_2}$) if and only if $\mathcal{F}_{\gamma_1} = \mathcal{F}_{\gamma_2}$, which implies that $\gamma_1 = \gamma_2$. Therefore, distinct curves near γ in Σ produce different 1-rulings of M and thus M has infinitely many 1-rulings.

Suppose that $\{\gamma_t : t \in \mathbb{R}\}$ is a one parameter family of distinct curves near γ in Σ, with $\gamma_0 = \gamma$. Each curve in the family defines a distinct 1-ruling (Σ_t, π_t), hence there exists $p \in M$ with M nonsingular at p such that $L_t = \pi_t^{-1}(\pi_t(p))$ is not constant as a line in $\mathbb{R}^7$. We therefore get a one parameter family of lines L_t in M through p with $\frac{dL_t}{dt} \neq 0$ at some point, i.e. such that L_t changes nontrivially. We have thus constructed a real analytic one-dimensional family of lines $\{L_t : t \in \mathbb{R}\}$ whose total space is a real analytic 2-fold N contained in M. Moreover, every line in M through p is a line in the affine associative 3-plane $p + T_pM$, so N is contained in $p + T_pM$. Then, since N has nonsingular points in the intersection between M and $p + T_pM$, Theorem 4.1.2 shows that M and $p + T_pM$ coincide on a component of M. Hence, M is planar, i.e. M is locally isomorphic to an affine associative 3-plane in $\mathbb{R}^7$. $\square$

In the course of the proof above, we have shown that a 1-ruled associative 3-fold which admits a one parameter family of distinct real analytic 1-rulings is planar. This is analogous to the result [7, Theorem 6 part 2], which relates to 1-ruled SL 3-folds.

We combine our results on 1-ruled associative 3-folds into the following theorem.

Theorem 4.5.6. *Let (M, Σ, π) be a non-planar, r-oriented, 1-ruled associative 3-fold in $\mathbb{R}^7$. There exist real analytic maps $\phi : \Sigma \to \mathcal{S}^6$ and $\psi : \Sigma \to \mathbb{R}^7$ such that:*

$$M = \{r\phi(\sigma) + \psi(\sigma) \,:\, r \in \mathbb{R},\, \sigma \in \Sigma\}\,;$$

$$\frac{\partial \phi}{\partial t} = \phi \times \frac{\partial \phi}{\partial s}\,;\ \textit{and} \tag{4.67}$$

$$\frac{\partial \psi}{\partial t} = \phi \times \frac{\partial \psi}{\partial s} + f\phi\,, \tag{4.68}$$

where (s,t) are oriented conformal coordinates on Σ and $f : \Sigma \to \mathbb{R}$ is some real analytic function.

Conversely, suppose $\phi : \Sigma \to \mathcal{S}^6$ and $\psi : \Sigma \to \mathbb{R}^7$ are real analytic maps satisfying (4.67) *and* (4.68) *on a connected real analytic 2-fold Σ. If M is defined as above, it is an r-oriented 1-ruled associative 3-fold wherever it is nonsingular.*

4.5.3 Holomorphic vector fields

We now follow [27, §6] and use a *holomorphic vector field* on a Riemann surface Σ to construct 1-ruled associative 3-folds. Note that here, and later in §5.4.5, we use the terminology 'holomorphic vector field' to describe what is, strictly speaking, the real part of a holomorphic vector field.

Proposition 4.5.7. *Let M_0 be an r-oriented, two-sided, associative cone in $\mathbb{R}^7$. We can then write M_0 in the form* (4.54) *for a real analytic map $\phi : \Sigma \to \mathcal{S}^6$, where Σ is a Riemann surface. Let w be a holomorphic vector field on Σ and define a map $\psi : \Sigma \to \mathbb{R}^7$ by $\psi = \mathcal{L}_w\phi$, where $\mathcal{L}_w$ is the Lie derivative with respect to w. If we define M by* (4.53)*, it is an r-oriented 1-ruled associative 3-fold in $\mathbb{R}^7$ with asymptotic cone M_0.*

Proof. We need only consider the case where w is not identically zero since the alternative is trivial. Then w has only isolated zeros and, since the fact that M is associative is a closed condition on the nonsingular part of M, it is sufficient to prove that (4.68) holds at any point $\sigma \in \Sigma$ such that $w(\sigma) \neq 0$. Suppose σ is such a point. Then, since w is a holomorphic vector field, there exists an open set in Σ containing σ on which oriented conformal coordinates (s,t) may be chosen such that $w = \frac{\partial}{\partial s}$. Hence, $\psi = \frac{\partial\phi}{\partial s}$ in a neighbourhood of σ and differentiating (4.67) gives:

$$\frac{\partial^2\phi}{\partial s\partial t} = \frac{\partial\phi}{\partial s} \times \frac{\partial\phi}{\partial s} + \phi \times \frac{\partial^2\phi}{\partial s^2}\,.$$

Interchanging the order of the partial derivatives on the left-hand side and noting that the cross product is alternating, we have that

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\phi}{\partial s\partial t} = \phi \times \frac{\partial\psi}{\partial s}\,.$$

The result follows from Theorem 4.5.6. □

Having proved a result which enables us to construct 1-ruled associative 3-folds given an associative cone on a Riemann surface Σ, we consider which choices for Σ will produce interesting examples. The only nontrivial vector spaces for holomorphic vector fields on a compact connected Riemann surface occur for genus zero or one. We therefore focus our attention upon the cases where we take Σ to be $\mathcal{S}^2$ or T^2. The space of holomorphic vector fields on $\mathcal{S}^2$ is (real) 6-dimensional, and on T^2 it is (real) 2-dimensional. The dimension of these spaces can be computed by knowing the degree of the holomorphic tangent bundle of the Riemann surface, which is itself obtained from the Riemann–Roch Theorem. In the SL case, any SL cone on $\mathcal{S}^2$ has to be an SL 3-plane [18, Theorem B]; Bryant [5, §4] shows that this is not true in the associative case and that, in fact, there are many nontrivial associative cones on $\mathcal{S}^2$.

Theorem 4.5.8. *Let M_0 be an r-oriented, two-sided, associative cone on a Riemann surface $\Sigma \cong \mathcal{S}^2$ (or T^2) with associated real analytic map $\phi : \Sigma \to \mathcal{S}^6$ as in (4.54). There exists a 6-dimensional (or 2-dimensional) family of distinct r-oriented 1-ruled associative 3-folds with asymptotic cone M_0, which are asymptotically conical to M_0 with rate -1 in the sense of Definition 1.2.3.*

Proof. If (s,t) are oriented conformal coordinates on Σ, we may write holomorphic vector fields on it in the form:

$$w = u(s,t)\frac{\partial}{\partial s} + v(s,t)\frac{\partial}{\partial t}\,, \tag{4.69}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy the Cauchy–Riemann equations. For each holomorphic vector field w, as written in (4.69), define a 3-fold M_w by:

$$M_w = \left\{ r\phi(s,t) + u(s,t)\frac{\partial \phi}{\partial s}(s,t) + v(s,t)\frac{\partial \phi}{\partial t}(s,t) \,:\, r \in \mathbb{R},\, (s,t) \in \Sigma \right\}.$$

By Proposition 4.5.7, M_w is an r-oriented 1-ruled associative 3-fold with asymptotic cone M_0, and it is clear that each holomorphic vector field w will give a distinct 3-fold.

We now construct a diffeomorphism Ψ as in Definition 1.2.3 satisfying (1.1) for $\lambda = -1$. Let $R > 0$, w be a holomorphic vector field as in (4.69) and let $\bar{B}_R$ denote the closed ball of radius R in $\mathbb{R}^7$. Define $\Psi : M_0 \setminus \bar{B}_R \to M_w$ by:

$$\Psi(r\phi(s,t)) = r\phi\left(s - \frac{u}{r}\,,\, t - \frac{v}{r}\right) + u\frac{\partial \phi}{\partial s}\left(s - \frac{u}{r}\,,\, t - \frac{v}{r}\right) + v\frac{\partial \phi}{\partial t}\left(s - \frac{u}{r}\,,\, t - \frac{v}{r}\right),$$

where $|r| > R$. Clearly, Ψ is a well-defined map with image in $M_w \setminus K$ for some compact subset K of M_w. Note that, by choosing R sufficiently large, we can expand the various terms defining Ψ in powers of r^{-1} as follows:

$$\begin{aligned}
\phi\left(s - \frac{u(s,t)}{r}\,,\, t - \frac{v(s,t)}{r}\right) &= \phi(s,t) - \frac{u(s,t)}{r}\frac{\partial \phi}{\partial s}(s,t) - \frac{v(s,t)}{r}\frac{\partial \phi}{\partial t} + O(r^{-2});\\
\frac{\partial \phi}{\partial s}\left(s - \frac{u(s,t)}{r}\,,\, t - \frac{v(s,t)}{r}\right) &= \frac{\partial \phi}{\partial s}(s,t) + O(r^{-1}); \text{ and}\\
\frac{\partial \phi}{\partial t}\left(s - \frac{u(s,t)}{r}\,,\, t - \frac{v(s,t)}{r}\right) &= \frac{\partial \phi}{\partial t}(s,t) + O(r^{-1}).
\end{aligned}$$

We deduce that

$$|\Psi(r\phi(s,t)) - r\phi(s,t)| = O(r^{-1}).$$

The other conditions in (1.1) can be derived similarly. We conclude that M_w is asymptotically conical to M_0 with rate -1. □

4.5.4 Examples

There are many examples of associative cones over T^2 given by the SL tori constructed by Haskins [18], Joyce [24], McIntosh [44] and others. However, by Theorem 4.4.7, applying Theorem 4.5.8 to them will only produce 1-ruled SL 3-folds and the result reduces to [27, Theorem 6.3]. However, we also have the associative cones over T^2 given by Theorem 4.2.6. This family of cones is determined by four real parameters, whereas the corresponding SL family, as discussed in [24, §7], is parameterised by one rational variable. Therefore, these cones are generically not SL and so we get the following examples of 1-ruled associative 3-folds.

Theorem 4.5.9. *Use the notation of Theorem 4.2.4 and suppose that* $\alpha_2 = \alpha_3 = -1$. *Let* M, *as given in Theorem 4.2.4, be an associative cone over* T^2, *which, by Theorem 4.2.6, occurs for generic choices of* $x_1(0)$, $z_1(0)$, $z_2(0)$ *and* $z_3(0)$. *Let* $u, v : \mathbb{R}^2 \to \mathbb{R}$ *be functions satisfying the Cauchy–Riemann equations and let* $M_0 = M \cup (-M) \cup \{0\}$. *The subset* $M_{u,v}$ *of* $\mathbb{R} \oplus \mathbb{C}^3$ *given by*

$$\begin{aligned} M_{u,v} = \Big\{ \Big(& r x_1(t) + v(s,t)\big(2|z_1(t)|^2 - |z_2(t)|^2 - |z_3(t)|^2\big),\, e^{2is}\big(r + 2iu(s,t) - 2v(s,t)x_1(t)\big)z_1(t), \\ & e^{-is}\big(\big(r - iu(s,t) + v(s,t)x_1(t)\big)z_2(t) - 3iv(s,t)\overline{z_3 z_1}\,\big), \\ & e^{-is}\big(\big(r - iu(s,t) + v(s,t)x_1(t)\big)z_3(t) + 3iv(s,t)\overline{z_1 z_2}\,\big)\Big) : r, s, t \in \mathbb{R} \Big\} \end{aligned}$$

is an r-oriented 1-ruled associative 3-fold in $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$. *Moreover,* $M_{u,v}$ *is asymptotically conical to* M_0 *with rate* -1 *in the sense of Definition 1.2.3.*

Proof. Define $\phi : \mathbb{R}^2 \to \mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$ by

$$\phi(s,t) = (x_1(t),\, e^{2is}z_1(t),\, e^{-is}z_2(t),\, e^{-is}z_3(t)).$$

Since $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, ϕ maps into $\mathcal{S}^6$ and we can write M_0 in the form (4.54). Define a holomorphic vector field w using u and v as in (4.69). Applying Proposition 4.5.7 and using equations (4.21)-(4.24) of Theorem 4.2.4 gives $M_{u,v}$ as stated. Since M_0 is a cone over T^2, Theorem 4.5.8 gives us the final line. $\square$

Note that, although M and hence M_0 is U(1)-invariant, $M_{u,v}$ will not be in general.

Chapter 5

Coassociative 4-folds in $\mathbb{R}^7$ and Cayley 4-folds in $\mathbb{R}^8$

The focus of this chapter is on constructing examples of coassociative and Cayley 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$ respectively. Our examples can be split into two categories. The first set, described in Sections 5.2 and 5.3, comprises 4-folds with *symmetries* which are constructed using *evolution equations*. The second consists of *2-ruled* submanifolds and forms the content of Section 5.4. The motivation for our study comes from [24], [27] and the work in the previous chapter.

5.1 Evolution Equations

To derive evolution equations for coassociative and Cayley 4-folds we need some results related to *real analyticity*. The first is an obvious corollary of Theorem 1.1.3.

Theorem 5.1.1. *A coassociative 4-fold in $\mathbb{R}^7$ or a Cayley 4-fold in $\mathbb{R}^8$ is real analytic wherever it is nonsingular.*

The next two results, [17, Theorem IV.4.3] and [17, Theorem IV.4.6], use real analyticity since their proofs rely upon the *Cartan–Kähler Theorem*, which is only applicable in the real analytic category.

Theorem 5.1.2. *Suppose P is a 3-dimensional real analytic submanifold of $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ such that $\varphi_0|_P \equiv 0$. There locally exists a real analytic coassociative 4-fold N in $\mathbb{R}^7$ which contains P. Moreover, N is locally unique.*

Notice here that, unlike Theorem 4.1.2 for associative 3-folds in $\mathbb{R}^7$, we have to impose an extra condition on the boundary submanifold P in order to extend it to a coassociative 4-fold in $\mathbb{R}^7$.

Theorem 5.1.3. *Suppose P is a 3-dimensional real analytic submanifold of $\mathbb{O} \cong \mathbb{R}^8$. There locally exists a real analytic Cayley 4-fold N in $\mathbb{R}^8$ which contains P. Moreover, N is locally unique.*

We now formulate our evolution equations, which are analogues of Theorems 3.2.2 and 4.1.3.

Theorem 5.1.4. *Let P be a compact, orientable, 3-dimensional, real analytic manifold, χ a real analytic nowhere vanishing section of $\Lambda^3 TP$ and $\psi : P \to \mathbb{R}^7$ a real analytic embedding (immersion) such that $\psi^*(\varphi_0) \equiv 0$ on P. There exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{R}^7$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^e = (\psi_t)_*(\chi)^{abc}(*\varphi_0)_{abcd}(g_0)^{de} \tag{5.1}$$

using index notation for tensors on $\mathbb{R}^7$, where $(g_0)^{ab}$ is the inverse of the Euclidean metric on $\mathbb{R}^7$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^7$ by $\Psi(t,p) = \psi_t(p)$. Then $M = \mathrm{Image}\,\Psi$ is a nonsingular embedded (immersed) coassociative 4-fold in $\mathbb{R}^7$.

Proof. The proof is again almost identical to that of Theorem 3.2.2, so we omit the details. Note that the condition $\psi^*(\varphi_0)|_P \equiv 0$ implies that φ_0 vanishes on the real analytic 3-fold $\psi(P)$ in $\mathbb{R}^7$. Hence, by Theorem 5.1.2, there locally exists a locally unique coassociative 4-fold N in $\mathbb{R}^7$ containing $\psi(P)$. The use of the Cauchy–Kowalesky Theorem (Theorem 1.1.4) allows us to construct ψ_t and thus M as stated. Since M contains $\psi(P)$, we may show that it agrees locally with N and so is coassociative. $\square$

Note that if $(\psi_t)_*(\chi) = \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$, the right-hand side of (5.1) is the triple cross product $\mathbf{x} \times \mathbf{y} \times \mathbf{z}$ defined by (2.11). Our result for Cayley 4-folds is proved entirely similarly using Theorems 5.1.1 and 5.1.3.

Theorem 5.1.5. *Let P be a compact, orientable, 3-dimensional, real analytic manifold, χ a real analytic nowhere vanishing section of $\Lambda^3 TP$ and $\psi : P \to \mathbb{R}^8$ a real analytic embedding (immersion). There exist $\epsilon > 0$ and a unique family $\{\psi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\psi_t : P \to \mathbb{R}^8$ with $\psi_0 = \psi$ satisfying*

$$\left(\frac{d\psi_t}{dt}\right)^e = (\psi_t)_*(\chi)^{abc}(\Phi_0)_{abcd}(g_0)^{de} \tag{5.2}$$

using index notation for tensors on $\mathbb{R}^8$, where $(g_0)^{ab}$ is the inverse of the Euclidean metric on $\mathbb{R}^8$. Define $\Psi : (-\epsilon, \epsilon) \times P \to \mathbb{R}^8$ by $\Psi(t,p) = \psi_t(p)$. Then $M = \mathrm{Image}\,\Psi$ is a nonsingular embedded (immersed) Cayley 4-fold in $\mathbb{R}^8$.

5.2 Coassociative 4-folds with Symmetries

We follow the construction method in §4.2 using symmetries, except we now want a Lie subgroup of $\mathrm{G}_2 \ltimes \mathbb{R}^7$ with a 3-dimensional orbit $\mathcal{O}$ such that φ_0 vanishes on $\mathcal{O}$.

5.2.1 $\mathrm{U}(1)^2$-invariant cones

We first consider coassociative 4-folds invariant both under the action of $\mathrm{U}(1)^2$ on the $\mathbb{C}^3$ component of $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ and under dilations.

Definition 5.2.1. Let $\mathbb{R}^+$ denote the group of positive real numbers under multiplication. Define an action of $\mathbb{R}^+ \times \mathrm{U}(1)^2$ on $\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ by:

$$(x_1, z_1, z_2, z_3) \longmapsto (rx_1,\, re^{i\phi_1}z_1,\, re^{i\phi_2}z_2,\, re^{-i(\phi_1+\phi_2)}z_3) \qquad r > 0,\ \phi_1, \phi_2 \in \mathbb{R}. \tag{5.3}$$

Define smooth maps $\psi_t : \mathbb{R}^+ \times \mathrm{U}(1)^2 \to \mathbb{R}^7$ by:

$$\psi_t(r, e^{i\phi_1}, e^{i\phi_2}) = (rx_1(t),\, re^{i\phi_1}z_1(t),\, re^{i\phi_2}z_2(t),\, re^{-i(\phi_1+\phi_2)}z_3(t)), \tag{5.4}$$

where $x_1(t)$, $z_1(t)$, $z_2(t)$ and $z_3(t)$ are smooth functions of t.

Using (5.4) we calculate the vectors tangential to the group action given in (5.3):

$$\mathbf{u} = (\psi_t)_*\left(\frac{\partial}{\partial r}\right) = x_1\frac{\partial}{\partial x_1} + z_1\frac{\partial}{\partial z_1} + \bar{z}_1\frac{\partial}{\partial \bar{z}_1} + z_2\frac{\partial}{\partial z_2} + \bar{z}_2\frac{\partial}{\partial \bar{z}_2} + z_3\frac{\partial}{\partial z_3} + \bar{z}_3\frac{\partial}{\partial \bar{z}_3}\,; \tag{5.5}$$

$$\mathbf{v} = (\psi_t)_*\left(\frac{\partial}{\partial \phi_1}\right) = i\left(z_1\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1}\right) - i\left(z_3\frac{\partial}{\partial z_3} - \bar{z}_3\frac{\partial}{\partial \bar{z}_3}\right); \text{ and} \tag{5.6}$$

$$\mathbf{w} = (\psi_t)_*\left(\frac{\partial}{\partial \phi_2}\right) = i\left(z_2\frac{\partial}{\partial z_2} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2}\right) - i\left(z_3\frac{\partial}{\partial z_3} - \bar{z}_3\frac{\partial}{\partial \bar{z}_3}\right). \tag{5.7}$$

If we set $\chi = \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \phi_1} \wedge \frac{\partial}{\partial \phi_2}$, then $(\psi_t)_*(\chi) = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$.

Using (5.5)-(5.7) and equation (2.7) for $*\varphi_0$ we find that

$$\begin{aligned}(\psi_t)_*(\chi)^{abc}(*\varphi_0)_{abcd}(g_0)^{de} = (\mathbf{u}\times\mathbf{v}\times\mathbf{w})^e &= -3\mathrm{Im}(z_1z_2z_3)\frac{\partial}{\partial x_1} \\ &+ \big(z_1(|z_2|^2 - |z_3|^2) + ix_1\overline{z_2z_3}\big)\frac{\partial}{\partial z_1} + \big(\bar{z}_1(|z_2|^2 - |z_3|^2) - ix_1z_2z_3\big)\frac{\partial}{\partial \bar{z}_1} \\ &+ \big(z_2(|z_3|^2 - |z_1|^2) + ix_1\overline{z_3z_1}\big)\frac{\partial}{\partial z_2} + \big(\bar{z}_2(|z_3|^2 - |z_1|^2) - ix_1z_3z_1\big)\frac{\partial}{\partial \bar{z}_2} \\ &+ \big(z_3(|z_1|^2 - |z_2|^2) + ix_1\overline{z_1z_2}\big)\frac{\partial}{\partial z_3} + \big(\bar{z}_3(|z_1|^2 - |z_2|^2) - ix_1z_1z_2\big)\frac{\partial}{\partial \bar{z}_3}.\end{aligned}$$

Using (5.4) with (5.5)-(5.7) and equation (2.6) for φ_0 we see that

$$(\psi_t)^*(\varphi_0)\cdot\chi = -3\,\mathrm{Re}(z_1z_2z_3).$$

Therefore $(\psi_t)^*(\varphi_0) \equiv 0$ is equivalent to $\mathrm{Re}(z_1 z_2 z_3) = 0$.

We also see that

$$\frac{d\psi_t}{dt} = \frac{dx_1}{dt}\frac{\partial}{\partial x_1} + \sum_{j=1}^{3} \frac{dz_j}{dt}\frac{\partial}{\partial z_j} + \sum_{j=1}^{3} \frac{d\bar{z}_j}{dt}\frac{\partial}{\partial \bar{z}_j}.$$

We then deduce our result from Theorem 5.1.4.

Theorem 5.2.2. *Let $x_1 : \mathbb{R} \to \mathbb{R}$ and $z_1, z_2, z_3 : \mathbb{R} \to \mathbb{C}$ be differentiable functions satisfying*

$$\frac{dx_1}{dt} = -3\mathrm{Im}(z_1 z_2 z_3), \tag{5.8}$$

$$\frac{dz_1}{dt} = z_1(|z_2|^2 - |z_3|^2) + ix_1\overline{z_2 z_3}, \tag{5.9}$$

$$\frac{dz_2}{dt} = z_2(|z_3|^2 - |z_1|^2) + ix_1\overline{z_3 z_1} \text{ and} \tag{5.10}$$

$$\frac{dz_3}{dt} = z_3(|z_1|^2 - |z_2|^2) + ix_1\overline{z_1 z_2}, \tag{5.11}$$

along with the condition

$$\mathrm{Re}(z_1 z_2 z_3) = 0 \tag{5.12}$$

at $t = 0$. The subset M of $\mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$ defined by

$$M = \Big\{ (rx_1(t),\, re^{i\phi_1} z_1(t),\, re^{i\phi_2} z_2(t),\, re^{-i(\phi_1+\phi_2)} z_3(t)) \;:\; r > 0,\, \phi_1, \phi_2, t \in \mathbb{R} \Big\}$$

is a coassociative 4-fold in $\mathbb{R}^7$. Moreover, (5.12) *holds for all $t \in \mathbb{R}$ and $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2$ is a constant which can be taken to be* 1.

Proof. It is immediate from (5.8)-(5.11) that $x_1^2 + |z_1|^2 + |z_2|^2 + |z_3|^2$ is a constant which can be chosen to be 1 without loss of generality. We may also calculate

$$\frac{d}{dt}(z_1 z_2 z_3) = ix_1(|z_2|^2|z_3|^2 + |z_3|^2|z_1|^2 + |z_1|^2|z_2|^2)$$

using (5.8)-(5.11) and deduce that $\mathrm{Re}(z_1 z_2 z_3)$ is a constant which has to be zero since (5.12) holds at $t = 0$. Theorem 5.1.4 only gives us that solutions to (5.8)-(5.11) exist for $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, but it is possible, as in §4.2.2, to show that solutions exist for all t as the functions involved are all bounded. □

5.2.2 SU(2) symmetry 1

There are three different natural actions of SU(2) on $\mathbb{R}^7$ in G_2. The first is where SU(2) acts on $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{C}^2$ with the standard SU(2) action on $\mathbb{C}^2$ and trivially on $\mathbb{R}^3$. The construction using this action produces an affine $\mathbb{C}^2 \subseteq \mathbb{R}^7$ as the coassociative 4-fold. The second has SU(2) acting on

$\mathbb{R}^7 \cong \mathbb{R} \oplus \mathbb{C}^3$ as SO(3) on $\mathbb{C}^3$ and trivially on $\mathbb{R}$. The construction then produces a complex surface in $\mathbb{R}^7$ as the coassociative 4-fold, which we may be written as follows:

$$\left\{(x_1, z_1, z_2, z_3) : z_1^2 + z_2^2 + z_3^2 = A,\ x_1 = B\right\}, \text{ where } A \in \mathbb{C} \text{ and } B \in \mathbb{R} \text{ are constants.}$$

The final action on $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{C}^2$, where the isomorphism $\mathbb{R}^4 \cong \mathbb{C}^2$ is given by $(x_4, x_5, x_6, x_7) \mapsto (x_4 + ix_6, x_5 + ix_7)$, has an SU(2) action on $\mathbb{C}^2$ and an action of $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$ on $\mathbb{R}^3$. Harvey and Lawson [17, IV.3] have already studied coassociative 4-folds invariant under this action by different means.

Definition 5.2.3. Let $X \in \mathrm{SU}(2)$ and let $(x_1, x_2, x_3, z_1, z_2) \in \mathbb{R}^3 \oplus \mathbb{C}^2 \cong \mathbb{R}^7$ with $z_1 = x_4 + ix_6$ and $z_2 = x_5 + ix_7$. There exist $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$ and

$$X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

We define the action of X on $(x_1, x_2, x_3, z_1, z_2)$ by:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} |a|^2 - |b|^2 & 2\mathrm{Re}(a\bar{b}) & -2\mathrm{Im}(a\bar{b}) \\ -2\mathrm{Re}(ab) & \mathrm{Re}(a^2 - \bar{b}^2) & -\mathrm{Im}(a^2 - \bar{b}^2) \\ -2\mathrm{Im}(ab) & \mathrm{Im}(a^2 + \bar{b}^2) & \mathrm{Re}(a^2 + \bar{b}^2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Denote this action by $X \cdot (x_1, x_2, x_3, z_1, z_2)$.

Define smooth maps $\psi_t : \mathrm{SU}(2) \to \mathbb{R}^7$ by:

$$\psi_t(X) = X \cdot (x_1(t), x_2(t), x_3(t), z_1(t), z_2(t)), \tag{5.13}$$

where $x_1(t), x_2(t), x_3(t)$ are smooth real-valued functions of t and $z_1(t) = x_4(t) + ix_6(t), z_2(t) = x_5(t) + ix_7(t)$ are smooth complex-valued functions of t.

Calculation shows that we can take the following three matrices as a basis for the Lie algebra of SU(2) acting on $\mathbb{R}^7$ in this way:

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}; \quad U_2 = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}; \text{ and}$$

$$U_3 = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

If we let $\mathbf{u}_j = (\psi_t)_*(U_j)$ for $j = 1, 2, 3$, (5.13) and Definition 5.2.3 imply that

$$\begin{aligned} \mathbf{u}_1 &= -2x_3\mathbf{e}_2 + 2x_2\mathbf{e}_3 + x_5\mathbf{e}_4 - x_4\mathbf{e}_5 + x_7\mathbf{e}_6 - x_6\mathbf{e}_7, \\ \mathbf{u}_2 &= -2x_1\mathbf{e}_3 + 2x_3\mathbf{e}_1 + x_6\mathbf{e}_4 - x_7\mathbf{e}_5 - x_4\mathbf{e}_6 + x_5\mathbf{e}_7 \text{ and} \\ \mathbf{u}_3 &= -2x_2\mathbf{e}_1 + 2x_1\mathbf{e}_2 - x_7\mathbf{e}_4 - x_6\mathbf{e}_5 + x_5\mathbf{e}_6 + x_4\mathbf{e}_7, \end{aligned}$$

where $\mathbf{e}_j = \frac{\partial}{\partial x_j}$. Hence, if $\chi = U_1 \wedge U_2 \wedge U_3$, then $(\psi_t)_*(\chi) = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$. Using the equations above for $\mathbf{u}_j$ and equation (2.11) for the triple cross product, we calculate:

$$\mathbf{u} \times \mathbf{v} \times \mathbf{w} = 4\sum_{j=1}^{3} x_j(x_4^2 + x_5^2 + x_6^2 + x_7^2)\mathbf{e}_j + \sum_{k=4}^{7} x_k(4x_1^2 + 4x_2^2 + 4x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2)\mathbf{e}_k. \tag{5.14}$$

We may also note that

$$\frac{d\psi_t}{dt} = \sum_{j=1}^{7} \frac{dx_j}{dt}\,\mathbf{e}_j. \tag{5.15}$$

Writing $z_1 = x_4 + ix_6$ and $z_2 = x_5 + ix_7$, we then equate (5.14) and (5.15) to get the following:

$$\frac{dx_j}{dt} = 4x_j(|z_1|^2 + |z_2|^2) \qquad \text{for } j = 1, 2, 3 \text{ and} \tag{5.16}$$

$$\frac{dz_k}{dt} = z_k\left(4(x_1^2 + x_2^2 + x_3^2) - (|z_1|^2 + |z_2|^2)\right) \qquad \text{for } k = 1, 2. \tag{5.17}$$

It is apparent from (5.16) that $x_j = c_j x$ for some function $x : \mathbb{R} \to \mathbb{R}$ and real constants c_j for $j = 1, 2, 3$. Similarly, from (5.17), $z_k = d_k z$ for some function $z : \mathbb{R} \to \mathbb{C}$ and complex constants d_k for $k = 1, 2$. By rescaling x and z, we may take $\mathbf{c} = (c_1, c_2, c_3)$ and $\mathbf{d} = (d_1, d_2)$ to be unit vectors. Hence, (5.16) and (5.17) become:

$$\frac{dx}{dt} = 4x|z|^2 \quad \text{and} \quad \frac{dz}{dt} = z(4x^2 - |z|^2). \tag{5.18}$$

If we let $r = |z|$ and use (5.18), we notice that z/r is constant and

$$\frac{dx}{dr} = \frac{4rx}{4x^2 - r^2},$$

which is the same differential equation that defines an SU(2)-invariant coassociative 4-fold as given in [17, Lemma IV.3.7]. The vanishing of φ_0 on the SU(2) orbit does not give any extra condition on x and r. Moreover, since the argument of z is constant, we may rotate it to equal r using SU(2). Thus Theorem 5.1.4 provides the following result.

Theorem 5.2.4. *For $c \in \mathbb{R}$ and unit vectors $\mathbf{c} \in \mathbb{R}^3$ and $\mathbf{d} \in \mathbb{C}^2$, the subset M_c of $\mathbb{R}^3 \oplus \mathbb{C}^2 \cong \mathbb{R}^7$ defined by*

$$M_c = \left\{ X \cdot (x\mathbf{c} + r\mathbf{d}) \,:\, x(4x^2 - 5r^2)^2 = c, \text{ for } x \in \mathbb{R},\, r \geq 0 \text{ and } X \in \mathrm{SU}(2) \right\},$$

where the action of SU(2) *is given in Definition 5.2.3, is a coassociative 4-fold in $\mathbb{R}^7$.*

We shall return to this example in §7.6.

5.2.3 SU(2) symmetry 2

There is another action of SU(2) on $\mathbb{R}^7$ which is described in [43, §3].

Definition 5.2.5. Let the Lie algebra of an action of SU(2) on $\mathbb{R}^7$ be spanned by the following three matrices:

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{pmatrix};$$

$$U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -2\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{10} & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{10} & 0 & -\sqrt{6} \\ 0 & \sqrt{10} & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{6} & 0 & \sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \end{pmatrix};\text{ and}$$

$$U_3 = \begin{pmatrix} 0 & 0 & 0 & 2\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{10} & 0 & -\sqrt{6} \\ 0 & 0 & 0 & -\sqrt{10} & 0 & -\sqrt{6} & 0 \\ -2\sqrt{6} & 0 & \sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to the induced action of SU(2) on $S^6\mathbb{C}^2$, the six-fold symmetric product of $\mathbb{C}^2$, from the usual action of SU(2) on $\mathbb{C}^2$ and then a suitable identification of $S^6\mathbb{C}^2$ with $\mathbb{R}^7 \otimes_{\mathbb{R}} \mathbb{C}$.

It is possible to follow the method of our previous subsections and derive ordinary differential equations defining coassociative 4-folds with this SU(2) symmetry group. However, this calculation is rather unwieldy and thus omitted.

Mashimo [43, Theorem 4.3] shows that are exactly two types of orbit of this SU(2) action which are *totally real* 3-dimensional submanifolds of $\mathcal{S}^6$; that is, submanifolds N such that the almost complex structure on $\mathcal{S}^6$ maps T_pN into the normal space $\nu_p(N)$ of N in $\mathcal{S}^6$ for all $p \in N$. By [43, Theorem 2.3], such submanifolds of $\mathcal{S}^6$ correspond to links of coassociative cones. One of the orbits has constant curvature 1/16 and Dillen et al. [12, p. 580] give an explicit formula for it in terms of harmonic polynomials of degree 6 on $\mathcal{S}^3$. This allows us to write an expression, albeit ugly, for a coassociative cone in $\mathbb{R}^7$.

Theorem 5.2.6. *For $K > 0$ let $\mathcal{S}^n(K) = \{(y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} y_i^2 = \frac{1}{K}\}$ and let $\psi : \mathbb{R}^+ \times \mathcal{S}^3(1/16) \to \mathbb{R}^7$ be given by $\psi(r, y_1, y_2, y_3, y_4) = (rx_1, rx_2, rx_3, rx_4, rx_5, rx_6, rx_7)$, where*

$$x_1(y_1, y_2, y_3, y_4) = 2^{-10}\sqrt{15}(y_1y_3 + y_2y_4)(y_1y_4 - y_2y_3)(y_1^2 + y_2^2 - y_3^2 - y_4^2),$$

$$x_2(y_1, y_2, y_3, y_4) = 2^{-12}\left(-\sum_{i=1}^{4} y_i^6 + \sum_{1\le i<j\le 4} 5y_i^2y_j^2(y_i^2 + y_j^2) - \sum_{1\le i<j<k\le 4} 30y_i^2y_j^2y_k^2\right),$$

$$x_3(y_1, y_2, y_3, y_4) = 2^{-10}\left(y_3y_4(y_3^2 - y_4^2)(y_3^2 + y_4^2 - 5y_1^2 - 5y_2^2) + y_1y_2(y_1^2 - y_2^2)(y_1^2 + y_2^2 - 5y_3^2 - 5y_4^2)\right),$$

$$x_4(y_1, y_2, y_3, y_4) = 2^{-12}\big(y_2y_4(y_2^4 + 3y_3^4 - y_4^4 - 3y_1^4) + y_1y_3(y_3^4 + 3y_2^4 - y_1^4 - 3y_4^4) + 2(y_1y_3 - y_2y_4)(y_1^2(y_2^2 + 4y_4^2) - y_3^2(y_4^2 + 4y_2^2))\big),$$

$$x_5(y_1, y_2, y_3, y_4) = x_4(y_2, -y_1, y_3, y_4),$$

$$x_6(y_1, y_2, y_3, y_4) = 2^{-12}\sqrt{6}\big(y_1y_3(y_1^4 + 5y_2^4 - y_3^4 - 5y_4^4) - y_2y_4(y_2^4 + 5y_1^4 - y_4^4 - 5y_3^4) + 10(y_1y_3 - y_2y_4)(y_3^2y_4^2 - y_1^2y_2^2)\big) \text{ and}$$

$$x_7(y_1, y_2, y_3, y_4) = x_6(y_2, -y_1, y_3, y_4).$$

Then $M = \psi(\mathbb{R}^+ \times \mathcal{S}^3(1/16))$ *is a coassociative cone in* $\mathbb{R}^7$*.*

Note that the choice of scaling in Theorem 5.2.6 ensures that $\psi(\{1\} \times \mathcal{S}^3(1/16)) \subseteq \mathcal{S}^6(1)$.

5.3 Cayley 4-folds with Symmetries

We continue to follow our now familiar construction method, using Lie subgroups of $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ with a 3-dimensional orbit.

5.3.1 $\mathrm{U}(1)^2$-invariant cones

Definition 5.3.1. Let $\mathrm{G} \subseteq \mathrm{U}(1)^4$ be defined by:

$$\mathrm{G} = \big\{(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}, e^{i\alpha_4}) \,:\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \text{ satisfying} \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \text{ and } a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0\big\}$$

for coprime integers a_1, a_2, a_3, a_4 with $a_1 + a_2 + a_3 + a_4 = 0$ and $a_1 \leq a_2 \leq a_3 \leq a_4$. This acts on $\mathbb{C}^4 \cong \mathbb{R}^8$ in the obvious way as a $\mathrm{U}(1)^2$ subgroup of $\mathrm{U}(1)^4$.

We consider G-invariant Cayley cones. Thus define embeddings $\psi_t : \mathbb{R}^+ \times \mathrm{G} \to \mathbb{C}^4 \cong \mathbb{R}^8$ by:

$$\psi_t\left(r, (e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}, e^{i\alpha_4})\right) = (re^{i\alpha_1}z_1(t), re^{i\alpha_2}z_2(t), re^{i\alpha_3}z_3(t), re^{i\alpha_4}z_4(t)), \tag{5.19}$$

where $z_1(t)$, $z_2(t)$, $z_3(t)$ and $z_4(t)$ are smooth functions of t.

We take our nowhere vanishing 3-vector χ to be:

$$\chi = \frac{r}{2}\frac{\partial}{\partial r} \wedge \sum_{1 \leq j < k \leq 4} (-1)^{j+k-1}(a_j - a_k)\,\partial_1 \wedge \ldots \wedge \partial_{j-1} \wedge \partial_{j+1} \wedge \ldots \wedge \partial_{k-1} \wedge \partial_{k+1} \wedge \partial_4, \tag{5.20}$$

where $\partial_j = \frac{\partial}{\partial \alpha_j}$. Using the formula (5.19) for ψ_t we find that

$$(\psi_t)_*\left(r\frac{\partial}{\partial r}\right) = \sum_{j=1}^{4}\left(z_j\frac{\partial}{\partial z_j} + \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right) \quad \text{and} \quad (\psi_t)_*(\partial_j) = iz_j\frac{\partial}{\partial z_j} - i\bar{z}_j\frac{\partial}{\partial \bar{z}_j} \quad \text{for } j = 1, 2, 3, 4.$$

We may then use these equations to calculate $(\psi_t)_*(\chi)$ from (5.20). Using equation (2.12) for Φ_0 we can calculate the right-hand side of (5.2) and the left-hand side can be written as:

$$\frac{d\psi_t}{dt} = \sum_{j=1}^{4}\left(\frac{dz_j}{dt}\frac{\partial}{\partial z_j} + \frac{d\bar{z}_j}{dt}\frac{\partial}{\partial \bar{z}_j}\right).$$

Equating both sides of (5.2) and implementing Theorem 5.1.5 gives the following.

Theorem 5.3.2. *Use the notation of Definition 5.3.1. Let $z_j : \mathbb{R} \to \mathbb{C}$ for $j = 1, 2, 3, 4$ be differentiable functions satisfying*

$$\frac{dz_1}{dt} = a_1\overline{z_2z_3z_4} + \frac{1}{2}\, z_1\big((a_4 - a_3)|z_2|^2 + (a_2 - a_4)|z_3|^2 + (a_3 - a_2)|z_4|^2\big), \tag{5.21}$$

$$\frac{dz_2}{dt} = a_2\overline{z_3z_4z_1} + \frac{1}{2}\, z_2\big((a_4 - a_1)|z_3|^2 + (a_1 - a_3)|z_4|^2 + (a_3 - a_4)|z_1|^2\big), \tag{5.22}$$

$$\frac{dz_3}{dt} = a_3\overline{z_4z_1z_2} + \frac{1}{2}\, z_3\big((a_2 - a_1)|z_4|^2 + (a_4 - a_2)|z_1|^2 + (a_1 - a_4)|z_2|^2\big) \textit{ and} \tag{5.23}$$

$$\frac{dz_4}{dt} = a_4\overline{z_1z_2z_3} + \frac{1}{2}\, z_4\big((a_2 - a_3)|z_1|^2 + (a_3 - a_1)|z_2|^2 + (a_1 - a_2)|z_3|^2\big). \tag{5.24}$$

The subset M of $\mathbb{C}^4 \cong \mathbb{R}^8$ given by

$$M = \big\{(re^{i\alpha_1}z_1(t), re^{i\alpha_2}z_2(t), re^{i\alpha_3}z_3(t), re^{i\alpha_4}z_4(t)) \,:\, r > 0,\, (e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}, e^{i\alpha_4}) \in \mathrm{G},\, t \in \mathbb{R}\big\}$$

is a Cayley 4-fold in $\mathbb{R}^8$. Moreover, $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2$ is a constant which can be taken to be 1 and $\mathrm{Im}(z_1z_2z_3z_4) = A$ for some real constant A.

Proof. Theorem 5.1.5 only gives existence of solutions of $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. However, as in §4.2.2, we can prove we have solutions that exist for all $t \in \mathbb{R}$ using the boundedness of the functions involved. It is clear from (5.21)-(5.24) that $|z_1|^2 + \ldots + |z_4|^2$ is a constant and that we can take this constant to be 1 without loss of generality. Furthermore,

$$\frac{d}{dt}\,(z_1z_2z_3z_4) = a_1|z_2z_3z_4|^2 + a_2|z_3z_4z_1|^2 + a_3|z_4z_1z_2|^2 + a_4|z_1z_2z_3|^2,$$

which is purely real. Therefore $\mathrm{Im}(z_1z_2z_3z_4) = A$ is constant. $\square$

5.3.2 SU(2) symmetry 1

We consider three different natural actions of SU(2) on $\mathbb{C}^4 \cong \mathbb{R}^8$ in Spin(7). The first is where SU(2) acts on $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ in the usual manner upon one $\mathbb{C}^2$ and trivially upon the other. The construction using this action gives an affine $\mathbb{C}^2 \subseteq \mathbb{C}^4$ as the Cayley 4-fold. The second is where SU(2) acts on $\mathbb{C}^4 \cong \mathbb{C}^3 \oplus \mathbb{C}$ as SO(3) on $\mathbb{C}^3$ and trivially on $\mathbb{C}$. The construction then produces a complex surface in $\mathbb{C}^4$ as the Cayley 4-fold, which may be written as follows:

$$\big\{(z_1, z_2, z_3, z_4) : z_1^2 + z_2^2 + z_3^2 = A,\, z_4 = B\big\}\,, \text{ where } A, B \in \mathbb{C} \text{ are constants.}$$

We therefore turn our attention to the diagonal action of SU(2).

Definition 5.3.3. Let

$$X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2),$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Then X acts on $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \cong \mathbb{R}^8$ as:

$$X \cdot (z_1, z_2, z_3, z_4) = (az_1 + bz_2, -\bar{b}z_1 + \bar{a}z_2, az_3 + bz_4, -\bar{b}z_3 + \bar{a}z_4).$$

Define smooth maps $\psi_t : \mathrm{SU}(2) \to \mathbb{C}^4 \cong \mathbb{R}^8$ by:

$$\psi_t(X) = X \cdot (z_1(t), z_2(t), z_3(t), z_4(t)),$$

where $z_1(t)$, $z_2(t)$, $z_3(t)$ and $z_4(t)$ are smooth functions of t.

Calculation shows that we may take the following three complex matrices as a basis for the Lie algebra of SU(2) acting in this way:

$$U_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}; \quad U_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; \quad \text{and} \quad U_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

If we let $\mathbf{u}_j = (\psi_t)_*(U_j)$ for $j = 1, 2, 3$,

$$\begin{aligned}
\mathbf{u}_1 &= i\left(z_1\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1} - z_2\frac{\partial}{\partial z_2} + \bar{z}_2\frac{\partial}{\partial \bar{z}_2} + z_3\frac{\partial}{\partial z_3} - \bar{z}_3\frac{\partial}{\partial \bar{z}_3} - z_4\frac{\partial}{\partial z_4} + \bar{z}_4\frac{\partial}{\partial \bar{z}_4} \right), \\
\mathbf{u}_2 &= z_2\frac{\partial}{\partial z_1} + \bar{z}_2\frac{\partial}{\partial \bar{z}_1} - z_1\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial \bar{z}_2} + z_4\frac{\partial}{\partial z_3} + \bar{z}_4\frac{\partial}{\partial \bar{z}_3} - z_3\frac{\partial}{\partial z_4} - \bar{z}_3\frac{\partial}{\partial \bar{z}_4} \qquad \text{and} \\
\mathbf{u}_3 &= i\left(z_2\frac{\partial}{\partial z_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_1} + z_1\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial \bar{z}_2} + z_4\frac{\partial}{\partial z_3} - \bar{z}_4\frac{\partial}{\partial \bar{z}_3} + z_3\frac{\partial}{\partial z_4} - \bar{z}_3\frac{\partial}{\partial \bar{z}_4} \right).
\end{aligned}$$

Thus, if we take $\chi = U_1 \wedge U_2 \wedge U_3$, $(\psi_t)_*(\chi) = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$. Using the equations above for $\mathbf{u}_j$ and the formula (2.12) for Φ_0, we may calculate the right-hand side of (5.2), which is $\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3$ as defined by (2.14). Moreover,

$$\frac{d\psi_t}{dt} = \sum_{j=1}^{4} \frac{dz_j}{dt}\frac{\partial}{\partial z_j} + \sum_{j=1}^{4} \frac{d\bar{z}_j}{dt}\frac{\partial}{\partial \bar{z}_j}.$$

Equating both sides of (5.2) and using Theorem 5.1.5 gives the following result.

Theorem 5.3.4. *Let $z_1(t), z_2(t), z_3(t), z_4(t)$ be smooth complex-valued functions of t satisfying*

$$\frac{dz_1}{dt} = z_1\left(|z_1|^2 + |z_2|^2 + |z_3|^2 - |z_4|^2\right) + 2(\overline{z_1 z_4 - z_2 z_3} + z_2 z_3)\bar{z}_4, \tag{5.25}$$

$$\frac{dz_2}{dt} = z_2\left(|z_4|^2 + |z_1|^2 + |z_2|^2 - |z_3|^2\right) - 2(\overline{z_1 z_4 - z_2 z_3} - z_1 z_4)\bar{z}_3, \tag{5.26}$$

$$\frac{dz_3}{dt} = z_3\left(|z_3|^2 + |z_4|^2 + |z_1|^2 - |z_2|^2\right) - 2(\overline{z_1 z_4 - z_2 z_3} - z_1 z_4)\bar{z}_2 \textit{ and} \tag{5.27}$$

$$\frac{dz_4}{dt} = z_4\left(|z_2|^2 + |z_3|^2 + |z_4|^2 - |z_1|^2\right) + 2(\overline{z_1 z_4 - z_2 z_3} + z_2 z_3)\bar{z}_1 \tag{5.28}$$

for all $t \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$. The subset M of $\mathbb{C}^4 \cong \mathbb{R}^8$ defined by

$$M = \{X \cdot (z_1(t), z_2(t), z_3(t), z_4(t)) \, : \, t \in (-\epsilon, \epsilon), \; X \in \mathrm{SU}(2)\},$$

where the action of $\mathrm{SU}(2)$ on $\mathbb{C}^4$ is given in Definition 5.3.3, is a Cayley 4-fold in $\mathbb{R}^8$.

We are able to give an explicit description of the Cayley 4-folds constructed in Theorem 5.3.4. Let $u(t)$ be a real-valued function satisfying

$$\frac{du}{dt} = 2u(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2). \tag{5.29}$$

We observe, using (5.25)-(5.28), that the following quadratics satisfy (5.29):

$$|z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2; \qquad z_1\bar{z}_2 + z_3\bar{z}_4;$$

$$\mathrm{Re}(z_1z_4 - z_2z_3); \quad \text{and} \qquad z_1\bar{z}_3 + z_2\bar{z}_4.$$

Hence, each of these quadratics is a constant multiple of u. The first two correspond to the moment maps of the SU(2) action and the latter two are SU(2)-invariant. The first two quadratics are not SU(2)-invariant, but

$$\begin{aligned} Q(z_1, z_2, z_3, z_4) &= (|z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2)^2 + 4|z_1\bar{z}_2 + z_3\bar{z}_4|^2 \\ &= (|z_1|^2 + |z_2|^2)^2 + (|z_3|^2 + |z_4|^2)^2 + 2|z_1\bar{z}_3 + z_2\bar{z}_4|^2 - 2|z_1z_4 - z_2z_3|^2 \end{aligned} \tag{5.30}$$

is SU(2)-invariant and is a constant multiple of u^2.

Using (5.25)-(5.28) we calculate:

$$\frac{d}{dt}\,\mathrm{Im}(z_1z_4 - z_2z_3) = -2\,\mathrm{Im}(z_1z_4 - z_2z_3)(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2).$$

Therefore, by (5.29), $\mathrm{Im}(z_1z_4 - z_2z_3)$ is a constant multiple of u^{-1} and it is an SU(2)-invariant quadratic. We then state our result, which is immediate from our discussion above.

Theorem 5.3.5. *Let A, B, C and D be real constants. Let $M \subseteq \mathbb{C}^4 \cong \mathbb{R}^8$ be defined by:*

$$M = \{X \cdot (z_1, z_2, z_3, z_4) : X \in \mathrm{SU}(2)\},$$

where the action of $X \in \mathrm{SU}(2)$ on $\mathbb{C}^4$ is given in Definition 5.3.3 and z_1, z_2, z_3, z_4 satisfy:

$$Q(z_1, z_2, z_3, z_4)\left(\mathrm{Im}(z_1z_4 - z_2z_3)\right)^2 = A; \tag{5.31}$$

$$\mathrm{Re}(z_1z_4 - z_2z_3)\,\mathrm{Im}(z_1z_4 - z_2z_3) = B; \tag{5.32}$$

$$\mathrm{Re}(z_1\bar{z}_3 + z_2\bar{z}_4)\,\mathrm{Im}(z_1z_4 - z_2z_3) = C; \text{ and} \tag{5.33}$$

$$\mathrm{Im}(z_1\bar{z}_3 + z_2\bar{z}_4)\,\mathrm{Im}(z_1z_4 - z_2z_3) = D, \tag{5.34}$$

where $Q(z_1, z_2, z_3, z_4)$ is given by (5.30). Then M is a Cayley 4-fold in $\mathbb{R}^8$.

The set of conditions (5.31)-(5.34) on the complex functions z_1, z_2, z_3, z_4 consists of setting one real octic and three real quartics to be constant, which defines a 4-dimensional subset of $\mathbb{C}^4$. Hence, Theorem 5.3.5 completely describes the SU(2)-invariant Cayley 4-folds given by Theorem 5.3.4.

5.3.3 SU(2) symmetry 2

We now consider another action of SU(2) on $\mathbb{C}^4$, which is also studied by Marshall [41, §3.4].

Definition 5.3.6. The usual action of SU(2) on $\mathbb{C}^2$ induces an action on $S^3\mathbb{C}^2$. We then identify $S^3\mathbb{C}^2$ with $\mathbb{C}^4$ in an appropriate way so as to define a subgroup G of SU(4) that is isomorphic to SU(2). Write the action of $X \in \mathrm{G}$ on $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \cong \mathbb{R}^8$ as $X \cdot (z_1, z_2, z_3, z_4)$.

Define smooth maps $\psi_t : \mathrm{G} \to \mathbb{C}^4 \cong \mathbb{R}^8$ by:

$$\psi_t(X) = X \cdot (z_1(t), z_2(t), z_3(t), z_4(t)),$$

where $z_1(t), z_2(t), z_3(t)$ and $z_4(t)$ are smooth functions of t.

Marshall [41, §3.4] shows that we may take the following three matrices as a basis for the Lie algebra of G:

$$U_1 = \begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -3i \end{pmatrix}; \; U_2 = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}; \text{ and } U_3 = \begin{pmatrix} 0 & \sqrt{3}i & 0 & 0 \\ \sqrt{3}i & 0 & 2i & 0 \\ 0 & 2i & 0 & \sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix}.$$

If $\mathbf{u}_j = (\psi_t)_*(U_j)$ for $j = 1, 2, 3$,

$$\mathbf{u}_1 = i\left(3z_1\frac{\partial}{\partial z_1} - 3\bar{z}_1\frac{\partial}{\partial \bar{z}_1} + z_2\frac{\partial}{\partial z_2} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2} - z_3\frac{\partial}{\partial z_3} + \bar{z}_3\frac{\partial}{\partial \bar{z}_3} - 3z_4\frac{\partial}{\partial z_4} + 3\bar{z}_4\frac{\partial}{\partial \bar{z}_4}\right),$$

$$\begin{aligned}\mathbf{u}_2 = {} & \sqrt{3}z_2\frac{\partial}{\partial z_1} + \sqrt{3}\bar{z}_2\frac{\partial}{\partial \bar{z}_1} + (-\sqrt{3}z_1 + 2z_3)\frac{\partial}{\partial z_2} + (-\sqrt{3}\bar{z}_1 + 2\bar{z}_3)\frac{\partial}{\partial \bar{z}_2} \\ & + (-2z_2 + \sqrt{3}z_4)\frac{\partial}{\partial z_3} + (-2\bar{z}_2 + \sqrt{3}\bar{z}_4)\frac{\partial}{\partial \bar{z}_3} - \sqrt{3}z_3\frac{\partial}{\partial z_4} - \sqrt{3}\bar{z}_3\frac{\partial}{\partial \bar{z}_4} \text{ and}\end{aligned}$$

$$\begin{aligned}\mathbf{u}_3 = i\Big(& \sqrt{3}z_2\frac{\partial}{\partial z_1} - \sqrt{3}\bar{z}_2\frac{\partial}{\partial \bar{z}_1} + (\sqrt{3}z_1 + 2z_3)\frac{\partial}{\partial z_2} - (\sqrt{3}\bar{z}_1 + 2\bar{z}_3)\frac{\partial}{\partial \bar{z}_2} \\ & + (2z_2 + \sqrt{3}z_4)\frac{\partial}{\partial z_3} - (2\bar{z}_2 + \sqrt{3}\bar{z}_4)\frac{\partial}{\partial \bar{z}_3} + \sqrt{3}z_3\frac{\partial}{\partial z_4} - \sqrt{3}\bar{z}_3\frac{\partial}{\partial \bar{z}_4}\Big).\end{aligned}$$

Therefore, if we set $\chi = U_1 \wedge U_2 \wedge U_3$, $(\psi_t)_*(\chi) = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$. Using the formulae above we may calculate $\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3$ as given in (2.14). Equating both sides of (5.2) and applying Theorem 5.1.5 as in previous subsections gives our result.

Theorem 5.3.7. *Let $z_1(t), z_2(t), z_3(t), z_4(t)$ be smooth complex-valued functions satisfying*

$$\begin{aligned}\frac{dz_1}{dt} &= \frac{1}{2}\, z_1(9|z_1|^2 + 9|z_2|^2 - 3|z_3|^2 - 9|z_4|^2) + 2\sqrt{3}\bar{z}_3(z_2^2 + 2\bar{z}_3^2) + 3z_2 z_3 \bar{z}_4 \\ &\quad + 18\bar{z}_4(\overline{z_1 z_4} - \overline{z_2 z_3}), && (5.35)\\ \frac{dz_2}{dt} &= \frac{1}{2}\, z_2(9|z_1|^2 + |z_2|^2 + 7|z_3|^2 - 3|z_4|^2) + 2\sqrt{3}z_3(2z_1\bar{z}_2 + z_3\bar{z}_4) + 3z_1\bar{z}_3 z_4 \\ &\quad + 12\bar{z}_2(\sqrt{3}\overline{z_2 z_4} - 2\bar{z}_3^2) - 18\bar{z}_3(\overline{z_1 z_4} - \overline{z_2 z_3}), && (5.36)\\ \frac{dz_3}{dt} &= \frac{1}{2}\, z_3(9|z_4|^2 + |z_3|^2 + 7|z_2|^2 - 3|z_1|^2) + 2\sqrt{3}z_2(2z_4\bar{z}_3 + z_2\bar{z}_1) + 3z_4\bar{z}_2 z_1 \\ &\quad + 12\bar{z}_3(\sqrt{3}\overline{z_3 z_1} - 2\bar{z}_2^2) - 18\bar{z}_2(\overline{z_1 z_4} - \overline{z_2 z_3}) \text{ and} && (5.37)\\ \frac{dz_4}{dt} &= \frac{1}{2}\, z_4(9|z_4|^2 + 9|z_3|^2 - 3|z_2|^2 - 9|z_1|^2) + 2\sqrt{3}\bar{z}_2(z_3^2 + 2\bar{z}_2^2) + 3z_2 z_3 \bar{z}_1 \\ &\quad + 18\bar{z}_1(\overline{z_1 z_4} - \overline{z_2 z_3}) && (5.38)\end{aligned}$$

for all $t \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$. The subset M of $\mathbb{C}^4 \cong \mathbb{R}^8$ defined by

$$M = \{X \cdot (z_1(t), z_2(t), z_3(t), z_4(t)) : t \in (-\epsilon, \epsilon),\ X \in \mathrm{SU}(2)\},$$

where the action of SU(2) *is described in Definition 5.3.6, is a Cayley 4-fold in $\mathbb{R}^8$. Moreover, there is a real constant A such that*

$$\mathrm{Im}\left(2\sqrt{3}(z_1 z_3^3 + z_2^3 z_4) - 9z_1 z_2 z_3 z_4 + \frac{9}{2} z_1^2 z_4^2 - \frac{3}{2} z_2^2 z_3^2\right) = A. \tag{5.39}$$

Proof. We need only derive (5.39). Using (5.35)-(5.38), we have that

$$\frac{d}{dt}\left(2\sqrt{3}(z_1 z_3^3 + z_2^3 z_4) - 9z_1 z_2 z_3 z_4 + \frac{9}{2} z_1^2 z_4^2 - \frac{3}{2} z_2^2 z_3^2\right)$$

is purely real. Thus the left-hand side of (5.39) is a real constant as claimed. □

5.3.4 Some U(1)-invariant examples

We finish this section with a description of a distinguished class of U(1)-invariant Cayley 4-folds. These are in fact *2-ruled* in the sense of §5.4.

Definition 5.3.8. Let $a_1, a_2, a_3, a_4 \in \mathbb{Z}$ be coprime such that $a_1 + a_2 + a_3 + a_4 = 0$. Define 4×4 complex matrices X and $Y(s)$, for $s \in \mathbb{R}$, by:

$$X = \begin{pmatrix} ia_1 & 0 & 0 & 0 \\ 0 & ia_2 & 0 & 0 \\ 0 & 0 & ia_3 & 0 \\ 0 & 0 & 0 & ia_4 \end{pmatrix} \quad \text{and} \quad Y(s) = \begin{pmatrix} e^{ia_1 s} & 0 & 0 & 0 \\ 0 & e^{ia_2 s} & 0 & 0 \\ 0 & 0 & e^{ia_3 s} & 0 \\ 0 & 0 & 0 & e^{ia_4 s} \end{pmatrix}.$$

Multiplication by $Y(s)$ on $\mathbb{C}^4$ corresponds to a U(1) action. Note that $X\mathbf{v}$ is orthogonal to $\mathbf{v}$ for any $\mathbf{v} \in \mathbb{C}^4$.

Define smooth maps $\psi_t : \mathbb{R}^3 \to \mathbb{C}^4 \cong \mathbb{R}^8$ by:

$$\psi_t(s, \lambda_1, \lambda_2) = Y(s)(\lambda_1 \mathbf{v}_1(t) + \lambda_2 \mathbf{v}_2(t) + \mathbf{w}(t)),$$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w} : \mathbb{R} \to \mathbb{C}^4$ are smooth functions.

If we take $\chi = \frac{\partial}{\partial s} \wedge \frac{\partial}{\partial \lambda_1} \wedge \frac{\partial}{\partial \lambda_2}$, (5.2) becomes:

$$\lambda_1 \frac{d\mathbf{v}_1}{dt} + \lambda_2 \frac{d\mathbf{v}_2}{dt} + \frac{d\mathbf{w}}{dt} = \mathbf{v}_1 \times \mathbf{v}_2 \times (\lambda_1 X\mathbf{v}_1 + \lambda_2 X\mathbf{v}_2 + X\mathbf{w})$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$, where the triple cross product is defined by (2.14). Applying Theorem 5.1.5 gives the result below.

Theorem 5.3.9. *Use the notation of Definition 5.3.8. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w} : \mathbb{R} \to \mathbb{C}^4$ be smooth maps, such that $\mathbf{v}_1(0), \mathbf{v}_2(0)$ are orthogonal unit vectors, satisfying*

$$\frac{d\mathbf{v}_1}{dt} = \mathbf{v}_1 \times \mathbf{v}_2 \times X\mathbf{v}_1, \qquad \frac{d\mathbf{v}_2}{dt} = \mathbf{v}_1 \times \mathbf{v}_2 \times X\mathbf{v}_2 \quad \textit{and} \quad \frac{d\mathbf{w}}{dt} = \mathbf{v}_1 \times \mathbf{v}_2 \times X\mathbf{w}, \tag{5.40}$$

for $t \in (-\epsilon, \epsilon)$, for some $\epsilon > 0$, where the triple cross product is defined by (2.14). The subset M of $\mathbb{C}^4 \cong \mathbb{R}^8$ given by

$$M = \{Y(s)(\lambda_1 \mathbf{v}_1(t) + \lambda_2 \mathbf{v}_2(t) + \mathbf{w}(t)) : \lambda_1, \lambda_2, s \in \mathbb{R}, t \in (-\epsilon, \epsilon)\}$$

is a Cayley 4-fold. Moreover, $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are orthogonal unit vectors for all $t \in (-\epsilon, \epsilon)$.

Proof. Theorem 5.1.5 gives existence of the solutions for $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Recall that the triple cross product is orthogonal to each of the vectors in the product by Proposition 2.4.2.

Clearly, from (5.40),

$$g_0\left(\mathbf{v}_1, \frac{d\mathbf{v}_1}{dt}\right) = 0 = g_0\left(\mathbf{v}_2, \frac{d\mathbf{v}_2}{dt}\right).$$

Thus $|\mathbf{v}_1|^2$ and $|\mathbf{v}_2|^2$ are constant and hence equal to 1 since $\mathbf{v}_1(0)$ and $\mathbf{v}_2(0)$ are unit vectors. Furthermore,

$$\frac{d}{dt} g_0(\mathbf{v}_1, \mathbf{v}_2) = g_0(\mathbf{v}_1 \times \mathbf{v}_2 \times X\mathbf{v}_1, \mathbf{v}_2) + g_0(\mathbf{v}_1, \mathbf{v}_1 \times \mathbf{v}_2 \times X\mathbf{v}_2) = 0,$$

which implies that $g_0(\mathbf{v}_1, \mathbf{v}_2)$ is constant, hence zero by the initial conditions as claimed. This completes the proof. $\square$

5.4 2-Ruled Calibrated 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$

Our study now turns to *2-ruled* 4-dimensional submanifolds of $\mathbb{R}^n$, which are analogous to 1-ruled 3-folds as given in Definition 4.5.1. We focus on providing constructions for Cayley examples, from which we derive methods for producing special Lagrangian and coassociative 2-ruled 4-folds.

We begin with the basic definitions.

Definition 5.4.1. Let M be a 4-dimensional submanifold of $\mathbb{R}^n$. A *2-ruling* of M is a pair (Σ, π), where Σ is a 2-dimensional manifold and $\pi : M \to \Sigma$ is a smooth map, such that $\pi^{-1}(\sigma)$ is an affine 2-plane in $\mathbb{R}^n$ for all $\sigma \in \Sigma$. The triple (M, Σ, π) is a *2-ruled* 4-fold in $\mathbb{R}^n$.

An *r-framing* for a 2-ruling (Σ, π) of M is a choice of oriented orthonormal basis, or frame, for the linear 2-plane associated to $\pi^{-1}(\sigma)$ given by the 2-ruling, for each $\sigma \in \Sigma$, which varies smoothly with σ. Then (M, Σ, π) with an r-framing is called *r-framed*.

Let (M, Σ, π) be an r-framed 2-ruled 4-fold in $\mathbb{R}^n$. For each $\sigma \in \Sigma$, define $(\phi_1(\sigma), \phi_2(\sigma))$ to be the oriented orthonormal basis for $\pi^{-1}(\sigma)$ given by the r-framing. Then $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^{n-1}$ are smooth maps. Define $\psi : \Sigma \to \mathbb{R}^n$ such that, for all $\sigma \in \Sigma$, $\psi(\sigma)$ is the unique vector in $\pi^{-1}(\sigma)$ orthogonal to $\phi_1(\sigma)$ and $\phi_2(\sigma)$. Then ψ is a smooth map and

$$M = \{r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma) \,:\, \sigma \in \Sigma,\, r_1, r_2 \in \mathbb{R}\}. \tag{5.41}$$

Define the *asymptotic cone* M_0 of a 2-ruled 4-fold M as the set of points in planes Π including the origin such that Π is parallel to $\pi^{-1}(\sigma)$ for some $\sigma \in \Sigma$. If M is r-framed,

$$M_0 = \{r_1\phi_1(\sigma) + r_2\phi_2(\sigma) \,:\, \sigma \in \Sigma,\, r_1, r_2 \in \mathbb{R}\} \tag{5.42}$$

and is usually a 4-dimensional cone; that is, whenever the map $\iota : \Sigma \times \mathcal{S}^1 \to \mathcal{S}^{n-1}$ given by $\iota(\sigma, e^{i\theta}) = \cos\theta\,\phi_1(\sigma) + \sin\theta\,\phi_2(\sigma)$ is an immersion.

In [19] Ionel et al. derive a method for constructing coassociative 4-folds in $\mathbb{R}^7$ and Cayley 4-folds in $\mathbb{R}^8$. Their technique produces examples which are 2-ruled. However, the Cayley 4-folds they produce are only either $\mathbb{R} \times L$ for some associative 3-fold L or lie in $\mathbb{R}^7$ and hence are coassociative. Explicit examples obtained from this construction are given in [19, §4].

Let (M, Σ, π) be a 2-ruled 4-fold in $\mathbb{R}^n$. Let

$$P = \{(\mathbf{v}, \sigma) \in \mathcal{S}^{n-1} \times \Sigma \,:\, \mathbf{v} \text{ is a unit vector parallel to } \pi^{-1}(\sigma),\, \sigma \in \Sigma\}$$

and let $\pi_P : P \to \Sigma$ be given by $\pi_P(\mathbf{v}, \sigma) = \sigma$. Clearly, $\pi_P : P \to \Sigma$ is an $\mathcal{S}^1$ bundle over Σ. Note that (M, Σ, π) admits an r-framing if and only if this bundle is trivializable. Therefore, if M is orientable and Σ is non-orientable, e.g. $\Sigma \cong \mathcal{K}$ where $\mathcal{K}$ is the Klein bottle, a 2-ruling (Σ, π) of

M cannot be r-framed. Moreover, if M is r-framed then M_0 is not necessarily 4-dimensional. For example, if we take $\Sigma = \mathbb{R}^2$ and define ϕ_1, ϕ_2 and ψ by $\phi_1(x,y) = (1,0,0,0)$, $\phi_2(x,y) = (0,1,0,0)$ and $\psi(x,y) = (0,0,x,y)$ for $x, y \in \mathbb{R}$, then M, as defined by (5.41), is an r-framed 2-ruled 4-fold since $M = \mathbb{R}^4$, but $M_0 = \mathbb{R}^2$. We also note that any r-framed 2-ruled 4-fold is defined by three maps ϕ_1, ϕ_2 and ψ as in (5.41). We may thus construct 2-ruled calibrated 4-folds by formulating partial differential equations for ϕ_1, ϕ_2 and ψ.

In an analogous manner to the 1-ruled case, one may show that an r-framed 2-ruled 4-fold is asymptotically conical with rate 0 to its asymptotic cone if Σ is compact.

5.4.1 The partial differential equations

We wish to construct 2-ruled calibrated 4-folds in $\mathbb{R}^7$ and $\mathbb{R}^8$ by solving partial differential equations for maps ϕ_1, ϕ_2, ψ. By Propositions 2.4.7 and 3.1.4, it is sufficient to consider the Cayley case.

Let Σ be a 2-dimensional, connected, real analytic manifold, let $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ be orthogonal real analytic maps such that $\iota : \Sigma \times \mathcal{S}^1 \to \mathcal{S}^7$ defined by $\iota(\sigma, e^{i\theta}) = \cos\theta\phi_1(\sigma) + \sin\theta\phi_2(\sigma)$ is an immersion and let $\psi : \Sigma \to \mathbb{R}^8$ be a real analytic map. Clearly, $\mathbb{R}^2 \times \Sigma$ is an r-framed 2-ruled 4-fold with 2-ruling (Σ, π), where $\pi(r_1, r_2, \sigma) = \sigma$. Let M be defined by (5.41). Then M is the image of the map $\iota_M : \mathbb{R}^2 \times \Sigma \to \mathbb{R}^8$ given by $\iota_M(r_1, r_2, \sigma) = r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma)$. Since ι is an immersion, ι_M is an immersion almost everywhere. Thus M is an r-framed 2-ruled 4-fold in $\mathbb{R}^8$, possibly with singularities.

Suppose M is Cayley and $p \in M$. There exist $(r_1, r_2) \in \mathbb{R}^2$ and $\sigma \in \Sigma$ such that $p = r_1\phi_1(\sigma) + r_2\phi_2(\sigma) + \psi(\sigma)$. Choose oriented coordinates (s,t) near σ in Σ. Then $T_pM = \langle \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \rangle_{\mathbb{R}}$, where

$$\mathbf{x} = \phi_1(\sigma), \quad \mathbf{y} = \phi_2(\sigma), \quad \mathbf{z} = r_1\frac{\partial\phi_1}{\partial s}(\sigma) + r_2\frac{\partial\phi_2}{\partial s}(\sigma) + \frac{\partial\psi}{\partial s}(\sigma) \text{ and } \mathbf{w} = r_1\frac{\partial\phi_1}{\partial t}(\sigma) + r_2\frac{\partial\phi_2}{\partial t}(\sigma) + \frac{\partial\psi}{\partial t}(\sigma).$$

The tangent space T_pM is a Cayley 4-plane. By Proposition 2.4.4 this is true if and only if $\mathrm{Im}(\mathbf{x} \times \mathbf{y} \times \mathbf{z} \times \mathbf{w}) = 0$, identifying $\mathbb{R}^8$ with $\mathbb{O}$. This implies that a quadratic in r_1 and r_2 must vanish for all $(r_1, r_2) \in \mathbb{R}^2$, so each coefficient in the quadratic is zero. Therefore, the following set of equations must hold in Σ:

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_1}{\partial s} \times \frac{\partial\phi_1}{\partial t}\right) = 0; \tag{5.43}$$

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_2}{\partial t}\right) = 0; \tag{5.44}$$

$$\mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_1}{\partial s} \times \frac{\partial\phi_2}{\partial t}\right) + \mathrm{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_1}{\partial t}\right) = 0; \tag{5.45}$$

$$\operatorname{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\psi}{\partial s} \times \frac{\partial\psi}{\partial t}\right) = 0; \tag{5.46}$$

$$\operatorname{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_1}{\partial s} \times \frac{\partial\psi}{\partial t}\right) + \operatorname{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\psi}{\partial s} \times \frac{\partial\phi_1}{\partial t}\right) = 0; \text{ and} \tag{5.47}$$

$$\operatorname{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\psi}{\partial t}\right) + \operatorname{Im}\left(\phi_1 \times \phi_2 \times \frac{\partial\psi}{\partial s} \times \frac{\partial\phi_2}{\partial t}\right) = 0. \tag{5.48}$$

If we do not suppose M to be Cayley but instead insist that (5.43)-(5.48) hold in Σ then, following the argument above, each tangent space to M must be Cayley and hence M is a Cayley 4-fold. Noting that (5.43)-(5.45) are precisely the conditions for the asymptotic cone M_0 of M to be Cayley, we deduce the following result.

Proposition 5.4.2. *The asymptotic cone of an r-framed 2-ruled Cayley 4-fold in $\mathbb{R}^8$ is Cayley provided it is 4-dimensional.*

It is apparent, from consideration of Proposition 2.4.7, that $\iota(\Sigma \times \mathcal{S}^1) \subseteq \mathcal{S}^7$ is the link of a Cayley cone if and only if it is an associative 3-fold in $\mathcal{S}^7$.

Clearly, M_0 is the image of the map $\iota_0 : \mathbb{R}^2 \times \Sigma \to \mathbb{R}^8$ given by $\iota_0(r_1, r_2, \sigma) = r_1\phi_1(\sigma) + r_2\phi_2(\sigma)$. Since we suppose that ι is an immersion, ι_0 is an immersion except at $(r_1, r_2) = (0, 0)$, so M_0 is nonsingular except at 0 and thus is a cone.

Note that Φ_0 is a nowhere vanishing 4-form on M_0 that defines its orientation, since M_0 is Cayley. Hence, if (s, t) are local coordinates on Σ, we can define them to be oriented by imposing

$$\Phi_0\left(\phi_1, \phi_2, r_1\frac{\partial\phi_1}{\partial s} + r_2\frac{\partial\phi_2}{\partial s}, r_1\frac{\partial\phi_1}{\partial t} + r_2\frac{\partial\phi_2}{\partial t}\right) > 0 \tag{5.49}$$

for all $(r_1, r_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. It follows that

$$\Phi_0\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_1}{\partial t}\right) > 0 \quad \text{and} \quad \Phi_0\left(\phi_1, \phi_2, \frac{\partial\phi_2}{\partial s}, \frac{\partial\phi_2}{\partial t}\right) > 0. \tag{5.50}$$

Consequently, $\{\phi_1, \phi_2, \frac{\partial\phi_j}{\partial s}, \frac{\partial\phi_j}{\partial t}\}$ is a linearly independent set for $j = 1, 2$. Moreover, (5.49) is equivalent to the condition that ι is an immersion.

We now construct a metric on Σ, under suitable conditions, using ϕ_1, ϕ_2 and the metric on $\mathbb{R}^8$. For a function $f : \Sigma \to \mathbb{R}^8$, we define $f^\perp : \Sigma \to \mathbb{R}^8$ by choosing $f^\perp(\sigma)$ to be the component of $f(\sigma)$ that lies in the orthogonal complement of $\langle\phi_1(\sigma), \phi_2(\sigma)\rangle_{\mathbb{R}}$. Since the fourfold cross product is alternating, (5.43)-(5.45) hold if and only if

$$\operatorname{Im}\left(\phi_1 \times \phi_2 \times \left(\cos\theta\frac{\partial\phi_1}{\partial s}^\perp + \sin\theta\frac{\partial\phi_2}{\partial s}^\perp\right) \times \left(\cos\theta\frac{\partial\phi_1}{\partial t}^\perp + \sin\theta\frac{\partial\phi_2}{\partial t}^\perp\right)\right) = 0 \tag{5.51}$$

for all $\theta \in \mathbb{R}$. Let $\sigma \in \Sigma$. From Proposition 2.4.4 and (5.49) we see that, for each $\theta \in \mathbb{R}$, the four terms in (5.51), evaluated at σ, form a basis for a Cayley 4-plane Π_θ. By Corollary 2.4.5, we may also take

$$\left(\phi_1(\sigma),\, \phi_2(\sigma),\, \cos\theta\frac{\partial\phi_1}{\partial s}^{\perp}(\sigma)+\sin\theta\frac{\partial\phi_2}{\partial s}^{\perp}(\sigma),\, \cos\theta\phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}^{\perp}(\sigma)+\sin\theta\phi_1\times\phi_2\times\frac{\partial\phi_2}{\partial s}^{\perp}(\sigma)\right)$$

as a basis for Π_θ. Therefore,

$$\begin{aligned}\cos\theta\frac{\partial\phi_1}{\partial t}^{\perp}(\sigma)+\sin\theta\frac{\partial\phi_2}{\partial t}^{\perp}(\sigma) &= A_\theta\left(\cos\theta\frac{\partial\phi_1}{\partial s}^{\perp}(\sigma)+\sin\theta\frac{\partial\phi_2}{\partial s}^{\perp}(\sigma)\right)\\ &\quad + B_\theta\left(\cos\theta\phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}^{\perp}(\sigma)+\sin\theta\phi_1\times\phi_2\times\frac{\partial\phi_2}{\partial s}^{\perp}(\sigma)\right) \qquad (5.52)\end{aligned}$$

for constants A_θ, B_θ depending on θ. We set $\theta = 0, \frac{\pi}{2}$ in (5.52) and substitute back in the expressions found for the t derivatives to obtain:

$$\begin{aligned}\cos\theta\left((A_0-A_\theta)\frac{\partial\phi_1}{\partial s}^{\perp}(\sigma)+(B_0-B_\theta)\phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}^{\perp}(\sigma)\right) &=\\ \sin\theta\left((A_\theta-A_{\frac{\pi}{2}})\frac{\partial\phi_2}{\partial s}^{\perp}(\sigma)+(B_\theta-B_{\frac{\pi}{2}})\phi_1\times\phi_2\times\frac{\partial\phi_2}{\partial s}^{\perp}(\sigma)\right)&. \qquad (5.53)\end{aligned}$$

To proceed in defining a metric on Σ we impose a condition on the dimension of

$$V_\sigma = \left\langle \frac{\partial\phi_1}{\partial s}^{\perp}(\sigma), \frac{\partial\phi_2}{\partial s}^{\perp}(\sigma), \phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}^{\perp}(\sigma), \phi_1\times\phi_2\times\frac{\partial\phi_2}{\partial s}^{\perp}(\sigma)\right\rangle_{\mathbb{R}}.$$

Let $W_\sigma = \langle\phi_1(\sigma),\phi_2(\sigma)\rangle_{\mathbb{R}}^{\perp} \subseteq \mathbb{R}^8$ and define $J_\sigma : W_\sigma \to W_\sigma$ by $J_\sigma(v) = \phi_1(\sigma)\times\phi_2(\sigma)\times v$. It is clear, through calculation in coordinates, that $J_\sigma^2 = -1$ on W_σ. Note that $V_\sigma \subseteq W_\sigma$ is closed under the action of J_σ, which can thus be considered as a form of complex structure on V_σ. Hence, V_σ is even-dimensional. Since the case $\dim V_\sigma = 0$ is excluded by (5.50), $\dim V_\sigma = 2$ or 4. Recall that Σ is real analytic and connected. Therefore $\{\sigma\in\Sigma : \dim V_\sigma = 2\}$ is a closed real analytic subset of Σ and consequently either coincides with Σ or is of zero measure in Σ.

Suppose that $\dim V_\sigma = 4$. The four vectors in (5.53) are then linearly independent and hence

$$(A_0-A_\theta)\cos\theta = (B_0-B_\theta)\cos\theta = (A_{\frac{\pi}{2}}-A_\theta)\sin\theta = (B_{\frac{\pi}{2}}-B_\theta)\sin\theta = 0$$

for all θ. This clearly forces A_θ and B_θ to be constant and equal to A and B, say, respectively. Define a metric g_Σ pointwise on Σ, up to scale, by the following equations:

$$g_\Sigma\left(\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right) = Ag_\Sigma\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right) \quad\text{and}\quad g_\Sigma\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) = (A^2+B^2)g_\Sigma\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right). \qquad (5.54)$$

Using (5.52) and the fact that $J_\sigma^2 = -1$ on V_σ,

$$\begin{pmatrix}\phi_1\times\phi_2\times\frac{\partial\phi_j}{\partial s}^{\perp}(\sigma)\\ \phi_1\times\phi_2\times\frac{\partial\phi_j}{\partial t}^{\perp}(\sigma)\end{pmatrix} = K\begin{pmatrix}\frac{\partial\phi_j}{\partial s}^{\perp}(\sigma)\\ \frac{\partial\phi_j}{\partial t}^{\perp}(\sigma)\end{pmatrix}$$

for $j = 1, 2$, where K is the 2×2 matrix given by:

$$K = \frac{1}{B} \begin{pmatrix} -A & 1 \\ -(A^2 + B^2) & A \end{pmatrix}.$$

If we change coordinates (s, t) to $(\tilde{s}, \tilde{t})$ with Jacobian matrix L, K transforms to $\tilde{K} = LKL^{-1}$. Calculation shows that the corresponding $\tilde{A}$ and $\tilde{B}$ defining $\tilde{K}$ satisfy (5.54) for the coordinates $(\tilde{s}, \tilde{t})$. Thus, g_Σ is a well-defined metric, up to scale, covariant under transformation of coordinates.

Having defined the metric g_Σ we can consider Σ as a *Riemannian* 2-fold, which has a natural orientation derived from the orientation on M and on the 2-planes $\langle \phi_1(\sigma), \phi_2(\sigma) \rangle_{\mathbb{R}}$. Therefore it has a natural *complex structure* which we denote as J. If we choose a local holomorphic coordinate $u = s + it$ on Σ, the corresponding real coordinates must satisfy $\frac{\partial}{\partial t} = J \frac{\partial}{\partial s}$. We say that local real coordinates (s, t) on Σ satisfying this condition are *oriented conformal coordinates* as in §4.5.1. This forces $A = 0$ and $B = 1$ in the notation of (5.54), since $B > 0$ by (5.50).

We now state and prove a theorem in this case.

Theorem 5.4.3. *Let Σ be a connected real analytic 2-fold, let $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ be orthogonal real analytic maps such that $\iota : \Sigma \times \mathcal{S}^1 \to \mathcal{S}^7$ defined by $\iota(\sigma, e^{i\theta}) = \cos\theta \phi_1(\sigma) + \sin\theta \phi_2(\sigma)$ is an immersion, and let $\psi : \Sigma \to \mathbb{R}^8$ be a real analytic map. Define M by* (5.41) *and suppose that* $\dim V_\sigma = 4$ *almost everywhere in Σ. Then M is Cayley if and only if*

$$\frac{\partial \phi_1}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s} + f\phi_2, \tag{5.55}$$

$$\frac{\partial \phi_2}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s} - f\phi_1, \tag{5.56}$$

for some function $f : \Sigma \to \mathbb{R}$, and ψ satisfies

$$\frac{\partial \psi}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \psi}{\partial s} + g_1\phi_1 + g_2\phi_2 \tag{5.57}$$

for some functions $g_1, g_2 : \Sigma \to \mathbb{R}$, where the triple cross product is defined in (2.14) *and (s, t) are oriented conformal coordinates on Σ. Moreover, sufficiency holds irrespective of* $\dim V_\sigma$.

Proof. Recalling that (5.43)-(5.48) correspond to the condition that M is Cayley, we show that (5.43)-(5.45) are equivalent to (5.55)-(5.56), and that (5.46)-(5.48) are equivalent to (5.57).

Let $\sigma \in \Sigma$. Since ϕ_1 maps to $\mathcal{S}^7$ it is clear that $\phi_1(\sigma)$ is orthogonal to $\frac{\partial \phi_1}{\partial s}(\sigma)$ and $\frac{\partial \phi_1}{\partial t}(\sigma)$. By (5.52) and the work above, there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$\frac{\partial \phi_1}{\partial t}(\sigma) = a_1 \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}(\sigma) + a_2\phi_2(\sigma) + a_3 \frac{\partial \phi_1}{\partial s}(\sigma). \tag{5.58}$$

We then calculate:

$$g_0\left(\frac{\partial \phi_1}{\partial t}^{\perp}(\sigma), \frac{\partial \phi_1}{\partial s}^{\perp}(\sigma) \right) = a_3 \left| \frac{\partial \phi_1}{\partial s}^{\perp}(\sigma) \right|^2.$$

The left-hand side is zero by (5.54) since (s,t) are oriented conformal coordinates, and hence $a_3 = 0$. Moreover, following a straightforward calculation,

$$\left|\frac{\partial\phi_1}{\partial t}^{\perp}(\sigma)\right|^2 = a_1^2\left|\phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}(\sigma)\right|^2 = a_1^2\left|\frac{\partial\phi_1}{\partial s}^{\perp}(\sigma)\right|^2$$

and thus $a_1^2 = 1$ by (5.54). Taking the inner product of (5.58) with the triple cross product gives:

$$\left|\phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}(\sigma)\right|^2 a_1 = g_0\left(\frac{\partial\phi_1}{\partial t}(\sigma), \phi_1\times\phi_2\times\frac{\partial\phi_1}{\partial s}(\sigma)\right) = \Phi_0\left(\phi_1(\sigma),\phi_2(\sigma),\frac{\partial\phi_1}{\partial s}(\sigma),\frac{\partial\phi_1}{\partial t}(\sigma)\right),$$

using equation (2.13). Therefore $a_1 > 0$ by equation (5.50). Hence $a_1 = 1$ and (5.55) holds at σ with $f(\sigma) = a_2$. If (5.55) holds at σ then, by Corollary 2.4.5, the 4-plane spanned by $\{\phi_1(\sigma),\phi_2(\sigma),\frac{\partial\phi_1}{\partial s}(\sigma),\frac{\partial\phi_1}{\partial t}(\sigma)\}$ is Cayley.

Similarly, we deduce that (5.44) holding at σ is equivalent to

$$\frac{\partial\phi_2}{\partial t} = \phi_1\times\phi_2\times\frac{\partial\phi_2}{\partial s} + f'\phi_1$$

at σ, for some function $f' : \Sigma\to\mathbb{R}$. However,

$$\frac{\partial}{\partial t}\, g_0(\phi_1,\phi_2) = g_0\left(\frac{\partial\phi_1}{\partial t},\phi_2\right) + g_0\left(\phi_1,\frac{\partial\phi_2}{\partial t}\right) = 0$$

and so $f' = -f$.

Let $(\mathbf{e}_1,\ldots,\mathbf{e}_8)$ be the standard orthonormal basis of $\mathbb{R}^8$, identified with $\mathbb{O}$ such that $\mathbf{e}_1$ corresponds to 1 and $(\mathbf{e}_2,\ldots,\mathbf{e}_8)$ corresponds to the basis of $\operatorname{Im}\mathbb{O}$ described in §2.1.1. It follows from [17, Theorem IV.1.38] that Spin(7) acts transitively upon oriented orthonormal bases of Cayley 4-planes. Hence, we can transform coordinates on $\mathbb{R}^8$ using Spin(7) such that a Cayley 4-plane has basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4)$. Moreover, any orthonormal pair can be mapped to $(\mathbf{e}_1,\mathbf{e}_2)$.

By the remarks above, transform coordinates on $\mathbb{R}^8$ using Spin(7) such that

$$\phi_1(\sigma) = \mathbf{e}_1, \quad \phi_2(\sigma) = \mathbf{e}_2, \quad \frac{\partial\phi_1}{\partial s}(\sigma) = b_1\mathbf{e}_1 + \ldots + b_8\mathbf{e}_8 \text{ and } \frac{\partial\phi_2}{\partial s}(\sigma) = b_1'\mathbf{e}_1 + \ldots + b_8'\mathbf{e}_8$$

for some real constants b_j and b_j'. If (5.55) and (5.56) hold, we may calculate $\frac{\partial\phi_1}{\partial t}(\sigma)$ and $\frac{\partial\phi_2}{\partial t}(\sigma)$. A straightforward calculation in coordinates then shows that (5.43)-(5.45) hold at σ. Since the triple cross product is invariant under Spin(7) by Definition 2.1.10, we conclude that (5.43)-(5.45) are equivalent to (5.55) and (5.56).

Suppose now that (5.46)-(5.48) hold at $\sigma\in\Sigma$. Using Spin(7), transform coordinates such that

$$\phi_1(\sigma) = \mathbf{e}_1, \qquad \phi_2(\sigma) = \mathbf{e}_2 \quad \text{and} \quad \frac{\partial\phi_1}{\partial s}(\sigma) = b_1\mathbf{e}_1 + \ldots + b_4\mathbf{e}_4,$$

where $b_1,\ldots,b_4$ are real constants, which we are free to do by (5.43). In these coordinates write

$$\frac{\partial\psi}{\partial s}(\sigma) = c_1\mathbf{e}_1 + \ldots + c_8\mathbf{e}_8 \quad \text{and} \quad \frac{\partial\psi}{\partial t}(\sigma) = d_1\mathbf{e}_1 + \ldots + d_8\mathbf{e}_8.$$

Calculating $\frac{\partial \phi_1}{\partial t}(\sigma)$ using (5.55), we then evaluate the terms in (5.47) as follows:

$$\begin{pmatrix} -b_4 & -b_3 \\ -b_3 & b_4 \end{pmatrix} \begin{pmatrix} d_5 + c_6 \\ d_6 - c_5 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} b_4 & b_3 \\ b_3 & -b_4 \end{pmatrix} \begin{pmatrix} d_7 + c_8 \\ d_8 - c_7 \end{pmatrix} = 0. \tag{5.59}$$

The values of the fourfold cross product required may be found in Proposition 2.4.6. The determinant of the matrices in (5.59) is $-b_3^2 - b_4^2 \neq 0$, since $\frac{\partial \phi_1}{\partial s}(\sigma) \notin \langle \phi_1(\sigma), \phi_2(\sigma) \rangle_{\mathbb{R}}$. Therefore

$$d_5 = -c_6, \qquad d_6 = c_5, \qquad d_7 = -c_8 \quad \text{and} \quad d_8 = c_7. \tag{5.60}$$

We may also evaluate (5.46):

$$c_5 d_8 + c_6 d_7 - c_7 d_6 - c_8 d_5 = 0; \qquad c_5 d_7 - c_6 d_8 - c_7 d_5 + c_8 d_6 = 0; \tag{5.61}$$

$$-c_3 d_8 + c_7 d_4 - c_4 d_7 + c_8 d_3 = 0; \qquad -c_3 d_7 + c_4 d_8 - c_8 d_4 + c_7 d_3 = 0; \tag{5.62}$$

$$c_3 d_6 + c_4 d_5 - c_5 d_4 - c_6 d_3 = 0; \text{ and} \qquad c_3 d_5 - c_4 d_6 - c_5 d_3 + c_6 d_4 = 0. \tag{5.63}$$

Again, calculation of the fourfold cross product may be found in Proposition 2.4.6. Substituting in (5.60), (5.61) are satisfied trivially and (5.62)-(5.63) become:

$$\begin{pmatrix} c_8 & c_7 \\ c_7 & -c_8 \end{pmatrix} \begin{pmatrix} d_3 + c_4 \\ d_4 - c_3 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} -c_6 & -c_5 \\ -c_5 & c_6 \end{pmatrix} \begin{pmatrix} d_3 + c_4 \\ d_4 - c_3 \end{pmatrix} = 0. \tag{5.64}$$

We deduce that the determinants of the matrices in (5.64) are zero, or the vector appearing in both equations is zero. Therefore,

(i) $d_3 = -c_4$ and $d_4 = c_3$

or

(ii) $c_5 = c_6 = c_7 = c_8 = 0$.

Condition (i) implies that (5.57) holds at σ with $g_1(\sigma) = d_1$ and $g_2(\sigma) = d_2$, by the definition of the triple cross product and its invariance under Spin(7). Condition (ii) corresponds to

$$\frac{\partial \psi}{\partial s}(\sigma), \frac{\partial \psi}{\partial t}(\sigma) \in \left\langle \phi_1(\sigma), \phi_2(\sigma), \frac{\partial \phi_j}{\partial s}(\sigma), \frac{\partial \phi_j}{\partial t}(\sigma) \right\rangle_{\mathbb{R}} \tag{5.65}$$

holding for $j = 1$. Thus, (5.46) and (5.47) are equivalent to (5.57) or (5.65) for $j = 1$ holding at σ. We similarly deduce that (5.46) and (5.48) are equivalent to (5.57) or (5.65) for $j = 2$.

We conclude that (5.43)-(5.48) are equivalent to (5.55), (5.56) and condition (5.57) or (5.65) for $j = 1, 2$ at each point $\sigma \in \Sigma$. Recall that Σ is connected and ϕ_1, ϕ_2, ψ and Σ are real analytic. Note that $\Sigma_1 = \{\sigma \in \Sigma : \dim V_\sigma = 4\}$ is an open subset of Σ whose complement is measure zero in Σ by hypothesis.

Let $\sigma \in \Sigma_1$ and suppose that (5.65) holds for $j = 1, 2$ at σ. Then there exist real constants C_{jk}, for $j = 1, 2$ and $1 \leq k \leq 4$, such that

$$\frac{\partial \psi}{\partial s}(\sigma) = C_{j1}\phi_1(\sigma) + C_{j2}\phi_2(\sigma) + C_{j3}\frac{\partial \phi_j}{\partial s}^{\perp}(\sigma) + C_{j4}\frac{\partial \phi_j}{\partial t}^{\perp}(\sigma).$$

Clearly, $C_{1k} = C_{2k}$ for $k = 1, 2$ by the definition of $g^{\perp}$ for a function g. Since $\dim V_\sigma = 4$ ensures the linear independence of the partial derivatives of ϕ_1 and ϕ_2, $C_{jk} = 0$ for $j = 1, 2$ and $k = 3, 4$. Hence, $\frac{\partial \psi}{\partial s}(\sigma)$ and, similarly, $\frac{\partial \psi}{\partial t}(\sigma)$ lie in $\langle \phi_1(\sigma), \phi_2(\sigma)\rangle_{\mathbb{R}}$ for almost all $\sigma \in \Sigma$. Thus ψ satisfies (5.57).

Consequently, (5.57) holds in Σ_1. Moreover, $\Sigma_2 = \{\sigma \in \Sigma : (5.57) \text{ holds at } \sigma\}$ is a closed real analytic subset of Σ and so must either coincide with Σ or be of zero measure in Σ. Since $\Sigma_1 \subseteq \Sigma_2$, Σ_2 cannot be measure zero and so must equal Σ. This completes the proof. $\square$

Note that (5.57) is a *linear* condition on ψ given ϕ_1 and ϕ_2, and that (5.55) and (5.56) are equivalent to the fact that the asymptotic cone M_0 of M is Cayley. Therefore, if we are given an r-framed 2-ruled Cayley cone M_0 defined by ϕ_1 and ϕ_2, any solution ψ of (5.57), together with ϕ_1 and ϕ_2, defines an r-framed 2-ruled Cayley 4-fold with asymptotic cone M_0. Moreover, (5.57) is unchanged if ϕ_1 and ϕ_2 are fixed and satisfy (5.55) and (5.56), but ψ is replaced by $\psi + \tilde{g}_1\phi_1 + \tilde{g}_2\phi_2$ for real analytic maps $\tilde{g}_1$ and $\tilde{g}_2$. We can thus locally transform ψ, if we relax the condition that ψ is orthogonal to ϕ_1 and ϕ_2, such that g_1 and g_2 are zero.

If we suppose instead that $\dim V_\sigma = 2$ for all $\sigma \in \Sigma$ then we are unable, in general, to define a suitable metric and hence oriented conformal coordinates on Σ. However, we shall show that if we exclude *planar* r-framed 2-ruled 4-folds, (5.55)-(5.57) of Theorem 5.4.3 characterize the Cayley condition on ϕ_1, ϕ_2 and ψ and there is a natural conformal structure on Σ.

5.4.2 Gauge transformations

Let ϕ_1 and ϕ_2 satisfy (5.55) and (5.56) in Theorem 5.4.3 for some map f. Taking the triple cross product of (5.55) and (5.56) with ϕ_1 and ϕ_2 gives:

$$\frac{\partial \phi_1}{\partial s} = -\phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial t} + f^{'}\phi_2 \quad \text{and} \quad \frac{\partial \phi_2}{\partial s} = -\phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial t} - f^{'}\phi_1, \tag{5.66}$$

for some function $f^{'} : \Sigma \to \mathbb{R}$ since $\phi_1 \times \phi_2 \times (\phi_1 \times \phi_2 \times v) = -v$ for any v orthogonal to ϕ_1 and ϕ_2.

We are allowed to perform a rotation $\Theta(\sigma)$ to the $(\phi_1(\sigma), \phi_2(\sigma))$-plane at each point $\sigma \in \Sigma$ as long as the function Θ is smooth. The choice of Θ will then alter f and $f^{'}$. We call such a transformation a *gauge transformation*.

We now show that under certain conditions there exists a gauge transformation such that $f =$

$f^{'} = 0$. Let $\Theta : \Sigma \to \mathbb{R}$ be a smooth function and define $\tilde{\phi}_1$ and $\tilde{\phi}_2$ by

$$\begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Then $\tilde{\phi}_1$ and $\tilde{\phi}_2$ satisfy (5.55) and (5.56) with f replaced by $\tilde{f} = f + \frac{\partial\Theta}{\partial t}$. Moreover, they satisfy (5.66) with f' replaced by $\tilde{f}' = f' + \frac{\partial\Theta}{\partial s}$. Therefore, locally, there exists a smooth function Θ such that $\tilde{f} = \tilde{f}' = 0$ if and only if $\frac{\partial f}{\partial s} = \frac{\partial f'}{\partial t}$.

If we differentiate (5.55) with respect to s and differentiate the first expression in (5.66) with respect to t we get

$$\frac{\partial^2\phi_1}{\partial s\partial t} = \phi_1 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_1}{\partial s} + \phi_1 \times \phi_2 \times \frac{\partial^2\phi_1}{\partial s^2} + \frac{\partial f}{\partial s}\phi_2 + f\frac{\partial\phi_2}{\partial s} \text{ and} \tag{5.67}$$

$$\frac{\partial^2\phi_1}{\partial t\partial s} = -\phi_1 \times \frac{\partial\phi_2}{\partial t} \times \frac{\partial\phi_1}{\partial t} - \phi_1 \times \phi_2 \times \frac{\partial^2\phi_1}{\partial t^2} + \frac{\partial f'}{\partial t}\phi_2 + f'\frac{\partial\phi_2}{\partial t}. \tag{5.68}$$

We must have that (5.67) and (5.68) are equal. In particular, the inner products of ϕ_2 with (5.67) and (5.68) must be equal. Note that, by (2.13),

$$g_0\left(\phi_2, \frac{\partial^2\phi_1}{\partial s\partial t}\right) = -\Phi_0\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial s}\right) + \frac{\partial f}{\partial s} \text{ and}$$

$$g_0\left(\phi_2, \frac{\partial^2\phi_1}{\partial t\partial s}\right) = \Phi_0\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial t}\right) + \frac{\partial f'}{\partial t},$$

and that

$$\Phi_0\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial s}\right) = g_0\left(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial s}\right) \text{ and } \Phi_0\left(\phi_1, \phi_2, \frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial t}\right) = -g_0\left(\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial t}\right). \tag{5.69}$$

However,

$$\Phi_0\left(\phi_1, \phi_2, \frac{\partial\phi_2}{\partial s}, \frac{\partial\phi_1}{\partial s}\right) = g_0\left(\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial t}\right).$$

Hence, since Φ_0 is alternating, $\frac{\partial f}{\partial s} = \frac{\partial f'}{\partial t}$ if and only if all the terms in (5.69) are zero.

We say that functions ϕ_1 and ϕ_2 satisfying (5.55), (5.56) and (5.66) with $f = f' = 0$ are in the *flat gauge*.

We now give a geometric interpretation of the flat gauge. Let (Σ, π) be a 2-ruling. Then, there is an $\mathcal{S}^1$ bundle $\pi_P : P \to \Sigma$ as described after Definition 5.4.1. An r-framing, which is equivalent to a choice of ϕ_1 and ϕ_2, gives a trivialization of P and we can consider it as a U(1) bundle. Define a connection ∇_P on P by a connection 1-form given by $d\theta - f'ds - fdt$, where θ corresponds to the U(1) direction. This connection is independent of the choice of r-framing, by the work above, and has curvature 2-form $\left(\frac{\partial f'}{\partial t} - \frac{\partial f}{\partial s}\right)ds \wedge dt$. Hence, the connection ∇_P defined by ϕ_1 and ϕ_2 is flat if and only if ϕ_1 and ϕ_2 can be put in the flat gauge by some gauge transformation locally.

5.4.3 Planar 2-ruled Cayley 4-folds

In this subsection we show that maps ϕ_1, ϕ_2, ψ which do not satisfy (5.55)-(5.57) for any local oriented coordinates (s,t) on Σ define a *planar* Cayley 4-fold.

The next result shows that (5.55)-(5.57) can be considered as *evolution equations* for ϕ_1, ϕ_2 and ψ. We recollect the definition, made at the start of §4.5.2, of a function being real analytic on a compact interval in $\mathbb{R}$.

Theorem 5.4.4. *Let I be a compact interval in $\mathbb{R}$, let s be a coordinate on I, let $\phi_1', \phi_2' : I \to \mathcal{S}^7$ be orthogonal real analytic maps and let $\psi' : I \to \mathbb{R}^8$ be real analytic. Let N be a neighbourhood of 0 in $\mathbb{R}$ and let $f : I \times N \to \mathbb{R}$ be a real analytic map. There exist $\epsilon > 0$ and unique real analytic maps $\phi_1, \phi_2 : I \times (-\epsilon, \epsilon) \to \mathcal{S}^7$, with ϕ_1, ϕ_2 orthogonal, and $\psi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^8$ satisfying $\phi_1(s,0) = \phi_1'(s)$, $\phi_2(s,0) = \phi_2'(s)$, $\psi(s,0) = \psi'(s)$ for all $s \in I$ and*

$$\frac{\partial\phi_1}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial\phi_1}{\partial s} + f\phi_2, \quad \frac{\partial\phi_2}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial\phi_2}{\partial s} - f\phi_1 \;\; \textit{and} \;\; \frac{\partial\psi}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial\psi}{\partial s}\,, \tag{5.70}$$

where t is a coordinate on $(-\epsilon, \epsilon)$ and the triple cross product is defined in (2.14)*. If M is given by*

$$M = \{r_1\phi_1(s,t) + r_2\phi_2(s,t) + \psi(s,t) : (r_1, r_2) \in \mathbb{R}^2,\ s \in I,\ t \in (-\epsilon, \epsilon)\},$$

it is an r-framed 2-ruled Cayley 4-fold in $\mathbb{R}^8$.

Proof. Since I is compact and $\phi_1', \phi_2', \psi', f$ are real analytic, we may apply the Cauchy–Kowalevsky Theorem (Theorem 1.1.4) to give unique functions $\phi_1, \phi_2, \psi : I \times (-\epsilon, \epsilon) \to \mathbb{R}^8$ satisfying the initial conditions and (5.70). We must now show that ϕ_1, ϕ_2 map to $\mathcal{S}^7$ and are orthogonal.

We first note that

$$\frac{\partial}{\partial t}\, g_0(\phi_1, \phi_2) = g_0\left(\frac{\partial\phi_1}{\partial t}\,, \phi_2\right) + g_0\left(\phi_1, \frac{\partial\phi_2}{\partial t}\right) = f\left(g_0(\phi_2, \phi_2) - g_0(\phi_1, \phi_1)\right) \text{ and}$$
$$\frac{\partial}{\partial t}\, g_0(\phi_1, \phi_1) = 2f g_0(\phi_1, \phi_2) = -\frac{\partial}{\partial t}\, g_0(\phi_2, \phi_2)$$

Then $g_0(\phi_j, \phi_k)$ for $j, k = 1, 2$ are real analytic functions satisfying this system of partial differential equations, together with the initial conditions

$$g_0(\phi_1, \phi_1) = g_0(\phi_2, \phi_2) = 1 \quad \text{and} \quad g_0(\phi_1, \phi_2) = \frac{\partial}{\partial t}\, g_0(\phi_j, \phi_k) = 0 \quad \text{at } t = 0$$

given by assumption. The functions $g_0(\phi_1, \phi_1) = g_0(\phi_2, \phi_2) \equiv 1$ and $g_0(\phi_1, \phi_2) \equiv 0$ also satisfy these equations and initial conditions. It therefore follows from Theorem 1.1.4 that these two solutions must be locally equal and hence, for $\epsilon > 0$ sufficiently small, $|\phi_1| = |\phi_2| = 1$ and ϕ_1 and ϕ_2 are orthogonal.

We conclude from Theorem 5.4.3 that M is an r-framed 2-ruled Cayley 4-fold. □

Let (Σ, π) and $(\tilde{\Sigma}, \tilde{\pi})$ be 2-rulings of a 4-fold in $\mathbb{R}^n$. We say that these 2-rulings are *distinct* if the families of affine 2-planes, $\mathcal{F}_\Sigma = \{\pi^{-1}(\sigma) : \sigma \in \Sigma\}$ and $\mathcal{F}_{\tilde{\Sigma}} = \{\tilde{\pi}^{-1}(\tilde{\sigma}) : \tilde{\sigma} \in \tilde{\Sigma}\}$, are different. If $\mathcal{F}^n$ is the family of all affine 2-planes in $\mathbb{R}^n$ we can consider (Σ, π) as a map from Σ to $\mathcal{F}^n$ given by $\sigma \mapsto \pi^{-1}(\sigma)$ with image $\mathcal{F}_\Sigma$.

We now give the result claimed at the start of the subsection, which is analogous to the results [27, Proposition 5.3] and Proposition 4.5.5 for 1-ruled SL and associative 3-folds respectively.

Proposition 5.4.5. *Any r-framed 2-ruled Cayley 4-fold (M, Σ, π) in $\mathbb{R}^8$ defined locally by maps ϕ_1, ϕ_2 and ψ which do not satisfy* (5.55)-(5.57) *for any local oriented coordinates (s, t) on Σ is locally isomorphic to an affine Cayley 4-plane in $\mathbb{R}^8$.*

Proof. We may take the 2-ruling (Σ, π) to be locally real analytic since M is real analytic by Theorem 5.1.1. Let $I = [0, 1]$ and let $\gamma : I \to \Sigma$ be a real analytic curve in Σ. If we set $\phi_1'(s) = \phi_1(\gamma(s))$, $\phi_2'(s) = \phi_2(\gamma(s))$ and $\psi'(s) = \psi(\gamma(s))$, then by Theorem 5.4.4 we construct $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\psi}$ defining an r-framed 2-ruled Cayley 4-fold $\tilde{M}$ satisfying (5.55)-(5.57) of Theorem 5.4.3. Thus M and $\tilde{M}$ coincide in the real analytic 3-fold $\pi^{-1}(\gamma(I))$ and hence, by Theorem 5.1.3, they are locally equal. Therefore, M locally admits a 2-ruling $(\tilde{\Sigma}, \tilde{\pi})$ satisfying (5.55)-(5.57) of Theorem 5.4.3.

The proof now follows in a similar manner to that of Proposition 4.5.5 so we omit the details. We can show that different real analytic curves near γ in Σ produce distinct 2-rulings. Hence, a one parameter family of such curves allows us to define, using the 2-rulings, a real analytic family of 2-planes through some point $p \in M$ whose total space is a real analytic 3-fold N in M. Moreover, each plane lies in $p + T_pM$, so $N \subseteq p + T_pM$. Thus M and $p + T_pM$ are locally equal by Theorem 5.1.3 and the result follows. $\square$

Note that in the proof of Theorem 5.4.3 the condition (5.57) on ψ was forced by the linear independence of the derivatives of ϕ_1 and ϕ_2. However, as we shall see in §5.4.6, non-planar 2-ruled 4-folds can be constructed when the derivatives of ϕ_1 and ϕ_2 are linearly dependent.

Proposition 5.4.5 tells us that for any *non-planar* 2-ruled Cayley 4-fold M defined by maps ϕ_1, ϕ_2 and ψ on Σ there exist locally oriented coordinates (s, t) on Σ such that (5.55)-(5.57) are satisfied. We shall see in the next subsection that there is therefore a natural conformal structure upon Σ, and (s, t) are oriented conformal coordinates with respect to this structure.

5.4.4 Main results

Our first theorem follows immediately from Theorem 5.4.3 and Proposition 5.4.5.

Theorem 5.4.6. *A non-planar, r-framed, 2-ruled 4-fold (M, Σ, π) in $\mathbb{R}^8$, defined by orthogonal real analytic maps $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ and a real analytic map $\psi : \Sigma \to \mathbb{R}^8$ as in* (5.41)*, is Cayley if and* *only if there exist locally oriented coordinates* (s,t) *on* Σ *such that* ϕ_1, ϕ_2 *and* ψ *satisfy* (5.55)-(5.57) *for some real analytic functions* $f, g_1, g_2 : \Sigma \to \mathbb{R}$.

We now prove the result claimed at the end of the last subsection.

Proposition 5.4.7. *Let* (M, Σ, π) *be a non-planar, r-framed, 2-ruled Cayley 4-fold in* $\mathbb{R}^8$. *There exists a unique conformal structure on* Σ *with respect to which* (s,t) *as given in Theorem 5.4.6 are oriented conformal coordinates.*

Proof. Let (s,t) be local oriented coordinates as given by Theorem 5.4.6. Define a complex structure J on Σ by requiring that $u = s + it$ is a holomorphic coordinate on Σ, i.e. that $\frac{\partial}{\partial t} = J\frac{\partial}{\partial s}$. Note that, by (5.55)-(5.56) and (5.66), ϕ_1 and ϕ_2 as given in Theorem 5.4.6 satisfy

$$\frac{\partial \phi_j}{\partial t}^{\perp} = \phi_1 \times \phi_2 \times \frac{\partial \phi_j}{\partial s}^{\perp} \quad \text{and} \quad \frac{\partial \phi_j}{\partial s}^{\perp} = -\phi_1 \times \phi_2 \times \frac{\partial \phi_j}{\partial t}^{\perp} \qquad \text{for } j = 1, 2. \tag{5.71}$$

Suppose that $(\tilde{s}, \tilde{t})$ are local oriented coordinates on Σ such that ϕ_1 and ϕ_2 also satisfy (5.55)-(5.56) in these coordinates. Hence, ϕ_1 and ϕ_2 satisfy (5.71) for the coordinates $(\tilde{s}, \tilde{t})$.

We calculate:

$$\frac{\partial \phi_j}{\partial \tilde{t}}^{\perp} = \frac{\partial s}{\partial \tilde{t}}\frac{\partial \phi_j}{\partial s}^{\perp} + \frac{\partial t}{\partial \tilde{t}}\frac{\partial \phi_j}{\partial t}^{\perp} = \phi_1 \times \phi_2 \times \left(\frac{\partial t}{\partial \tilde{t}}\frac{\partial \phi_j}{\partial s}^{\perp} - \frac{\partial s}{\partial \tilde{t}}\frac{\partial \phi_j}{\partial t}^{\perp}\right)$$

and

$$\frac{\partial \phi_j}{\partial \tilde{s}}^{\perp} = \frac{\partial s}{\partial \tilde{s}}\frac{\partial \phi_j}{\partial s}^{\perp} + \frac{\partial t}{\partial \tilde{s}}\frac{\partial \phi_j}{\partial t}^{\perp}.$$

Note that, from (5.71), $\frac{\partial \phi_j}{\partial t}^{\perp}$ is orthogonal to $\frac{\partial \phi_j}{\partial s}^{\perp}$ and, moreover, that $\frac{\partial \phi_j}{\partial t}^{\perp} \neq 0$ if and only if $\frac{\partial \phi_j}{\partial s}^{\perp} \neq 0$ by the definition of $f^{\perp}$ for a function $f : \Sigma \to \mathbb{R}$ and the properties of the triple cross product. Using (5.71) for $(\tilde{s}, \tilde{t})$ we deduce that

$$\frac{\partial s}{\partial \tilde{s}} = \frac{\partial t}{\partial \tilde{t}} \quad \text{and} \quad \frac{\partial s}{\partial \tilde{t}} = -\frac{\partial t}{\partial \tilde{s}}, \tag{5.72}$$

since not both $\frac{\partial \phi_1}{\partial s}^{\perp}$ and $\frac{\partial \phi_2}{\partial s}^{\perp}$ are zero.

Therefore, using (5.72),

$$\frac{\partial}{\partial \tilde{t}} = \frac{\partial s}{\partial \tilde{t}}\frac{\partial}{\partial s} + \frac{\partial t}{\partial \tilde{t}}\frac{\partial}{\partial t} = -\frac{\partial t}{\partial \tilde{s}}\frac{\partial}{\partial s} + \frac{\partial s}{\partial \tilde{s}}\frac{\partial}{\partial t} = J\left(\frac{\partial t}{\partial \tilde{s}}\frac{\partial}{\partial t} + \frac{\partial s}{\partial \tilde{s}}\frac{\partial}{\partial s}\right) = J\frac{\partial}{\partial \tilde{s}}.$$

Hence we have the result. □

It is clear that the conformal structure given by Proposition 5.4.7 coincides with the one given by the metric as described in the preamble to Theorem 5.4.3.

We now use Propositions 2.4.7 and 3.1.4 to prove analogous results for coassociative 4-folds in $\mathbb{R}^7$ and SL 4-folds in $\mathbb{C}^4$. We begin with the SL case, for which we recall the triple cross product on $\mathbb{C}^4$ described in Definition 3.1.8.

Theorem 5.4.8. *A non-planar, r-framed, 2-ruled 4-fold (M, Σ, π) in $\mathbb{C}^4 \cong \mathbb{R}^8$, defined by orthogonal real analytic maps $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ and a real analytic map $\psi : \Sigma \to \mathbb{R}^8$ as in (5.41), is SL if and only if $\omega_4(\phi_1, \phi_2) \equiv 0$ and there exist locally oriented coordinates (s,t) on Σ such that:*

$$\omega_4\left(\phi_j, \frac{\partial \phi_k}{\partial s}\right) \equiv 0 \quad \text{for } j,k = 1,2; \qquad \omega_4\left(\phi_j, \frac{\partial \psi}{\partial s}\right) \equiv 0 \quad \text{for } j = 1,2; \tag{5.73}$$

$$\frac{\partial \phi_1}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s} + f\phi_2; \tag{5.74}$$

$$\frac{\partial \phi_2}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s} - f\phi_1; \text{ and} \tag{5.75}$$

$$\frac{\partial \psi}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \psi}{\partial s} + g_1\phi_1 + g_2\phi_2, \tag{5.76}$$

where the triple cross product is defined by (3.5) and $f, g_1, g_2 : \Sigma \to \mathbb{R}$ are some real analytic functions.

It is worth making clear that (5.74)-(5.76) are not the same as (5.55)-(5.57) because of the different definitions of the triple cross product.

Proof. By Proposition 3.1.4, M is SL if and only if M is Cayley and $\omega_4|_M \equiv 0$. We thus conclude from Theorem 5.4.6 that M is SL if and only if ϕ_1, ϕ_2 and ψ satisfy (5.55)-(5.57) and $\omega_4|_{T_pM} \equiv 0$ for all $p \in M$. Therefore ω_4 vanishes on $\langle \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\rangle_{\mathbb{R}}$, where

$$\mathbf{x} = \phi_1(\sigma), \quad \mathbf{y} = \phi_2(\sigma), \quad \mathbf{z} = r_1\frac{\partial \phi_1}{\partial s}(\sigma) + r_2\frac{\partial \phi_2}{\partial s}(\sigma) + \frac{\partial \psi}{\partial s}(\sigma) \text{ and } \mathbf{w} = r_1\frac{\partial \phi_1}{\partial t}(\sigma) + r_2\frac{\partial \phi_2}{\partial t}(\sigma) + \frac{\partial \psi}{\partial t}(\sigma)$$

for all $(r_1, r_2) \in \mathbb{R}^2$ and $\sigma \in \Sigma$. Hence, the equations that must be satisfied are: $\omega_4(\phi_1, \phi_2) \equiv 0$;

$$\omega_4\left(\phi_j\,, \frac{\partial \phi_k}{\partial s}\right) = \omega_4\left(\phi_j\,, \frac{\partial \phi_k}{\partial t}\right) \equiv 0, \qquad \text{for } j,k = 1,2; \tag{5.77}$$

$$\omega_4\left(\phi_j\,, \frac{\partial \psi}{\partial s}\right) = \omega_4\left(\phi_j\,, \frac{\partial \psi}{\partial t}\right) \equiv 0, \qquad \text{for } j = 1,2; \tag{5.78}$$

$$\omega_4\left(\frac{\partial \phi_j}{\partial s}\,, \frac{\partial \phi_j}{\partial t}\right) = \omega_4\left(\frac{\partial \psi}{\partial s}\,, \frac{\partial \psi}{\partial t}\right) \equiv 0, \qquad \text{for } j = 1,2; \tag{5.79}$$

$$\omega_4\left(\frac{\partial \phi_1}{\partial s}\,, \frac{\partial \phi_2}{\partial t}\right) + \omega_4\left(\frac{\partial \phi_2}{\partial s}\,, \frac{\partial \phi_1}{\partial t}\right) \equiv 0; \qquad \text{and} \tag{5.80}$$

$$\omega_4\left(\frac{\partial \phi_j}{\partial s}\,, \frac{\partial \psi}{\partial t}\right) + \omega_4\left(\frac{\partial \psi}{\partial s}\,, \frac{\partial \phi_j}{\partial t}\right) \equiv 0, \qquad \text{for } j = 1,2. \tag{5.81}$$

However, if ϕ_1, ϕ_2 and ψ satisfy (5.77)-(5.81) and (5.55)-(5.57), they satisfy (5.74)-(5.76). Hence, it is enough to show that the conditions in the theorem force (5.77)-(5.81) to hold to prove the result.

If $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ are vectors in $\mathbb{C}^4$ such that ω_4 vanishes on $\langle \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\rangle_{\mathbb{R}}$, direct calculation in coordinates shows that

$$\omega_4(\mathbf{x} \times \mathbf{y} \times \mathbf{z}, \mathbf{w}) = \mathrm{Im}(\epsilon_{abcd}\mathbf{x}^a\mathbf{y}^b\mathbf{z}^c\mathbf{w}^d), \tag{5.82}$$

where ϵ_{abcd} is the permutation symbol. Noting that $\omega_4(\phi_1, \phi_2) \equiv 0$, that conditions (5.73) are satisfied, and the relationship between the triple cross products on $\mathbb{C}^4$ and $\mathbb{R}^8$, we see that (5.74)-(5.76) hold. Hence $\omega_4(\phi_j, \frac{\partial \phi_k}{\partial t}) = 0$ for all j, k using (5.74)-(5.75) and (5.82). Therefore (5.77) is satisfied. Moreover, (5.76) and (5.82) imply that (5.78) is satisfied. If we use (5.73), (5.74)-(5.76) and (5.82) again, we have that (5.79) holds.

Calculation using (5.73), (5.74)-(5.75) and (5.82) gives:

$$\begin{aligned}\omega_4\left(\frac{\partial \phi_1}{\partial s}, \frac{\partial \phi_2}{\partial t}\right) + \omega_4\left(\frac{\partial \phi_2}{\partial s}, \frac{\partial \phi_1}{\partial t}\right) &= \omega_4\left(\frac{\partial \phi_1}{\partial s}, \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s}\right) + \omega_4\left(\frac{\partial \phi_2}{\partial s}, \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s}\right) \\ &= \mathrm{Im}\left(\epsilon_{abcd} \frac{\partial \phi_1}{\partial s}^a \phi_1^b \phi_2^c \frac{\partial \phi_2}{\partial s}^d\right) + \mathrm{Im}\left(\epsilon_{abcd} \frac{\partial \phi_2}{\partial s}^a \phi_1^b \phi_2^c \frac{\partial \phi_1}{\partial s}^d\right) \\ &= \mathrm{Im}\left((\epsilon_{abcd} + \epsilon_{dbca}) \frac{\partial \phi_1}{\partial s}^a \phi_1^b \phi_2^c \frac{\partial \phi_2}{\partial s}^d\right) \equiv 0\end{aligned}$$

by the definition of the permutation symbol. Hence (5.80) is satisfied. An entirely similar argument using (5.73)-(5.76) and (5.82) gives that (5.81) is satisfied. □

For the coassociative case we recall the triple cross product on $\mathbb{R}^7$ given in Definition 2.3.3.

Theorem 5.4.9. *A non-planar, r-framed, 2-ruled 4-fold (M, Σ, π) in $\mathbb{R}^7$, defined by orthogonal real analytic maps $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^6$ and a real analytic map $\psi : \Sigma \to \mathbb{R}^7$ as in* (5.41), *is coassociative if and only if there exist locally oriented coordinates (s, t) on Σ such that:*

$$\varphi_0\left(\phi_1, \phi_2, \frac{\partial \phi_j}{\partial s}\right) \equiv 0 \quad \text{for } j = 1, 2; \qquad \varphi_0\left(\phi_1, \phi_2, \frac{\partial \psi}{\partial s}\right) \equiv 0; \tag{5.83}$$

$$\frac{\partial \phi_1}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_1}{\partial s} + f\phi_2; \tag{5.84}$$

$$\frac{\partial \phi_2}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_2}{\partial s} - f\phi_1; \text{ and} \tag{5.85}$$

$$\frac{\partial \psi}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \psi}{\partial s} + g_1\phi_1 + g_2\phi_2, \tag{5.86}$$

where the triple cross product is defined by (2.11) *and $f, g_1, g_2 : \Sigma \to \mathbb{R}$ are some real analytic functions.*

Proof. By Proposition 2.4.7, M is coassociative if and only if $\{0\} \times M \subseteq \mathbb{R} \oplus \mathbb{R}^7 \cong \mathbb{R}^8$ is Cayley. We may deduce from Theorem 5.4.6 that $\{0\} \times M$ is Cayley if and only if there exist locally coordinates (s, t) such that ϕ_1, ϕ_2 and ψ satisfy (5.55)-(5.57). We then note that (5.83) and the relationship between the triple cross products on $\mathbb{R}^7$ and $\mathbb{R}^8$ ensure that (5.84)-(5.86) are equivalent to (5.55)-(5.57). The result follows. □

5.4.5 Holomorphic vector fields

We give a means of constructing r-framed 2-ruled calibrated 4-folds from r-framed 2-ruled calibrated cones using *holomorphic vector fields*, which is analogous to [27, Theorem 6.1] and Proposition 4.5.7. Again, we take the term 'holomorphic vector field' to refer to the real part of a holomorphic vector field as in §4.5.3.

Suppose that M_0 is an r-framed, 2-ruled, Cayley cone in $\mathbb{R}^8$ defined by maps $\phi_1, \phi_2 : \Sigma \to \mathbb{R}^8$ as in (5.42). Proposition 5.4.7 gives us a conformal structure on Σ related to ϕ_1, ϕ_2 and hence we can consider Σ as a *Riemann surface.* Therefore Σ has a natural complex structure J and we may define oriented conformal coordinates (s, t) on Σ. Suppose further that ϕ_1 and ϕ_2 are in the flat gauge. Hence the equations ϕ_1 and ϕ_2 satisfy are:

$$\frac{\partial \phi_j}{\partial t} = \phi_1 \times \phi_2 \times \frac{\partial \phi_j}{\partial s} \quad \text{for } j = 1, 2 \quad \text{and} \quad \frac{\partial \phi_j}{\partial s} = -\phi_1 \times \phi_2 \times \frac{\partial \phi_j}{\partial t} \quad \text{for } j = 1, 2. \tag{5.87}$$

These equations indicate a correspondence between '$\phi_1 \times \phi_2 \times$' and the complex structure J on Σ.

Theorem 5.4.10. *Let M_0 be an r-framed, 2-ruled, Cayley cone in $\mathbb{R}^8$ defined by maps $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ in the flat gauge, where Σ is a Riemann surface. Let w be a holomorphic vector field on Σ and define a map $\psi : \Sigma \to \mathbb{R}^8$ by $\psi = \mathcal{L}_w \phi_1 + \mathcal{L}_{iw} \phi_2$, where $\mathcal{L}_w$ and $\mathcal{L}_{iw}$ denote the Lie derivatives with respect to w and iw respectively. If M is defined by ϕ_1, ϕ_2 and ψ as in* (5.41)*, it is an r-framed 2-ruled Cayley 4-fold in $\mathbb{R}^8$.*

Proof. We need to show ψ as defined satisfies (5.57). If w is identically zero, ψ trivially satisfies (5.57). Therefore we need only consider the case where w has isolated zeros. Since the condition for M to be Cayley is a closed condition on M, it is sufficient to prove that (5.57) holds at any point $\sigma \in \Sigma$ with $w(\sigma) \neq 0$.

Let $\sigma \in \Sigma$ be such a point. Then, since w is a holomorphic vector field, there exists an open set in Σ containing σ with oriented conformal coordinates (s, t) such that $w = \frac{\partial}{\partial s}$, so that $iw = \frac{\partial}{\partial t}$. Hence $\psi = \frac{\partial \phi_1}{\partial s} + \frac{\partial \phi_2}{\partial t}$ in a neighbourhood of σ.

Let $(\mathbf{e}_1, \ldots, \mathbf{e}_8)$ be an oriented orthonormal basis for $\mathbb{R}^8 \cong \mathbb{O}$ such that $\mathbf{e}_1$ corresponds to 1 and $(\mathbf{e}_2, \ldots, \mathbf{e}_8)$ corresponds to the basis of $\operatorname{Im} \mathbb{O}$ described in §2.1.1. Let $A = |\frac{\partial \phi_1}{\partial s}(\sigma)|$. We transform coordinates on $\mathbb{R}^8$ using $\mathrm{Spin}(7)$ such that

$$\phi_1(\sigma) = \mathbf{e}_1, \quad \phi_2(\sigma) = \mathbf{e}_2, \quad \frac{\partial \phi_1}{\partial s}(\sigma) = A\mathbf{e}_3 \ \text{ and } \ \frac{\partial \phi_2}{\partial s}(\sigma) = a_1\mathbf{e}_1 + \ldots + a_8\mathbf{e}_8,$$

for some real constants a_j. Clearly, by (5.87), $\frac{\partial \phi_1}{\partial t}(\sigma) = A\mathbf{e}_4$ and hence $a_1 = a_2 = a_4 = 0$ by the orthogonality conditions imposed on $\frac{\partial\phi_2}{\partial s}$ in the flat gauge. Differentiating (5.87) gives:

$$\frac{\partial^2\phi_1}{\partial s\partial t} = \phi_1 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_1}{\partial s} + \phi_1 \times \phi_2 \times \frac{\partial^2\phi_1}{\partial s^2} \text{ and}$$
$$\frac{\partial^2\phi_2}{\partial t^2} = \frac{\partial\phi_1}{\partial t} \times \phi_2 \times \frac{\partial\phi_2}{\partial s} + \phi_1 \times \frac{\partial\phi_2}{\partial t} \times \frac{\partial\phi_2}{\partial s} + \phi_1 \times \phi_2 \times \frac{\partial^2\phi_2}{\partial t\partial s}.$$

Therefore,

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial\phi_1}{\partial s} + \frac{\partial\phi_2}{\partial t}\right)\\ &= \phi_1 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_1}{\partial s} + \frac{\partial\phi_1}{\partial t} \times \phi_2 \times \frac{\partial\phi_2}{\partial s} + \phi_1 \times \frac{\partial\phi_2}{\partial t} \times \frac{\partial\phi_2}{\partial s} + \phi_1 \times \phi_2 \times \frac{\partial}{\partial s}\left(\frac{\partial\phi_1}{\partial s} + \frac{\partial\phi_2}{\partial t}\right).\end{aligned}$$

Calculation using (2.14) and (5.87) shows that

$$\phi_1 \times \frac{\partial\phi_2}{\partial s} \times \frac{\partial\phi_1}{\partial s}(\sigma) = A(a_7\mathbf{e}_5 - a_8\mathbf{e}_6 - a_5\mathbf{e}_7 + a_6\mathbf{e}_8),$$
$$\frac{\partial\phi_1}{\partial t} \times \phi_2 \times \frac{\partial\phi_2}{\partial s}(\sigma) = A(-a_3\mathbf{e}_1 - a_7\mathbf{e}_5 + a_8\mathbf{e}_6 + a_5\mathbf{e}_7 - a_6\mathbf{e}_8) \text{ and}$$
$$\phi_1 \times \frac{\partial\phi_2}{\partial t} \times \frac{\partial\phi_2}{\partial s}(\sigma) = -(a_3^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2)\mathbf{e}_2.$$

We conclude that (5.57) holds for ψ at σ with $g_1(\sigma) = -Aa_3 = -g_0\big(\frac{\partial\phi_1}{\partial s}(\sigma), \frac{\partial\phi_2}{\partial s}(\sigma)\big)$ and $g_2(\sigma) = -|\frac{\partial\phi_2}{\partial s}(\sigma)|^2$. By the invariance of g_0 and the triple cross product under Spin(7), and the discussion above, ψ satisfies (5.57) for some $g_1, g_2 : \Sigma \to \mathbb{R}$. Applying Theorem 5.4.6, the proof is complete. □

This result does not generalise to the SL case in the way we might expect. The construction starting with a 2-ruled SL cone M_0 will generally produce a 2-ruled Cayley, but not SL, 4-fold M. The fact that M is Cayley follows trivially from Theorem 5.4.10, but if we impose the condition $\omega_4|_M \equiv 0$, ϕ_1 and ϕ_2 must also satisfy

$$\omega_4\left(\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial s}\right) = -\omega_4\left(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial t}\right) = 0 \quad \text{and} \quad \omega_4\left(\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial t}\right) = \omega_4\left(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial s}\right) = 0$$

wherever $w \neq 0$. At such a point, either all the derivatives of ϕ_1 and ϕ_2 are zero or at least one is nonzero. In the first case both ϕ_1 and ϕ_2 are locally constant. Otherwise, suppose without loss of generality that $\frac{\partial\phi_1}{\partial s} \neq 0$ at a point σ such that $w(\sigma) \neq 0$. Then $\langle\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_1}{\partial t}\rangle_{\mathbb{C}} = \mathbb{C}^2$ and it is orthogonal to $\langle\phi_1, \phi_2\rangle_{\mathbb{C}} = \mathbb{C}^2$ since M is SL and ϕ_1, ϕ_2 are in the flat gauge. Therefore $\frac{\partial\phi_2}{\partial s} \in \langle\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_1}{\partial t}\rangle_{\mathbb{C}}$. Note that $g_4(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial s}) = \omega_4(\frac{\partial\phi_1}{\partial t}, \frac{\partial\phi_2}{\partial s}) = 0$ and $\omega_4(\frac{\partial\phi_1}{\partial s}, \frac{\partial\phi_2}{\partial s}) = 0$. Hence there exists $\theta \in \mathbb{R}$ such that $\cos\theta\frac{\partial\phi_1}{\partial s} + \sin\theta\frac{\partial\phi_2}{\partial s} = 0$. Using (5.87), $\cos\theta\frac{\partial\phi_1}{\partial t} + \sin\theta\frac{\partial\phi_2}{\partial t} = 0$. Therefore, $\cos\theta\phi_1 + \sin\theta\phi_2$ is constant on a neighbourhood of σ and thus on the component of Σ containing σ. We deduce the following.

Theorem 5.4.11. *Let M_0 be an r-framed, 2-ruled, SL cone in $\mathbb{C}^4 \cong \mathbb{R}^8$ defined by $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^7$ in the flat gauge, where Σ is a Riemann surface. Let w be a holomorphic vector field on Σ and define $\psi : \Sigma \to \mathbb{C}^4$ by $\psi = \mathcal{L}_w\phi_1 + \mathcal{L}_{iw}\phi_2$, where $\mathcal{L}_w$ and $\mathcal{L}_{iw}$ denote the Lie derivatives with respect* *to w and iw respectively. If M is defined by ϕ_1, ϕ_2 and ψ as in* (5.41)*, it is an r-framed 2-ruled SL 4-fold in $\mathbb{C}^4$ if and only if $w \equiv 0$ or there exists $\theta \in \mathbb{R}$ for each component K of Σ such that* $\cos\theta\phi_1 + \sin\theta\phi_2$ *is constant on K.*

We do, however, have a similar result to Theorem 5.4.10 for coassociative 4-folds.

Theorem 5.4.12. *Let M_0 be an r-framed, 2-ruled, coassociative cone in $\mathbb{R}^7$ defined by $\phi_1, \phi_2 : \Sigma \to \mathcal{S}^6$ in the flat gauge, where Σ is a Riemann surface. Let w be a holomorphic vector field on Σ and define $\psi : \Sigma \to \mathbb{R}^7$ by $\psi = \mathcal{L}_w\phi_1 + \mathcal{L}_{iw}\phi_2$, where $\mathcal{L}_w$ and $\mathcal{L}_{iw}$ denote the Lie derivatives with respect to w and iw respectively. If M is defined by ϕ_1, ϕ_2 and ψ as in* (5.41)*, it is an r-framed 2-ruled coassociative 4-fold in $\mathbb{R}^7$.*

Proof. This follows immediately from Theorem 5.4.10 since $\{0\} \times M \subseteq \mathbb{R} \oplus \mathbb{R}^7 \cong \mathbb{R}^8$ is Cayley and therefore coassociative by Proposition 2.4.7. □

5.4.6 Examples

We conclude this section and the chapter with three sets of examples of 2-ruled 4-folds.

U(1)-invariant 2-Ruled Cayley 4-folds

Note that the 4-folds constructed in Theorem 5.3.9 provide U(1)-invariant examples of r-framed 2-ruled Cayley 4-folds with, using the notation of that result, $\Sigma = \mathbb{R}^2$, $\phi_1(s,t) = Y(s)\mathbf{v}_1(t)$, $\phi_2(s,t) = Y(s)\mathbf{v}_2(t)$ and $\psi(s,t) = Y(s)\mathbf{w}(t)$. It is straightforward to check that ϕ_1 and ϕ_2 are in the flat gauge if $\Phi_0(\mathbf{v}_1(0), \mathbf{v}_2(0), X\mathbf{v}_1(0), X\mathbf{v}_2(0)) = 0$, so we may apply Theorem 5.4.10 to the cone defined when $\mathbf{w}(t) \equiv 0$ to give the following examples, noting that they are *not* U(1)-invariant in general.

Theorem 5.4.13. *Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions satisfying the Cauchy–Riemann equations. Use the notation of Definition 5.3.8 and Theorem 5.3.9. Suppose further that*

$$\Phi_0(\mathbf{v}_1(0), \mathbf{v}_2(0), X\mathbf{v}_1(0), X\mathbf{v}_2(0)) = 0.$$

The subset M of $\mathbb{C}^4 \cong \mathbb{R}^8$ defined by

$$\begin{aligned} M = \big\{ Y(s)\big((\lambda_1 + u(s,t)X)\mathbf{v}_1(t) + (\lambda_2 - v(s,t)X)\mathbf{v}_2(t) \\ + \mathbf{v}_1(t) \times \mathbf{v}_2(t) \times (v(s,t)X\mathbf{v}_1(t) + u(s,t)X\mathbf{v}_2(t))\big) : \lambda_1, \lambda_2, s, t \in \mathbb{R} \big\} \end{aligned}$$

is an r-framed 2-ruled Cayley 4-fold in $\mathbb{R}^8$.

Proof. Using the notation above,

$$\frac{\partial\phi_j}{\partial s} = Y(s)X\mathbf{v}_j \quad \text{and} \quad \frac{\partial\phi_j}{\partial t} = Y(s)(\mathbf{v}_1 \times \mathbf{v}_2 \times X\mathbf{v}_j)$$

for $j = 1, 2$. Further, we may write a holomorphic vector field w, and hence iw, on $\mathbb{R}^2$ as:

$$w = u(s,t)\frac{\partial}{\partial s} + v(s,t)\frac{\partial}{\partial t} \quad \text{and} \quad iw = -v(s,t)\frac{\partial}{\partial s} + u(s,t)\frac{\partial}{\partial t}\,.$$

The result follows from Theorem 5.4.10. $\square$

Now consider the family of SL 4-folds in $\mathbb{C}^4$ given in [17, Theorem III.3.1]. Let $\mathbf{c} = (c_1, c_2, c_3, c_4)$ in $\mathbb{R}^4$ be constant and define $M_{\mathbf{c}} \subseteq \mathbb{C}^4$ by:

$$M_{\mathbf{c}} = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \mathrm{Re}(z_1 z_2 z_3 z_4) = c_1 \text{ and } |z_1|^2 - |z_j|^2 = c_j \text{ for } j = 2, 3, 4\}.$$

Then $M_{\mathbf{c}}$ is an SL 4-fold in $\mathbb{C}^4$ invariant under $\mathrm{U}(1)^3$.

Taking $\mathbf{c} = 0$, M_0 is an r-framed 2-ruled SL cone in $\mathbb{C}^4$ with three different 2-rulings. For each of the distinct 2-rulings we apply the holomorphic vector field result of Theorem 5.4.11 to obtain families of r-framed 2-ruled Cayley 4-folds which are invariant under U(1).

Theorem 5.4.14. *Let $w : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then*

$$\begin{aligned} M_1 = \Big\{\tfrac{1}{2}\Big(& ie^{is}\left(re^{i\theta} + i\bar{w}(s+it)\right), e^{-is}\left(re^{i\theta} - i\bar{w}(s+it)\right), \\ & e^{it}\left(re^{-i\theta} + w(s+it)\right), e^{-it}\left(re^{-i\theta} - w(s+it)\right)\Big) : r, s, t, \theta \in \mathbb{R}\Big\}, \\ M_2 = \Big\{\tfrac{1}{2}\Big(& ie^{is}\left(re^{i\theta} + i\bar{w}(s+it)\right), e^{-it}\left(re^{-i\theta} - w(s+it)\right), \\ & e^{it}\left(re^{-i\theta} + w(s+it)\right), e^{-is}\left(re^{i\theta} - i\bar{w}(s+it)\right)\Big) : r, s, t, \theta \in \mathbb{R}\Big\} \text{ and} \\ M_3 = \Big\{\tfrac{1}{2}\Big(& ie^{is}\left(re^{i\theta} + i\bar{w}(s+it)\right), e^{it}\left(re^{-i\theta} + w(s+it)\right), \\ & e^{-is}\left(re^{i\theta} - i\bar{w}(s+it)\right), e^{-it}\left(re^{-i\theta} - w(s+it)\right)\Big) : r, s, t, \theta \in \mathbb{R}\Big\} \end{aligned}$$

are r-framed 2-ruled Cayley 4-folds in $\mathbb{R}^8 \cong \mathbb{C}^4$.

Proof. We only prove the result for M_1 as the proof for the other two is similar. In this example we define M_0 by functions $\phi_1, \phi_2 : \mathbb{R}^2 \to \mathcal{S}^7 \subseteq \mathbb{C}^4$ given by:

$$\phi_1(s,t) = \frac{1}{2}(ie^{is}, e^{-is}, e^{it}, e^{-it}) \quad \text{and} \quad \phi_2(s,t) = \frac{i}{2}(ie^{is}, e^{-is}, -e^{it}, -e^{-it}).$$

Thus, M_0 is 2-ruled by planes of the form:

$$\begin{aligned} \Pi_{r,\theta} &= \{r\cos\theta\,\phi_1(s,t) + r\sin\theta\,\phi_2(s,t) : s, t \in \mathbb{R}\} \\ &= \left\{\frac{r}{2}\left(ie^{i(\theta+s)}, e^{i(\theta-s)}, e^{-i(\theta-t)}, e^{-i(\theta+t)}\right) : s, t \in \mathbb{R}\right\}. \end{aligned}$$

We verify through direct calculation that $g_4(\phi_1, \phi_2) = \omega_4(\phi_1, \phi_2) = 0$, the first equation in (5.73) holds, (5.74)-(5.75) with $f = 0$ are satisfied, and that ϕ_1 and ϕ_2 are in the flat gauge.

Let $w(s+it) = u(s,t) + iv(s,t)$ for functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. Then $\psi = \mathcal{L}_w \phi_1 + \mathcal{L}_{iw} \phi_2$ is:

$$\begin{aligned}\psi(s,t) &= u(s,t)\frac{\partial \phi_1}{\partial s} + v(s,t)\frac{\partial \phi_1}{\partial t} - v(s,t)\frac{\partial \phi_2}{\partial s} + u(s,t)\frac{\partial \phi_2}{\partial t} \\ &= \frac{1}{2}\left(i\bar{w}(s+it)ie^{is}, -i\bar{w}(s+it)e^{-is}, w(s+it)e^{it}, -w(s+it)e^{-it}\right).\end{aligned}$$

We check that ψ satisfies (5.57) with $g_1 = 0$ and $g_2 = -\frac{1}{2}$, which agrees with the calculations in the proof of Theorem 5.4.10. Therefore, by Theorem 5.4.10, M_1 is an r-framed 2-ruled Cayley 4-fold. □

Note, from the proof above, that $2\omega_4(\phi_1, \frac{\partial \psi}{\partial s}) = v$ and $2\omega_4(\phi_1, \frac{\partial \psi}{\partial s}) = u$, so that (5.73) of Theorem 5.4.8 is satisfied if and only if $w \equiv 0$. Therefore, if w is not zero, Theorem 5.4.11 shows that M_1 is an r-framed 2-ruled Cayley 4-fold which is not SL. Similarly for M_2 and M_3.

An interesting special case is when w in Theorem 5.4.14 is taken to be constant. Here, calculation shows that, each M_j is invariant under a $\mathrm{U}(1)^2$ subgroup of $\mathrm{U}(1)^3$. Moreover, they are asymptotically conical to M_0 with rate -1, in the sense of Definition 1.2.3.

1-Ruled Associative and Special Lagrangian 3-folds

We can construct examples of 2-ruled 4-folds from 1-ruled associative 3-folds in $\mathbb{R}^7$ and SL 3-folds in $\mathbb{C}^3$, as described in §4.5 and [27] respectively. Recall the notation of Definition 4.5.1.

Suppose that (N, Σ, π) is a 1-ruled 3-fold. Let $M = \mathbb{R} \times N$ and let $\tilde{\pi} : M \to \Sigma$ be given by $\tilde{\pi}(r, p) = \pi(p)$ for all $p \in N$. Clearly, $(M, \Sigma, \tilde{\pi})$ is a 2-ruled 4-fold since $\tilde{\pi}^{-1}(\sigma) = \mathbb{R} \times \pi^{-1}(\sigma)$ for all $\sigma \in \Sigma$. Suppose further that (N, Σ, π) is r-oriented. Using the r-orientation, we have a natural choice of oriented orthonormal basis for the plane $\tilde{\pi}^{-1}(\sigma)$, which varies smoothly with σ. Therefore, $(M, \Sigma, \tilde{\pi})$ is r-framed.

We now state and prove the following theorem.

Theorem 5.4.15. *(a) If $N \subseteq \mathbb{R}^7$ is an (r-oriented) 1-ruled associative 3-fold, $\mathbb{R} \times N \subseteq \mathbb{R} \oplus \mathbb{R}^7 \cong \mathbb{R}^8$ is an (r-framed) 2-ruled Cayley 4-fold.*

(b) If $L \subseteq \mathbb{C}^3$ is an (r-oriented) 1-ruled SL 3-fold with phase $-i$, $\mathbb{R} \times L \subseteq \mathbb{R} \oplus \mathbb{C}^3 \cong \mathbb{R}^7$ is an (r-framed) 2-ruled coassociative 4-fold.

Proof. Let N be an associative 3-fold in $\mathbb{R}^7$. By Proposition 2.4.7, $\mathbb{R} \times N$ is Cayley. The comments before the theorem then give the result (a). Similarly, (b) follows from Proposition 3.1.3 and the comments above. □

Complex Cones

Define a *complex cone* C in $\mathbb{C}^4$ by:

$$C = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : P(z_1, z_2, z_3, z_4) = Q(z_1, z_2, z_3, z_4) = 0\},$$

where P and Q are homogeneous complex polynomials such that C is non-planar and nonsingular except at 0. Define a projection $\tilde{\pi}$ from $C \setminus 0$ to $\mathbb{CP}^3$ by $\tilde{\pi}((z_1, z_2, z_3, z_4)) = [(z_1, z_2, z_3, z_4)]$ and let Σ be the image of $\tilde{\pi}$. Let

$$M_0 = \{(z_1, z_2, z_3, z_4, \sigma) \in C \times \Sigma : (z_1, z_2, z_3, z_4) \in \sigma\}$$

and define $\iota : M_0 \to \mathbb{C}^4$ by $\iota(z_1, z_2, z_3, z_4, \sigma) = (z_1, z_2, z_3, z_4)$. Then ι is an immersion except at 0 and thus M_0 is an immersed submanifold of $\mathbb{C}^4$ which is only singular at 0. Let $\pi : M_0 \to \Sigma$ be given by $\pi(z_1, z_2, z_3, z_4, \sigma) = \sigma$. Clearly, M_0 is 2-ruled by complex lines $\pi^{-1}(\sigma)$ in $\mathbb{C}^4$. Since any complex surface in $\mathbb{C}^4 \cong \mathbb{R}^8$ is Cayley by [17, §IV.2.C], (M_0, Σ, π) is a 2-ruled Cayley 4-fold.

We can define a local holomorphic coordinate w, hence oriented conformal coordinates (s, t), in Σ by $w \mapsto [(z_1, z_2, z_3, z_4)](w)$ in some open set U in Σ. Suppose without loss of generality that $z_4 \neq 0$ in U. Then, we may rescale so that $z_4 = 1$ and define maps $\phi_1, \phi_2 : U \to \mathcal{S}^7$ by:

$$\phi_1(s,t) = \left(\frac{z_1(s,t)}{r}, \frac{z_2(s,t)}{r}, \frac{z_3(s,t)}{r}, \frac{1}{r}\right) \quad \text{and} \quad \phi_2(s,t) = i\phi_1(s,t),$$

where $r = (1 + |z_1(s,t)|^2 + |z_2(s,t)|^2 + |z_3(s,t)|^2)^{\frac{1}{2}}$. We can thus write M_0 locally in the form (5.42), where ϕ_1 and ϕ_2 satisfy (5.55)-(5.56), since C and hence M_0 is non-planar. If we define M by (5.41) where ψ satisfies (5.57) then, from Theorem 5.4.6, M is a non-planar, r-framed, 2-ruled Cayley 4-fold in $\mathbb{R}^8$.

Part II

Noncompact Coassociative Deformations

Abstractness, sometimes hurled as a reproach at mathematics, is its chief glory and its surest title to practical usefulness. It is also the source of such beauty as may spring from mathematics.

– Eric Temple Bell

Chapter 6

Analysis on Noncompact Riemannian Manifolds

Various analytic techniques may be employed on certain classes of noncompact Riemannian manifold. A key idea, presented in §6.2, is the introduction of *weighted* versions of Sobolev and Hölder spaces. This enables the study of Fredholm and index theory of *uniformly elliptic* operators between these Banach spaces in §6.3 and the description of elliptic *regularity* results in §6.4.

6.1 AC and CS Manifolds

We are interested in two types of noncompact Riemannian manifolds: asymptotically conical manifolds and manifolds with conical singularities.

Definition 6.1.1. Let (M, g) be a Riemannian manifold with $\dim M = n$. Then M is *asymptotically conical* (AC) (with rate λ) if there exist constants $R > 0$ and $\lambda < 1$, a compact $(n-1)$-dimensional Riemannian manifold (Σ, h), a compact set $K \subseteq M$ and a diffeomorphism $\Psi : (R, \infty) \times \Sigma \to M \setminus K$ such that

$$|\nabla^j(\Psi^*(g) - g_{\text{cone}})| = O(r^{\lambda-1-j}) \qquad \text{for } j \in \mathbb{N} \text{ as } r \to \infty, \tag{6.1}$$

where r is the coordinate on $(0, \infty)$ on the cone $C = (0, \infty) \times \Sigma$, $g_{\text{cone}} = dr^2 + r^2 h$ is the conical metric on C, ∇ is the Levi–Civita connection derived from g_{cone} and $|.|$ is calculated using g_{cone}. We call C the *asymptotic cone* of M and define the *ends* M_∞ of M to be the components of $M \setminus K$.

The choice of $\lambda < 1$ ensures that the metric on M converges to the conical metric on C at infinity by (6.1). It is also clear, from Definition 1.2.3, that a Riemannian submanifold of $\mathbb{R}^n$ which is AC with rate λ to a cone M_0 is an AC Riemannian manifold with rate λ, where $\Sigma = M_0 \cap \mathcal{S}^{n-1}$.

Definition 6.1.2. Let M be an AC manifold and use the notation of Definition 6.1.1. A *radius function* $\rho : M \to [1, \infty)$ on M is a smooth function such that there exist positive constants $c_1 < 1$ and $c_2 > 1$ with $c_1 r < \Psi^*(\rho) < c_2 r$ on $(R, \infty) \times \Sigma$.

If M is AC we may define a radius function $\rho : M \to [1, \infty)$ on it using r, by requiring that ρ is equal to r on $\Psi((R + 1, \infty) \times \Sigma)$ and then extending ρ smoothly to a function on M.

Definition 6.1.3. Let M be a connected Hausdorff topological space and let $z_1, \ldots, z_s \in M$. Suppose that $\hat{M} = M \setminus \{z_1, \ldots, z_s\}$ has the structure of a (nonsingular) n-dimensional Riemannian manifold, with Riemannian metric g, compatible with its topology. Then M is a *manifold with conical singularities* (at $z_1, \ldots, z_s$ with rate λ) if there exist constants $\epsilon > 0$ and $\lambda > 1$, a compact $(n-1)$-dimensional Riemannian manifold (Σ_i, h_i), an open set $U_i \ni z_i$ in M with $U_i \cap U_j = \emptyset$ for $j \neq i$ and a diffeomorphism $\Psi_i : (0, \epsilon) \times \Sigma_i \to U_i \setminus \{z_i\} \subseteq \hat{M}$, for $i = 1, \ldots, s$, such that

$$|\nabla_i^j (\Psi_i^*(g) - g_i)| = O(r_i^{\lambda - 1 - j}) \qquad \text{for } j \in \mathbb{N} \text{ as } r_i \to 0, \tag{6.2}$$

where r_i is the coordinate on $(0, \infty)$ on the cone $C_i = (0, \infty) \times \Sigma_i$, $g_i = dr_i^2 + r_i^2 h_i$ is the conical metric on C_i, ∇_i is the Levi–Civita connection derived from g_i and $|.|$ is calculated using g_i. We call C_i the *cone* at the singularity z_i and let the *ends* $\hat{M}_\infty$ of $\hat{M}$ be the disjoint union

$$\hat{M}_\infty = \bigsqcup_{i=1}^{s} U_i \setminus \{z_i\}.$$

We say that M is *CS* or a *CS manifold* (with rate λ) if it is a manifold with conical singularities which have rate λ and it is compact as a topological space. In these circumstances it may be written as the disjoint union

$$M = K \sqcup \bigsqcup_{i=1}^{s} U_i,$$

where K is compact as it is closed in M.

The condition $\lambda > 1$ guarantees that the metric on $\hat{M}$ genuinely converges to the conical metric on C_i, as is evident from (6.2). Since M is supposed to be Hausdorff, the set $U_i \setminus \{z_i\}$ is open in $\hat{M}$ for all i. Moreover, the condition that the U_i are disjoint may be easily satisfied since, if $i \neq j$, z_i and z_j may be separated by two disjoint open sets and, by hypothesis, there are only a finite number of singularities.

It is also important to note that $\hat{M}$ is a *noncompact* manifold.

Definition 6.1.4. Let M be a CS manifold. Using the notation of Definition 6.1.3, a *radius function* on $\hat{M}$ is a smooth function $\rho : \hat{M} \to (0, 1]$, bounded below by a positive constant on $\hat{M} \setminus \hat{M}_\infty$, such that there exist positive constants $c_1 < 1$ and $c_2 > 1$ with

$$c_1 r_i < \Psi_i^*(\rho) < c_2 r_i$$

on $(0,\epsilon)\times\Sigma_i$ for $i=1,\ldots,s$.

If M is CS we may construct a radius function on $\hat{M}$ as follows. Let $\rho(x)=1$ for all $x\in\hat{M}\setminus\hat{M}_\infty$. Define $\rho_i:\Psi_i((0,\epsilon/2)\times\Sigma_i)\to(0,1)$ to be equal to r_i for $i=1,\ldots,s$ and then define ρ by interpolating smoothly between its definition on $\hat{M}\setminus\hat{M}_\infty$ and ρ_i on each of the disjoint sets $\Psi_i((\epsilon/2,\epsilon)\times\Sigma_i)$.

For the purposes of this chapter only, we shall henceforth, when referring to a CS manifold, mean the nonsingular part $\hat{M}$ of M, in the notation of Definition 6.1.3.

6.2 Weighted Banach Spaces

We define *weighted* Banach spaces of forms as in [3, §1], as well as the usual 'unweighted' spaces.

Definition 6.2.1. Let (M,g) be a Riemannian n-fold, let $p\geq 1$ and let $k,m\in\mathbb{N}$ with $m\leq n$. The *Sobolev space* $L^p_k(\Lambda^m T^*M)$ is the set of m-forms ξ on M which are k times weakly differentiable and such that the norm

$$\|\xi\|_{L^p_k}=\left(\sum_{j=0}^{k}\int_M|\nabla^j\xi|^p\,dV_g\right)^{\frac{1}{p}}\tag{6.3}$$

is finite. The normed vector space $L^p_k(\Lambda^m T^*M)$ is a Banach space for all $p\geq 1$ and $L^2_k(\Lambda^m T^*M)$ is a Hilbert space.

We introduce the space of m-forms

$$L^p_{k,\,\mathrm{loc}}(\Lambda^m T^*M)=\{\xi\,:\,f\xi\in L^p_k(\Lambda^m T^*M)\text{ for all }f\in C^\infty_{\mathrm{cs}}(M)\}$$

where $C^\infty_{\mathrm{cs}}(M)$ is the space of smooth functions on M with compact support.

Suppose further that M is AC or CS. Let $\mu\in\mathbb{R}$ and let ρ be a radius function on M. The *weighted Sobolev space* $L^p_{k,\,\mu}(\Lambda^m T^*M)$ of m-forms ξ on M is the subspace of $L^p_{k,\,\mathrm{loc}}(\Lambda^m T^*M)$ such that the norm

$$\|\xi\|_{L^p_{k,\,\mu}}=\left(\sum_{j=0}^{k}\int_M|\rho^{j-\mu}\nabla^j\xi|^p\rho^{-n}\,dV_g\right)^{\frac{1}{p}}\tag{6.4}$$

is finite. Then $L^p_{k,\,\mu}(\Lambda^m T^*M)$ is a Banach space and $L^2_{k,\,\mu}(\Lambda^m T^*M)$ is a Hilbert space.

We may note here, trivially, that $L^p_0(\Lambda^m T^*M)$ is equal to the standard L^p-space of m-forms on M. Further, by comparing equations (6.3) and (6.4) for the respective norms, $L^p(\Lambda^m T^*M)=L^p_{0,\,-\frac{n}{p}}(\Lambda^m T^*M)$. In particular,

$$L^2(\Lambda^m T^*M)=L^2_{0,\,-\frac{n}{2}}(\Lambda^m T^*M).\tag{6.5}$$

For the following two definitions we take $C^k_{\mathrm{loc}}(\Lambda^m T^*M)$ to be the vector space of k times continuously differentiable m-forms.

Definition 6.2.2. Let (M,g) be an n-dimensional AC or CS manifold and let ρ be a radius function on M. Let $\mu \in \mathbb{R}$ and let $k,m \in \mathbb{N}$ with $m \le n$. The *weighted C^k-space* $C^k_\mu(\Lambda^m T^*M)$ of m-forms ξ on M is the subspace of $C^k_{\mathrm{loc}}(\Lambda^m T^*M)$ such that the norm

$$\|\xi\|_{C^k_\mu} = \sum_{j=0}^{k} \sup_M |\rho^{j-\mu} \nabla^j \xi|$$

is finite. We also define

$$C^\infty_\mu(\Lambda^m T^*M) = \bigcap_{k \ge 0} C^k_\mu(\Lambda^m T^*M).$$

Then $C^k_\mu(\Lambda^m T^*M)$ is a Banach space but in general $C^\infty_\mu(\Lambda^m T^*M)$ is not.

In the next definition we refer to the usual normed vector space $C^k(\Lambda^m T^*M)$ of k times continuously differentiable m-forms such that the following norm is finite:

$$\|\xi\|_{C^k} = \sum_{j=0}^{k} \sup_M |\nabla^j \xi|.$$

Definition 6.2.3. Let (M,g) be an n-dimensional AC or CS manifold, let $d(x,y)$ be the geodesic distance between points $x,y \in M$ and let ρ be a radius function on M. Let $a \in (0,1)$ and let $k,m \in \mathbb{N}$ with $m \le n$. Let

$$\begin{aligned} H = \{(x,y) \in M \times M \ : x \ne y,\ c_1\rho(x) \le \rho(y) \le c_2\rho(x) \text{ and} \\ \text{there exists a geodesic in } M \text{ of length } d(x,y) \text{ from } x \text{ to } y\}, \end{aligned}$$

where $0 < c_1 < 1 < c_2$ are constant. A section s of a vector bundle V on M is *Hölder continuous* (with *exponent* a) if

$$[s]^a = \sup_{(x,y)\in H} \frac{|s(x) - s(y)|_V}{d(x,y)^a} < \infty.$$

We understand the quantity $|s(x) - s(y)|_V$ as follows. Given $(x,y) \in H$, there exists a geodesic γ of length $d(x,y)$ connecting x and y. Parallel translation along γ using the connection on V identifies the fibres over x and y and the metrics on them. Thus, with this identification, $|s(x) - s(y)|_V$ is well-defined.

The *Hölder space* $C^{k,a}(\Lambda^m T^*M)$ is the set of $\xi \in C^k(\Lambda^m T^*M)$ such that $\nabla^k \xi$ is Hölder continuous (with exponent a) and the norm

$$\|\xi\|_{C^{k,a}} = \|\xi\|_{C^k} + [\nabla^k \xi]^a$$

is finite. The normed vector space $C^{k,a}(\Lambda^m T^*M)$ is a Banach space.

We also introduce the notation

$$C^{k,\,a}_{\mathrm{loc}}(\Lambda^m T^*M) = \{\xi \in C^k_{\mathrm{loc}}(\Lambda^m T^*M) : f\xi \in C^{k,\,a}(\Lambda^m T^*M) \text{ for all } f \in C^\infty_{\mathrm{cs}}(M)\}.$$

Let $\mu \in \mathbb{R}$. The *weighted Hölder space* $C^{k,\,a}_\mu(\Lambda^m T^*M)$ of m-forms ξ on M is the subspace of $C^{k,\,a}_{\mathrm{loc}}(\Lambda^m T^*M)$ such that the norm

$$\|\xi\|_{C^{k,\,a}_\mu} = \|\xi\|_{C^k_\mu} + [\xi]^{k,\,a}_\mu$$

is finite, where

$$[\xi]^{k,\,a}_\mu = [\rho^{k+a-\mu}\nabla^k\xi]^a.$$

Then $C^{k,\,a}_\mu(\Lambda^m T^*M)$ is a Banach space. It is clear that we have an embedding $C^{k,\,a}_\mu(\Lambda^m T^*M) \hookrightarrow C^l_\mu(\Lambda^m T^*M)$ whenever $l \leq k$.

We shall need the analogue of the Sobolev Embedding Theorem for weighted spaces, which is adapted from [37, Lemma 7.2] and [3, Theorem 1.2]. It is dependent on whether M is AC or CS.

Theorem 6.2.4 (**Weighted Sobolev Embedding Theorem**). *Let M be an n-dimensional AC (or CS) manifold. Let $p, q \geq 1$, $a \in (0,1)$, $\mu, \nu \in \mathbb{R}$ and $k, l, m \in \mathbb{N}$ with $m \leq n$.*

(a) *If $k \geq l$, $k - \frac{n}{p} \geq l - \frac{n}{q}$ and either*

  (i) *$p \leq q$ and $\mu \leq \nu$ if M is AC (or $\mu \geq \nu$ if M is CS) or*

  (ii) *$p > q$ and $\mu < \nu$ if M is AC (or $\mu > \nu$ if M is CS),*

  there is a continuous embedding $L^p_{k,\,\mu}(\Lambda^m T^*M) \hookrightarrow L^q_{l,\,\nu}(\Lambda^m T^*M)$.

(b) *If $k - \frac{n}{p} \geq l + a$, there is a continuous embedding $L^p_{k,\,\mu}(\Lambda^m T^*M) \hookrightarrow C^{l,\,a}_\mu(\Lambda^m T^*M)$.*

We shall also require an Implicit Function Theorem for Banach spaces, which follows immediately from [34, Chapter 6, Theorem 2.1].

Theorem 6.2.5 (**Implicit Function Theorem**). *Let X and Y be Banach spaces and let $W \subseteq X$ be an open neighbourhood of 0. Let $\mathcal{G} : W \to Y$ be a C^k map ($k \geq 1$) such that $\mathcal{G}(0) = 0$. Suppose further that $d\mathcal{G}|_0 : X \to Y$ is surjective with kernel K such that $X = K \oplus A$ for some closed subspace A of X. There exist open sets $V \subseteq K$ and $V' \subseteq A$, both containing 0, with $V \times V' \subseteq W$, and a unique C^k map $\mathcal{V} : V \to V'$ such that*

$$\operatorname{Ker}\mathcal{G} \cap (V \times V') = \{(x, \mathcal{V}(x)) \,:\, x \in V\}$$

in $X = K \oplus A$.

6.3 Uniformly Elliptic AC and CS Operators

We must begin by defining the operators that we shall be concerned with.

Definition 6.3.1. Let V and W be bundles of forms over an AC or CS manifold M, let $l \in \mathbb{N}$, let $\nu \in \mathbb{R}$ and let P be a linear differential operator of order l from V to W. Use the notation of Definitions 6.1.1 and 6.1.3. Transform coordinates (r, σ) on the ends of M to *cylindrical coordinates* (t, σ); that is, $r = Re^t$ in the AC case and $r = \epsilon e^{-t}$ in the CS case. Let $P^\nu = e^{\nu t} P$ be the operator on the cylindrical ends $(0, \infty) \times \Sigma$ of M, where we take $\Sigma = \sqcup_{i=1}^s \Sigma_i$ for the CS case. Let $\pi : (0, \infty) \times \Sigma \to \Sigma$ be the projection map and let V_Σ and W_Σ be the bundles of forms over Σ such that $V_\infty = \pi^*(V_\Sigma)$ and $W_\infty = \pi^*(W_\Sigma)$ are bundles over the ends corresponding to V and W respectively. Write

$$P^\nu \xi = \sum_{i=0}^{l} P_i^\nu \nabla^i \xi$$

for $\xi \in C_{\text{loc}}^l(V_\infty)$, where the P_i^ν are tensors taking values in $V_\infty^* \otimes W_\infty$ and ∇ is the Levi–Civita connection of the *cylindrical metric*. We say that P^ν is *asymptotically cylindrical* if there exists a linear differential operator P_∞ of order l from V_∞ to W_∞, which may be written as

$$P_\infty \xi = \sum_{i=0}^{l} P_{i,\infty} \nabla^i \xi,$$

such that $P_{i,\infty}$ is translation invariant on the ends of M in cylindrical coordinates and

$$|\nabla^j (P_i^\nu - P_{i,\infty})| \to 0 \qquad \text{for } j \in \mathbb{N} \text{ as } t \to \infty$$

for all i. If P^ν is asymptotically cylindrical then P is *AC* or *CS* with *rate* ν as appropriate.

Note that an AC or CS operator with rate ν reduces the growth rate of a form on the ends of M by ν. Examples of AC or CS operators abound: d and d^* are first order operators with rate 1 and the Laplacian is a second order operator with rate 2.

Definition 6.3.2. Use the notation of Definition 6.3.1. We say that an AC or CS operator P is *uniformly elliptic* if it is elliptic and P_∞ is elliptic.

We observe that the definition of uniformly elliptic above implies uniform ellipticity in the sense of global bounds on the coefficients of the symbol. The operators $d + d^*$ and the Laplacian are examples of uniformly elliptic operators.

6.3.1 Fredholm theory

We have a general Fredholm result adapted from [37, Theorem 1.1 & Theorem 6.1].

Theorem 6.3.3. *Let V and W be bundles of forms over an AC or CS manifold M, let $p \geq 1$, let $\mu, \nu \in \mathbb{R}$, let $k, l \in \mathbb{N}$ and let $P : L^p_{k+l,\,\mu}(V) \to L^p_{k,\,\mu-\nu}(W)$ be a uniformly elliptic AC or CS operator, respectively, of order l and rate ν. There exists a countable discrete set $\mathcal{D}(P) \subseteq \mathbb{R}$, depending only on P_∞ as in Definition 6.3.1, such that P is Fredholm if and only if $\mu \notin \mathcal{D}(P)$.*

We shall be primarily concerned with the uniformly elliptic map

$$d + d^* : L^p_{k+1,\,\mu}(\Lambda^2_+ T^*M \oplus \Lambda^4 T^*M) \to L^p_{k,\,\mu-1}(\Lambda^3 T^*M), \tag{6.6}$$

where M is 4-dimensional. We give an explicit description of the set $\mathcal{D}_{\mathrm{AC}}$ for which (6.6) is not Fredholm in the AC case, following [42, §6.1.2].

Recall the notation of Definition 6.1.1. Transform the asymptotically conical metric on M to a conformally equivalent asymptotically *cylindrical* metric; that is, if (t, σ) are coordinates on $(0, \infty) \times \Sigma$, the metric is asymptotic to $dt^2 + g_\Sigma$. With respect to this new metric, $d + d^*$ corresponds to

$$(d + d^*)_\infty = e^{-mt}(d + e^{2t} d^*) e^{mt}$$

acting on m-forms on M. If $\pi : (0, \infty) \times \Sigma \to \Sigma$ is the natural projection map, the action of $(d+d^*)_\infty$ on $\pi^*(\Lambda^2 T^*\Sigma) \oplus \pi^*(\Lambda^{\mathrm{odd}} T^*\Sigma)$ is:

$$(d + d^*)_\infty = \begin{pmatrix} d + d^* & -(\frac{\partial}{\partial t} + 3 - m) \\ \frac{\partial}{\partial t} + m & -(d + d^*) \end{pmatrix} \tag{6.7}$$

where m denotes the operator which multiplies m-forms by a factor m. However, we wish only to consider elements of $\Lambda^1 T^*\Sigma \oplus \Lambda^2 T^*\Sigma$ which correspond to self-dual 2-forms on M, so we define $V_\Sigma \subseteq \Lambda^2 T^*\Sigma \oplus \Lambda^{\mathrm{odd}} T^*\Sigma$ by

$$V_\Sigma = \{(\alpha, *\alpha + \beta) \,:\, \alpha \in \Lambda^2 T^*\Sigma,\, \beta \in \Lambda^3 T^*\Sigma\}.$$

Then $\pi^*(V_\Sigma)$ corresponds to $\Lambda^2_+ T^*M \oplus \Lambda^4 T^*M$.

For $w \in \mathbb{C}$ define a map $(d + d^*)_\infty(w)$ by:

$$(d + d^*)_\infty(w) = \begin{pmatrix} d + d^* & -(w + 3 - m) \\ w + m & -(d + d^*) \end{pmatrix} \tag{6.8}$$

acting on $V_\Sigma \otimes \mathbb{C}$. Notice that we have formally substituted w for $\frac{\partial}{\partial t}$ in (6.7).

Let

$$W_\Sigma = \{(*\alpha + \beta, \alpha) \,:\, \alpha \in \Lambda^2 T^*\Sigma,\, \beta \in \Lambda^3 T^*\Sigma\} \subseteq \Lambda^{\mathrm{odd}} T^*\Sigma \oplus \Lambda^2 T^*\Sigma.$$

Define $\mathcal{C} \subseteq \mathbb{C}$ as the set of w for which the map

$$(d + d^*)_\infty(w) : L^p_{k+1}(V_\Sigma \otimes \mathbb{C}) \to L^p_k(W_\Sigma \otimes \mathbb{C})$$

is *not* an isomorphism. By the proof of [37, Theorem 1.1], $\mathcal{D}_{\mathrm{AC}} = \{\operatorname{Re} w : w \in \mathcal{C}\}$. In fact, $\mathcal{C} \subseteq \mathbb{R}$ by [42, Lemma 6.1.13], which shows that the corresponding sets $\mathcal{C}(\Delta^m)$ are all real, where Δ^m is the Laplacian on m-forms. Hence $\mathcal{C} = \mathcal{D}_{\mathrm{AC}}$.

The symbol, hence the index ind_w, of $(d + d^*)_\infty(w)$ is independent of w. Furthermore, $(d + d^*)_\infty(w)$ is an isomorphism for generic values of w since $\mathcal{D}_{\mathrm{AC}}$ is countable and discrete. Therefore $\mathrm{ind}_w = 0$ for all $w \in \mathbb{C}$; that is,

$$\dim \mathrm{Ker}(d + d^*)_\infty(w) = \dim \mathrm{Coker}(d + d^*)_\infty(w),$$

so that (6.8) is not an isomorphism precisely when it is not injective.

The condition $(d + d^*)_\infty(w) = 0$, using (6.8), corresponds to the existence of $\alpha \in C^\infty(\Lambda^2 T^*\Sigma)$ and $\beta \in C^\infty(\Lambda^3 T^*\Sigma)$ satisfying

$$d\alpha = w\beta \quad \text{and} \quad d{*}\alpha + d^*\beta = (w + 2)\alpha. \tag{6.9}$$

We first note that (6.9) implies that

$$dd^*\beta = \Delta\beta = w(w + 2)\beta.$$

Since eigenvalues of the Laplacian on Σ must necessarily be positive, $\beta = 0$ if $w \in (-2, 0)$. If $w = 0$ and we take $\alpha = 0$, (6.9) forces β to be coclosed. As there are non-trivial coclosed 3-forms on Σ, $(d + d^*)_\infty(0)$ is not injective and hence $0 \in \mathcal{D}_{\mathrm{AC}}$.

Suppose that $w = -2$ lies in $\mathcal{D}_{\mathrm{AC}}$. Then (6.9) gives $[\beta] = 0$ in $H^3_{\mathrm{dR}}(\Sigma)$. We know that β is harmonic so, by Hodge theory, $\beta = 0$. Therefore $-2 \in \mathcal{D}_{\mathrm{AC}}$ if and only if there exists a nonzero closed and coclosed 2-form on Σ.

We state a proposition which follows from the work above noting that, with minor adjustment, the same arguments will apply to the CS case.

Proposition 6.3.4. *(a) Let M be a 4-dimensional AC manifold. Use the notation of Definition 6.1.1. Let $D(\mu) = \{(\alpha, \beta) \in C^\infty(\Lambda^2 T^*\Sigma \oplus \Lambda^3 T^*\Sigma) : d\alpha = \mu\beta,\ d{*}\alpha + d^*\beta = (\mu + 2)\alpha\}$. The set $\mathcal{D}_{\mathrm{AC}}$ of real numbers μ for which* (6.6) *is not Fredholm is given by:*

$$\mathcal{D}_{\mathrm{AC}} = \{\mu \in \mathbb{R} : D(\mu) \neq 0\}.$$

(b) Let M be a 4-dimensional CS manifold. Use the notation of Definition 6.1.3. For $i = 1, \ldots, s$ let $D(\mu, i) = \{(\alpha, \beta) \in C^\infty(\Lambda^2 T^*\Sigma_i \oplus \Lambda^3 T^*\Sigma_i) : d\alpha = \mu\beta,\ d{*}\alpha + d^*\beta = (\mu - 2)\alpha\}$. The set $\mathcal{D}_{\mathrm{CS}}$ of real numbers μ such that (6.6) *is not Fredholm is given by:*

$$\mathcal{D}_{\mathrm{CS}} = \bigcup_{i=1}^{s} \{\mu \in \mathbb{R} : D(\mu, i) \neq 0\}.$$

A perhaps more illuminating way to characterise $D(\mu)$ and $D(\mu,i)$ is by:

$$\begin{aligned}(\alpha,\beta)\in D(\mu) &\iff \xi=(r^{\mu+2}\alpha+r^{\mu+1}dr\wedge *\alpha, r^{\mu+3}dr\wedge\beta)\\ &\quad\text{is an } O(r^\mu) \text{ solution of } (d+d^*)\xi=0 \text{ on } C \quad \text{and}\\ (\alpha,\beta)\in D(\mu,i) &\iff \xi=(r^{\mu-2}\alpha+r^{\mu-1}dr\wedge *\alpha, r^{\mu-3}dr\wedge\beta)\\ &\quad\text{is an } O(r^\mu) \text{ solution of } (d+d^*)\xi=0 \text{ on } C_i,\end{aligned}$$

using the notation of Definitions 6.1.1 and 6.1.3.

6.3.2 Index theory

We begin with some definitions following [37].

Definition 6.3.5. Use the notation of Definition 6.3.1 and Theorem 6.3.3. Let $\mu\in\mathcal{D}(P)$. Define $\mathrm{d}(\mu)$ to be the dimension of the vector space of solutions of $P_\infty\xi=0$ of the form

$$\xi(t,\sigma)=e^{\mu t}p\,(t,\sigma)$$

where $p\,(t,\sigma)$ is a polynomial in $t\in(0,\infty)$ with coefficients in $C^\infty(V_\Sigma\otimes\mathbb{C})$. We also define, for $\lambda,\lambda'\notin\mathcal{D}(P)$ with $\lambda'\leq\lambda$, the quantity

$$N(\lambda,\lambda')=\sum_{\mu\in\mathcal{D}(P)\,\cap(\lambda',\,\lambda)}\mathrm{d}(\mu).$$

The next result is immediate from [37, Theorem 1.2].

Theorem 6.3.6. *Use the notation of Theorem 6.3.3. Let $\lambda,\lambda'\notin\mathcal{D}(P)$ with $\lambda'\leq\lambda$. For any $\mu\notin\mathcal{D}(P)$ let $\mathrm{ind}_\mu(P)$ denote the Fredholm index of $P:L^p_{k+l,\,\mu}(V)\to L^p_{k,\,\mu-\nu}(W)$. If M is AC,*

$$\mathrm{ind}_\lambda(P)-\mathrm{ind}_{\lambda'}(P)=N(\lambda,\lambda').$$

If M is CS,

$$\mathrm{ind}_{\lambda'}(P)-\mathrm{ind}_\lambda(P)=N(\lambda,\lambda').$$

We make a key observation, which shall be used on a number of occasions in Chapters 7 and 8.

Proposition 6.3.7. *Use the notation of Theorem 6.3.3. Let $\lambda,\lambda'\in\mathbb{R}$ such that $\lambda'\leq\lambda$ and $[\lambda',\lambda]\cap\mathcal{D}(P)=\emptyset$. The kernels, and cokernels, of $P:L^p_{k+l,\,\mu}(V)\to L^p_{k,\,\mu-\nu}(W)$ when $\mu=\lambda$ and $\mu=\lambda'$ are equal.*

Proof. Denote the dimensions of the kernel and cokernel of $P:L^p_{k+l,\,\mu}(V)\to L^p_{k,\,\mu-\nu}(W)$, for $\mu\notin\mathcal{D}(P)$, by $k(\mu)$ and $c(\mu)$ respectively. Since $[\lambda',\lambda]\cap\mathcal{D}(P)=\emptyset$, $k(\lambda)-c(\lambda)=k(\lambda')-c(\lambda')$ and hence

$$k(\lambda)-k(\lambda')=c(\lambda)-c(\lambda'). \tag{6.10}$$

Suppose that M is an n-dimensional AC manifold. We therefore know that $k(\lambda) \geq k(\lambda')$ because $L^p_{k+1,\,\lambda'} \hookrightarrow L^p_{k+1,\,\lambda}$ by Theorem 6.2.4(a) as $\lambda \geq \lambda'$. Similarly, since $c(\mu)$ is equal to the dimension of the kernel of the formal adjoint operator acting on a Sobolev space with weight $-n-(\mu-\nu)$, $c(\lambda) \leq c(\lambda')$. Noting that the left-hand side of (6.10) is non-negative and the right-hand side is less than or equal to zero, we conclude that both must be zero. The result for the AC case follows from the fact that the kernel of P in $L^p_{k+1,\,\lambda'}$ is contained in the kernel of P in $L^p_{k+1,\,\lambda}$, and vice versa for the cokernels.

If M is a CS manifold, we need only note that the direction of embeddings of Sobolev spaces is reversed from the AC scenario, so that the signs of each side of (6.10) are swapped. The same deductions as above may thus be made and the proposition is proved. $\square$

We now go further and give a more explicit description of the quantity $\mathrm{d}(\mu)$ in Definition 6.3.5 for the set $\mathcal{D}_{\mathrm{AC}}$ given in Proposition 6.3.4. We assume M is AC, but a similar argument will hold if M is CS.

Use the notation of Proposition 6.3.4 and the work preceding it and Definition 6.3.5. Let $p(t,\sigma)$ be a polynomial in t of degree m written as

$$p(t,\sigma) = \sum_{j=0}^{m} p_j t^j + \sum_{j=0}^{m} (*p_j + q_j)t^j,$$

where $p_j \in C^\infty(\Lambda^2 T^*\Sigma)$ and $q_j \in C^\infty(\Lambda^3 T^*\Sigma)$ for $j = 0,\dots,m$, with p_m and q_m not both zero, and let $\xi(t,\sigma) = e^{\mu t}p(t,\sigma)$. Using (6.7), $(d+d^*)_\infty \xi = 0$ is equivalent to

$$\sum_{j=0}^{m} t^j(dp_j - \mu q_j) - \sum_{j=0}^{m} jt^{j-1}q_j = 0 \quad \text{and} \quad \sum_{j=0}^{m} t^j((\mu+2)p_j - d{*}p_j - d^*q_j) + \sum_{j=0}^{m} jt^{j-1}p_j = 0. \tag{6.11}$$

Comparing coefficients of t^m we deduce that $(p_m, q_m) \in D(\mu)$. We know for $\mu \in \mathcal{D}_{\mathrm{AC}}$ that $D(\mu) \neq 0$, so we may take $m = 0$.

Suppose that $m \geq 1$. Comparing coefficients of t^{m-1} in (6.11):

$$dp_{m-1} - \mu q_{m-1} = mq_m \quad \text{and} \quad d{*}p_{m-1} + d^*q_{m-1} - (\mu+2)p_{m-1} = mp_m. \tag{6.12}$$

We then compute using (6.12):

$$\begin{aligned}
m\langle p_m, p_m\rangle_{L^2} &= \langle p_m, d{*}p_{m-1} + d^*q_{m-1} - (\mu+2)p_{m-1}\rangle_{L^2} \\
&= \langle d{*}p_m - (\mu+2)p_m, p_{m-1}\rangle_{L^2} + \langle dp_m, q_{m-1}\rangle_{L^2} \\
&= \langle -d^*q_m, p_{m-1}\rangle_{L^2} + \langle \mu q_m, q_{m-1}\rangle_{L^2} \\
&= -\langle q_m, dp_{m-1} - \mu q_{m-1}\rangle_{L^2} \\
&= -m\langle q_m, q_m\rangle_{L^2}.
\end{aligned}$$

Hence,

$$m(\|p_m\|_{L^2}^2 + \|q_m\|_{L^2}^2) = 0$$

and so $p_m = q_m = 0$, which is a contradiction.

We deduce the following.

Proposition 6.3.8. *Use the notation of Proposition 6.3.4 and Definition 6.3.5. If M is AC,* $\mathrm{d}(\mu) = \dim D(\mu)$ *for* $\mu \in \mathcal{D}_{\mathrm{AC}}$*. If M is CS,* $\mathrm{d}(\mu) = \sum_{i=1}^{s} \dim D(\mu, i)$ *for* $\mu \in \mathcal{D}_{\mathrm{CS}}$*.*

6.4 Elliptic Regularity

We now wish to discuss the *regularity* of solutions ξ to $P\xi = \eta$, where P is a uniformly elliptic AC or CS operator.

Theorem 6.4.1. *Suppose M is AC or CS and let V and W be bundles of forms on M. Let P be a smooth uniformly elliptic AC or CS operator as appropriate from V to W of order l and rate ν.*

(a) Let $p > 1$, let $k \in \mathbb{N}$ and let $\mu \in \mathbb{R}$. Suppose that $P\xi = \eta$ holds for $\xi \in L^1_{l,\,\mathrm{loc}}(V)$ and $\eta \in L^1_{0,\,\mathrm{loc}}(W)$. If $\xi \in L^p_{0,\,\mu}(V)$ and $\eta \in L^p_{k,\,\mu-\nu}(W)$ then $\xi \in L^p_{k+l,\,\mu}(V)$ and

$$\|\xi\|_{L^p_{k+l,\,\mu}} \leq c\left(\|\eta\|_{L^p_{k,\,\mu-\nu}} + \|\xi\|_{L^p_{0,\,\mu}}\right)$$

for some constant $c > 0$ independent of ξ and η.

(b) Let $a \in (0,1)$, let $k \in \mathbb{N}$ and let $\mu \in \mathbb{R}$. Suppose that $P\xi = \eta$ holds for $\xi \in C^l_{\mathrm{loc}}(V)$ and $\eta \in C^0_{\mathrm{loc}}(W)$. If $\xi \in C^0_\mu(V)$ and $\eta \in C^{k,\,a}_{\mu-\nu}(W)$ then $\xi \in C^{k+l,\,a}_\mu(V)$ and

$$\|\xi\|_{C^{k+l,\,a}_\mu} \leq c'\left(\|\eta\|_{C^{k,\,a}_{\mu-\nu}} + \|\xi\|_{C^0_\mu}\right)$$

for some constant $c' > 0$ independent of ξ and η. Moreover, these estimates hold if the coefficients of P only lie in $C^{k,\,a}_{\mathrm{loc}}$.

These results are given in [42, §6.1.1] for AC manifolds and can be easily adapted for the CS case.

We have a useful corollary concerning the kernel of uniformly elliptic operators.

Corollary 6.4.2. *In the notation of Theorem 6.4.1, if $\xi \in C^l_\mu(V)$ satisfies $P\xi = 0$ then $\xi \in C^\infty_\mu(V)$.*

This follows from Theorem 6.4.1(b), taking $\eta = 0$. In particular, if $\xi \in C^2_\mu(\Lambda^m T^*M)$ satisfies $\Delta\xi = 0$, it lies in $C^\infty_\mu(\Lambda^m T^*M)$.

Thus far we have only considered linear elliptic operators. In Chapters 7 and 8 we need to consider *nonlinear* ones. In general we do not have regularity results of the strength of Theorem 6.4.1 for nonlinear operators. However, we do have a result which follows from [46, Theorem 6.8.1].

Theorem 6.4.3. *Let M be AC or CS and let V be a bundle of forms on M. Let $P(\xi, \nabla\xi) = 0$ be a smooth nonlinear elliptic equation on $\xi \in C^1_{\mathrm{loc}}(V)$ and $\nabla\xi$. Then $\xi \in C^\infty(V)$.*

6.5 Hodge Theory

Definition 6.5.1. Let (M, g) be a Riemannian manifold. Define

$$\mathcal{H}^m = \{\xi \in L^2(\Lambda^m T^*M) \,:\, d\xi = d^*\xi = 0\}.$$

If M is compact, $\mathcal{H}^m$ is equal to the space of smooth harmonic m-forms and *Hodge's Theorem* shows that these forms uniquely represent cohomology classes in the mth *de Rham cohomology group*

$$H^m_{\mathrm{dR}}(M) = \frac{\mathrm{Ker}\,\{d : C^\infty(\Lambda^m T^*M) \to C^\infty(\Lambda^{m+1}T^*M)\}}{\mathrm{Image}\,\{d : C^\infty(\Lambda^{m-1}T^*M) \to C^\infty(\Lambda^m T^*M)\}}\,.$$

Recall that we may define the *compactly supported* cohomology groups by:

$$H^m_{\mathrm{cs}}(M) = \frac{\mathrm{Ker}\,\{d : C^\infty_{\mathrm{cs}}(\Lambda^m T^*M) \to C^\infty_{\mathrm{cs}}(\Lambda^{m+1}T^*M)\}}{\mathrm{Image}\,\{d : C^\infty_{\mathrm{cs}}(\Lambda^{m-1}T^*M) \to C^\infty_{\mathrm{cs}}(\Lambda^m T^*M)\}}\,.$$

Define $\jmath : H^m_{\mathrm{cs}}(M) \to H^m_{\mathrm{dR}}(M)$ by $\jmath([\xi]) = [\xi]$ for a closed compactly supported m-form ξ.

We now give a generalisation of Hodge's Theorem for AC and CS manifolds.

Theorem 6.5.2. *Let M be an n-dimensional AC or CS manifold.*

(a) If M is AC,

$$\mathcal{H}^m \cong \begin{cases} H^m_{\mathrm{dR}}(M) & \text{if } m > n/2, \\ \jmath(H^m_{\mathrm{cs}}(M)) & \text{if } m = n/2 \text{ and} \\ H^m_{\mathrm{cs}}(M) & \text{if } m < n/2. \end{cases}$$

(b) If M is CS,

$$\mathcal{H}^m \cong \begin{cases} H^m_{\mathrm{cs}}(M) & \text{if } m > n/2, \\ \jmath(H^m_{\mathrm{cs}}(M)) & \text{if } m = n/2 \text{ and} \\ H^m_{\mathrm{dR}}(M) & \text{if } m < n/2. \end{cases}$$

Moreover, the isomorphism when $m \geq n/2$ if M is AC or $m \leq n/2$ if M is CS is $\xi \mapsto [\xi]$.

Proof. Part (a) follows from [36, Example (0.15)] for $m \geq n/2$ and Poincaré duality for $m < n/2$. Similarly, (b) follows from [36, Example (0.16)] for $m \leq n/2$ and Poincaré duality for $m > n/2$. □

In the last two chapters we shall be particularly interested in closed L^2 2-forms on a 4-dimensional AC or CS manifold M which are self-dual. Such forms are automatically coclosed. Hence, we now consider $\mathcal{H}^2$ in this situation.

Example 6.5.3. Let M be a 4-dimensional AC or CS manifold. The Hodge star maps $\mathcal{H}^2$ into itself, so there is a splitting $\mathcal{H}^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$ where

$$\mathcal{H}^2_\pm = \mathcal{H}^2 \cap C^\infty(\Lambda^2_\pm T^*M).$$

Let $\mathcal{J} = \jmath\big(H^2_{\mathrm{cs}}(M)\big)$. If $\alpha, \beta \in \mathcal{J}$, there exist compactly supported closed 2-forms ξ and η such that $\alpha = [\xi]$ and $\beta = [\eta]$. We define a product on $\mathcal{J} \times \mathcal{J}$ by

$$\alpha \cup \beta = \int_M \xi \wedge \eta. \tag{6.13}$$

Suppose that ξ' and η' are also compactly supported with $\alpha = [\xi']$ and $\beta = [\eta']$. Then there exist 1-forms χ and ζ such that $\xi - \xi' = d\chi$ and $\eta - \eta' = d\zeta$. Therefore,

$$\begin{aligned}\int_M \xi' \wedge \eta' &= \int_M (\xi - d\chi) \wedge (\eta - d\zeta) = \int_M \xi \wedge \eta - d\chi \wedge \eta - \xi' \wedge d\zeta \\ &= \int_M \xi \wedge \eta - d(\chi \wedge \eta) - d(\xi' \wedge \zeta) = \int_M \xi \wedge \eta,\end{aligned}$$

as both $\chi \wedge \eta$ and $\xi' \wedge \zeta$ have compact support. The product (6.13) on $\mathcal{J} \times \mathcal{J}$ is thus well-defined and is a symmetric topological product with a signature (a, b). By Theorem 6.5.2, $\dim \mathcal{H}^2_+ = a$ and hence is a topological number.

Chapter 7

Deformation Theory of Asymptotically Conical Coassociative 4-folds

In this chapter we study *deformations* of coassociative 4-folds N in $\mathbb{R}^7$ which are *asymptotically conical* (AC) with rate λ to some fixed cone C. We formulate a local description of the moduli space, in §7.1, as the kernel of a nonlinear first order differential operator F. Consideration of the elliptic map $d+d^*$, related to the linearisation of F, allows us to prove, in §7.2, that the deformation theory is unobstructed for $\lambda > -2$. Section 7.3 contains the main result: when λ takes generic values in $(-2, 1)$, N admits a *smooth* moduli space $\mathcal{M}(N, \lambda)$ of coassociative deformations of N which are AC to C with rate λ. Moreover, the theory of Chapter 6 provides us with the tools necessary to calculate the dimension of $\mathcal{M}(N, \lambda)$ in Section 7.4. The treatment by Marshall [42] of the deformation theory of AC special Lagrangian m-folds inspires the material presented here, though these SL deformations were in fact studied earlier by Pacini [47] using different methods.

Throughout the chapter we shall use a common notation. Let $N \subseteq \mathbb{R}^7$ be a coassociative 4-fold which is AC to a cone $C \subseteq \mathbb{R}^7$ with rate λ in the sense of Definition 1.2.3. Let $C \cong (0, \infty) \times \Sigma$, where $\Sigma = C \cap \mathcal{S}^6$, with coordinates (r, σ). Using the notation of Definition 1.2.3, we have a smooth map $\Psi : (R, \infty) \times \Sigma \to N \setminus K$, for some $R > 0$ and compact subset K of N, satisfying (1.1) and an inclusion map $\iota : (0, \infty) \times \Sigma \to \mathbb{R}^7$ given by $\iota(r, \sigma) = r\sigma$. Note that Ψ satisfies (6.1) and thus N, considered as a Riemannian manifold, is AC with rate λ as in Definition 6.1.1. We therefore have a radius function $\rho : N \to [1, \infty)$ on N as in Definition 6.1.2.

7.1 The Deformation Map

7.1.1 Preliminaries

We wish to discuss *deformations* of N; that is, coassociative submanifolds that are 'near' to N in $\mathbb{R}^7$. We define this formally.

Definition 7.1.1. The *moduli space of deformations* $\mathcal{M}(N,\lambda)$ is the set of coassociative 4-folds $N' \subseteq \mathbb{R}^7$ which are AC to C with rate λ such that there exists a diffeomorphism $h : N \to N'$ isotopic to the identity.

The first result we need is immediate from the proof of [33, Chapter IV, Theorem 9].

Theorem 7.1.2. *Let P be a closed embedded submanifold of a Riemannian manifold M. There exist an open subset V of the normal bundle $\nu(P)$ of P in M, containing the zero section, and an open set S in M containing P, such that the exponential map $\exp|_V : V \to S$ is a diffeomorphism.*

The proof of this result relies entirely on the observation that $\exp|_{\nu(P)}$ is a local isomorphism upon the zero section. This information provides us with a useful corollary.

Corollary 7.1.3. *Choose $\Psi : (R,\infty)\times\Sigma \to N \setminus K \subseteq \mathbb{R}^7$ uniquely by imposing the condition that $\Psi(r,\sigma)-\iota(r,\sigma) \in (T_{r\sigma}C)^{\perp}$ for all $(r,\sigma)\in(R,\infty)\times\Sigma$, which can be achieved by making R and K larger if necessary. Let $P=\iota((R,\infty)\times\Sigma)$, $Q = N\setminus K$ and define $n_P : \nu(P)\to\mathbb{R}^7$ by $n_P(r\sigma,v) = v+\Psi(r,\sigma)$. There exist an open subset V of $\nu(P)$ in $\mathbb{R}^7$, containing the zero section, and an open set S in $\mathbb{R}^7$ containing Q, such that $n_P|_V : V \to S$ is a diffeomorphism. Moreover, V and S can be chosen to grow like r on $(R,\infty)\times\Sigma$ and such that $P\subseteq S$.*

Proof. Note that n_P takes the zero section of $\nu(P)$ to Q. By the definition of Ψ, n_P is a local isomorphism upon the zero section. Thus, the proof of Theorem 7.1.2 gives open sets V and S such that $n_P|_V : V\to S$ is a diffeomorphism.

Since $\Psi-\iota$ is orthogonal to $(R,\infty)\times\Sigma$, it can be identified with a small section of the normal bundle. Hence P lies in S as long as S grows with order $O(r)$ as $r\to\infty$. As we can form S and V in a translation equivariant way because we are working on a portion of the cone C, we can construct our sets with this growth rate at infinity and such that they do not collapse near R. $\square$

Recall that, by Proposition 1.2.4, C is coassociative. Therefore, since $\Psi(r,\sigma)-\iota(r,\sigma)$ lies in $(T_{r\sigma}C)^{\perp}\cong\nu_{r\sigma}(C)$ for $r>R$, $\Psi-\iota$ can be identified with the graph of an element α_C of $\Lambda^2_+T^*C$ on $(R,\infty)\times\Sigma$ by Proposition 2.3.14. Then

$$|\nabla^j\alpha_C| = O(r^{\lambda-j}) \quad \text{for } j\in\mathbb{N} \text{ as } r\to\infty, \tag{7.1}$$

since N is AC to C with rate λ, where ∇ and $|.|$ are calculated using the conical metric. Thus, α_C lies in $C^\infty_\lambda(\Lambda^2_+T^*C)$. Moreover, we have a decomposition:

$$\mathbb{R}^7 = T_{\Psi(r,\sigma)}N \oplus \nu_{r\sigma}(C)$$

at $\Psi(r,\sigma)$ for large r. We can therefore identify $\nu_{\Psi(r,\sigma)}(N)$ with $\nu_{r\sigma}(C)$ and hence we may identify $\Lambda^2_+T^*N$ and $\Lambda^2_+T^*C$ near infinity. Formally, we have the following.

Proposition 7.1.4. *Use the notation of Corollary 7.1.3 and let $\jmath_C$ and $\jmath_N$ be the isomorphisms given by Proposition 2.3.14 applied to C and N respectively. There exists a diffeomorphism $\Upsilon : \nu(P) \to \nu(Q)$, with $\Upsilon(0) = 0$, and hence a diffeomorphism $\tilde{\Upsilon} : \Lambda^2_+T^*P \to \Lambda^2_+T^*Q$ given by $\tilde{\Upsilon} = \jmath_N \circ \Upsilon \circ \jmath_C^{-1}$.*

Proposition 7.1.5. *Use the notation of Corollary 7.1.3 and Proposition 7.1.4. There exists an open set $U \subseteq \Lambda^2_+T^*N$ containing the zero section and $W = (\jmath_N \circ \Upsilon)(V)$, a tubular neighbourhood T of N in $\mathbb{R}^7$ containing S, and a diffeomorphism $\delta : U \to T$, affine on the fibres, that takes the zero section of $\Lambda^2_+T^*N$ to N and such that the following diagram commutes:*

$$\begin{tikzcd} W \arrow[r, "\tilde{\Upsilon}^{-1}"] \arrow[d, "\delta"'] & \jmath_C(V) \arrow[d, "\jmath_C^{-1}"] \\ S & V. \arrow[l, "n_P"'] \end{tikzcd} \tag{7.2}$$

Moreover, we may choose both T and U to grow with order $O(\rho)$ as $\rho \to \infty$.

Proof. Define the diffeomorphism $\delta|_W : W \to S$ by (7.2). Interpolating smoothly over the compact set K, we extend S to T, W to U and $\delta|_W$ to δ as required. Furthermore, by Corollary 7.1.3, V and S can be chosen to grow at order $O(r)$ as $r \to \infty$, and hence U and T can be chosen to have the growth rate at infinity as claimed. □

7.1.2 The map F and the associated map G

We introduce the notation $C^k_\lambda(U) = \{\alpha \in C^k_\lambda(\Lambda^2_+T^*N) : \alpha \in U\}$, where U is given by Proposition 7.1.5. The fact that U grows with order $O(\rho)$ as $\rho \to \infty$ ensures that $C^k_\lambda(U)$ is an open subset of $C^k_\lambda(\Lambda^2_+T^*N)$, since $\lambda < 1$. We use similar conventions to define subsets of the Banach spaces discussed in §6.2. We may now describe our *deformation map*.

Definition 7.1.6. Use the notation of Proposition 7.1.5. For $\alpha \in C^1_{\text{loc}}(U)$ let $\pi_\alpha : N \to \Gamma_\alpha \subseteq U$, where Γ_α is the graph of α in $\Lambda^2_+T^*N$, be given by $\pi_\alpha(x) = (x, \alpha(x))$. Let $f_\alpha = \delta \circ \pi_\alpha$ and let $N_\alpha = \delta(\Gamma_\alpha) = f_\alpha(N)$, so that N_α is the deformation of N corresponding to α. Then $f_\alpha^*(\varphi_0|_{N_\alpha}) \in C^0_{\text{loc}}(\Lambda^3T^*N)$, so we define $F : C^1_{\text{loc}}(U) \to C^0_{\text{loc}}(\Lambda^3T^*N)$ by

$$F(\alpha) = f_\alpha^*(\varphi_0|_{N_\alpha}).$$

By Proposition 2.3.6, the space of coassociative deformations of N corresponds to $\operatorname{Ker} F$.

However, we wish only to consider coassociative deformations N_α of N, for $\alpha \in C^1_{\mathrm{loc}}(U)$, which are AC to C with rate λ. If N_α is such a deformation, there exists a diffeomorphism $\Psi_\alpha : (R,\infty)\times\Sigma \to N_\alpha \setminus K_\alpha$, where K_α is a compact subset of N_α. We may define Ψ_α such that $\Psi_\alpha(r,\sigma) - \iota(r,\sigma)$ is orthogonal to $T_{r\sigma}C$ for all $\sigma \in \Sigma$ and $r > R$.

Before Proposition 7.1.4, using the notation there, we showed that $\Psi - \iota$ can be identified with the graph of $\alpha_C \in C^\infty_\lambda(\Lambda^2_+T^*P)$ and therefore with the graph of $\tilde{\Upsilon}(\alpha_C) \in C^\infty_\lambda(\Lambda^2_+T^*Q)$. Similarly, $\Psi_\alpha - \iota$ can be identified with the graph of $\alpha + \tilde{\Upsilon}(\alpha_C) \in C^\infty_\lambda(\Lambda^2_+T^*Q)$. Hence, $\alpha \in C^\infty_\lambda(U)$.

We conclude that N_α is AC to C with rate λ if and only if $\alpha \in C^\infty_\lambda(U) \subseteq C^\infty_\lambda(\Lambda^2_+T^*N)$.

Proposition 7.1.7. *Using the notation of Definitions 7.1.1 and 7.1.6, $\mathcal{M}(N,\lambda)$ is locally homeomorphic to the kernel of $F : C^\infty_\lambda(U) \to C^\infty(\Lambda^3T^*N)$.*

This is immediate from our work above.

Definition 7.1.8. Use the notation of Definition 7.1.6. Note that $dF|_0(\alpha) = d\alpha$ by [45, p. 731]. Define $G : C^1_{\mathrm{loc}}(U) \times C^1_{\mathrm{loc}}(\Lambda^4T^*N) \to C^0_{\mathrm{loc}}(\Lambda^3T^*N)$ by:

$$G(\alpha,\beta) = F(\alpha) + d^*\beta.$$

Then G is a first order elliptic operator at $(0,0)$ since

$$dG|_{(0,0)} = d + d^* : C^1_{\mathrm{loc}}(\Lambda^2_+T^*N \oplus \Lambda^4T^*N) \longrightarrow C^0_{\mathrm{loc}}(\Lambda^3T^*N).$$

It is clear that $d\alpha + d^*\beta$ lies in $C^\infty_{\lambda-1}(\Lambda^3T^*N)$ if $(\alpha,\beta) \in C^\infty_\lambda(U) \times C^\infty_\lambda(\Lambda^4T^*N)$, but showing that G maps C^∞_λ into $C^\infty_{\lambda-1}$ requires the following proposition.

Proposition 7.1.9. *The map F given in Definition 7.1.6 can be written as*

$$F(\alpha)(x) = d\alpha(x) + P_F(x,\alpha(x),\nabla\alpha(x)) \tag{7.3}$$

for $x \in N$, where $P_F : \{(x,y,z) : (x,y) \in U,\ z \in T^*_xN \otimes \Lambda^2_+T^*_xN)\} \to \Lambda^3T^*N$ is a smooth map such that $P_F(x,y,z) \in \Lambda^3T^*_xN$. For $\alpha \in C^\infty_\lambda(U)$ with $\|\alpha\|_{C^1_1}$ sufficiently small, denoting $P_F(x,\alpha(x),\nabla\alpha(x))$ by $P_F(\alpha)(x)$, $P_F(\alpha) \in C^\infty_{2\lambda-2}(\Lambda^3T^*N) \subseteq C^\infty_{\lambda-1}(\Lambda^3T^*N)$, as $\lambda < 1$. Moreover, for each $k \in \mathbb{N}$, if $\alpha \in C^{k+1}_\lambda(U)$ and $\|\alpha\|_{C^1_1}$ is sufficiently small, $P_F(\alpha) \in C^k_{2\lambda-2}(\Lambda^3T^*N)$ and there exists a constant $c_k > 0$ such that

$$\|P_F(\alpha)\|_{C^k_{2\lambda-2}} \leq c_k\|\alpha\|^2_{C^{k+1}_\lambda}.$$

Proof. Firstly, by the definition of F, $F(\alpha)(x)$ relates to the tangent space to Γ_α at $\pi_\alpha(x)$. Note that $T_{\pi_\alpha(x)}\Gamma_\alpha$ depends on both $\alpha(x)$ and $\nabla\alpha(x)$ and hence so must $F(\alpha)(x)$. We may then define P_F by (7.3) such that it is a smooth function of its arguments as claimed.

We argued after Corollary 7.1.3 that we may identify the displacement $\Psi - \iota$ of N from C, at least outside some compact subset, with $\alpha_C \in C^\infty_\lambda(\Lambda^2_+T^*C)$. Using the notation of Proposition 7.1.4, define a function $F_C(\alpha+\alpha_C)$, for $\alpha \in C^\infty_\lambda(\Lambda^2_+T^*P)$, on $(R,\infty)\times\Sigma$ by

$$F_C(\alpha+\alpha_C)(r,\sigma) = F(\tilde{\Upsilon}(\alpha))(\Psi(r,\sigma)). \tag{7.4}$$

Define a smooth function P_C by an equation analogous to (7.3):

$$F_C(\alpha+\alpha_C)(r,\sigma) = d(\alpha+\alpha_C)(r,\sigma) + P_C((r,\sigma),(\alpha+\alpha_C)(r,\sigma),\nabla(\alpha+\alpha_C)(r,\sigma)). \tag{7.5}$$

We notice that F_C and P_C are only dependent on the cone C and, rather trivially, on R. Therefore, because of this fact and our choice of δ in Proposition 7.1.5, these functions have scale equivariance properties. We may therefore derive equations and inequalities on $\{R\}\times\Sigma$ and deduce the result on all of $(R,\infty)\times\Sigma$ by introducing an appropriate scaling factor of r.

Now, since $\alpha=0$ corresponds to our coassociative 4-fold N, $F(\tilde{\Upsilon}(0)) = F(0) = 0$. So, by (7.4),

$$F_C(\alpha_C) = d\alpha_C + P_C(\alpha_C) = 0, \tag{7.6}$$

adopting similar notation for $P_C(\alpha_C)$ as for $P_F(\tilde{\Upsilon}(\alpha))$. Using (7.3)-(7.6), we deduce that

$$\begin{aligned} P_F(\tilde{\Upsilon}(\alpha)) &= d\alpha_C + P_C(\alpha+\alpha_C) = d\alpha_C + P_C(\alpha+\alpha_C) - (d\alpha_C + P_C(\alpha_C)) \\ &= P_C(\alpha+\alpha_C) - P_C(\alpha_C). \end{aligned} \tag{7.7}$$

We then calculate

$$\begin{aligned} P_C(\alpha+\alpha_C) - P_C(\alpha_C) &= \int_0^1 \frac{d}{dt}\,P_C(t\alpha+\alpha_C)\,dt \\ &= \int_0^1 \alpha\cdot\frac{\partial P_C}{\partial y}(t\alpha+\alpha_C) + \nabla\alpha\cdot\frac{\partial P_C}{\partial z}(t\alpha+\alpha_C)\,dt, \end{aligned} \tag{7.8}$$

recalling that P_C is a function of three variables x, y and z. Using Taylor's Theorem,

$$P_C(\alpha+\alpha_C) = P_C(\alpha_C) + \alpha\cdot\frac{\partial P_C}{\partial y}(\alpha_C) + \nabla\alpha\cdot\frac{\partial P_C}{\partial z}(\alpha_C) + O(r^{-2}|\alpha|^2 + |\nabla\alpha|^2) \tag{7.9}$$

when $|\alpha|$ and $|\nabla\alpha|$ are small. Since $dF|_0(\tilde{\Upsilon}(\alpha)) = d(\tilde{\Upsilon}(\alpha))$, $dF_C|_{\alpha_C}(\alpha+\alpha_C) = d\alpha$ and hence $dP_C|_{\alpha_C} = 0$. Thus, the first derivatives of P_C with respect to y and z must vanish at α_C by (7.9). Therefore, given small $\epsilon > 0$ there exists a constant $A_0 > 0$ such that

$$\left|\frac{\partial P_C}{\partial y}(t\alpha+\alpha_C)\right| \leq A_0(r^{-2}|\alpha| + r^{-1}|\nabla\alpha|) \quad \text{and} \quad \left|\frac{\partial P_C}{\partial z}(t\alpha+\alpha_C)\right| \leq A_0(r^{-1}|\alpha| + |\nabla\alpha|) \tag{7.10}$$

for $t \in [0,1]$ whenever

$$r^{-1}|\alpha|,\, r^{-1}|\alpha_C|,\, |\nabla\alpha| \text{ and } |\nabla\alpha_C| \leq \epsilon. \tag{7.11}$$

By (7.1), $r^{-1}|\alpha_C|$ and $|\nabla\alpha_C|$ tend to zero as $r \to \infty$. We can thus ensure that (7.11) is satisfied by the α_C components by making R larger. Hence, (7.11) holds if $\|\alpha\|_{C^1_1} \leq \epsilon$. Therefore, putting estimates (7.10) in (7.8) and using (7.7),

$$|P_F(\tilde{\Upsilon}(\alpha))| = |P_C(\alpha+\alpha_C) - P_C(\alpha_C)| \leq A_0(r^{-1}|\alpha| + |\nabla\alpha|)^2 \tag{7.12}$$

whenever $\|\alpha\|_{C^1_1} \leq \epsilon$. As $r \to \infty$ the terms in the bracket on the right-hand side of (7.12) are of order $O(r^{\lambda-1})$ by (7.1). Thus, $|P_F(\tilde{\Upsilon}(\alpha))|$ is of order $O(r^{2\lambda-2})$, hence $O(r^{\lambda-1})$ since $\lambda < 1$, as $r \to \infty$.

Similar calculations give analogous results to (7.12) for derivatives of P_F, but we shall explain the method by considering the first derivative. From (7.8) we calculate

$$\begin{aligned}
\nabla(P_C(\alpha+\alpha_C) - P_C(\alpha_C)) &= \int_0^1 \nabla\left(\alpha\cdot\frac{\partial P_C}{\partial y}(t\alpha+\alpha_C) + \nabla\alpha\cdot\frac{\partial P_C}{\partial z}(t\alpha+\alpha_C)\right) dt \\
&= \int_0^1 \nabla\alpha\cdot\frac{\partial P_C}{\partial y} + \alpha\cdot\left(\nabla(t\alpha+\alpha_C)\cdot\frac{\partial^2 P_C}{\partial y^2} + \nabla^2(t\alpha+\alpha_C)\cdot\frac{\partial^2 P_C}{\partial y\partial z}\right) \\
&\quad + \nabla^2\alpha\cdot\frac{\partial P_C}{\partial z} + \nabla\alpha\cdot\left(\nabla(t\alpha+\alpha_C)\cdot\frac{\partial^2 P_C}{\partial z\partial y} + \nabla^2(t\alpha+\alpha_C)\cdot\frac{\partial^2 P_C}{\partial z^2}\right) dt.
\end{aligned}$$

Whenever $\|\alpha\|_{C^1_1} \leq \epsilon$ there exists a constant $A_1 > 0$ such that (7.10) holds with A_0 replaced by A_1 and, for $t \in [0,1]$,

$$\left|\frac{\partial^2 P_C}{\partial y^2}(t\alpha+\alpha_C)\right|,\ \left|\frac{\partial^2 P_C}{\partial y\partial z}(t\alpha+\alpha_C)\right| \text{ and } \left|\frac{\partial^2 P_C}{\partial z^2}(t\alpha+\alpha_C)\right| \leq A_1,$$

since the second derivatives of P_C are continuous functions defined on the closed bounded set given by $\|\alpha\|_{C^1_1} \leq \epsilon$. We deduce that

$$\left|\nabla\big(P_F(\tilde{\Upsilon}(\alpha))\big)\right| = |\nabla(P_C(\alpha+\alpha_C) - P_C(\alpha_C))| \leq A_1\left(\sum_{i=0}^{2} r^{i-2}|\nabla^i\alpha|\right)^2$$

whenever $\|\alpha\|_{C^1_1} \leq \epsilon$. Therefore $\left|\nabla\big(P_F(\tilde{\Upsilon}(\alpha))\big)\right|$ is of order $O(r^{2\lambda-3})$, hence $O(r^{\lambda-2})$, as $r \to \infty$.

In general we have the estimate

$$\left|\nabla^j\big(P_F(\tilde{\Upsilon}(\alpha))\big)\right| \leq A_j\left(\sum_{i=0}^{j+1} r^{i-(j+1)}|\nabla^i\alpha|\right)^2$$

for some $A_j > 0$ whenever $\|\alpha\|_{C^1_1} \leq \epsilon$. The result follows. □

It is immediate from Proposition 7.1.9 that G maps $(\alpha,\beta) \in C^\infty_\lambda(U) \times C^\infty_\lambda(\Lambda^4 T^*N)$, with α sufficiently near 0 in C^1_1, into $C^\infty_{\lambda-1}(\Lambda^3 T^*N)$.

Further, if $G(\alpha,\beta) = 0$ then

$$d(G(\alpha,\beta)) = d(F(\alpha)) + dd^*\beta = \Delta\beta = 0,$$

since $F(\alpha)$ is exact because φ_0 is exact near N in $\mathbb{R}^7$. If β decays with order $O(\rho^\lambda)$ as $\rho \to \infty$, where $\lambda < 0$, then $*\beta$ is a harmonic function on N which tends to zero as $\rho \to \infty$. The Maximum Principle (Theorem 1.2.5) allows us to deduce that $*\beta = 0$ and conclude that $\beta = 0$. Thus, the kernel of F in $C^\infty_\lambda(U)$ is isomorphic to the kernel of G in $C^\infty_\lambda(U) \times C^\infty_\lambda(\Lambda^4 T^*N)$ for $\lambda < 0$.

7.1.3 Regularity

We consider the regularity of solutions to the *nonlinear* elliptic equation $G(\alpha, \beta) = 0$ near $(0, 0)$.

Suppose that $(\alpha, \beta) \in L^p_{k+1,\,\lambda}(U) \times L^p_{k+1,\,\lambda}(\Lambda^4 T^*N)$ for some $p > 4$ and $k \geq 2$. Then α and β lie in C^1_{loc} by Theorem 6.2.4, since $\frac{k}{4} > \frac{1}{p}$.

Suppose further that $G(\alpha, \beta) = 0$ and that $\|\alpha\|_{C^1_1}$ is sufficiently small. Since F smoothly depends on α and $\nabla\alpha$, G is a smooth function of $\alpha, \beta, \nabla\alpha$ and $\nabla\beta$. We apply Theorem 6.4.3 to conclude that α and β are smooth. However, we want more than this: the derivatives of α and β must decay at the required rates.

Recall the observation made at the end of §7.1.2 that $G(\alpha, \beta) = 0$ implies that $\Delta\beta = 0$. By Theorem 6.2.4 and Corollary 6.4.2, $\beta \in C^\infty_\lambda(\Lambda^4 T^*N)$.

For the following argument we find it useful to work with weighted Hölder spaces. By Theorem 6.2.4, $\alpha \in C^{k,\,a}_\lambda(U)$ with $a = 1 - 4/p \in (0, 1)$ since $p > 4$. We also know that $d^*(G(\alpha, \beta)) = d^*(F(\alpha)) = 0$, which is a nonlinear elliptic equation on α. Using the notation and results of Proposition 7.1.9, $d^*d\alpha + d^*(P_F(\alpha)) = 0$ and $d^*(P_F(\alpha)) \in C^{k-2,\,a}_{2\lambda-3}(\Lambda^2 T^*N)$. We see that

$$d^*(F(\alpha))(x) = R_F(x, \alpha(x), \nabla\alpha(x))\nabla^2\alpha(x) + E_F(x, \alpha(x), \nabla\alpha(x)),$$

where $R_F(x, \alpha(x), \nabla\alpha(x))$ and $E_F(x, \alpha(x), \nabla\alpha(x))$ are smooth functions of their arguments. Define

$$S_\alpha(\gamma)(x) = R_F(x, \alpha(x), \nabla\alpha(x))\nabla^2\gamma(x)$$

for $\gamma \in C^2_{\mathrm{loc}}(\Lambda^2_+ T^*N)$. Then S_α is a smooth *linear uniformly elliptic* second order operator, if $\|\alpha\|_{C^1_1}$ is sufficiently small, whose coefficients depend on x, $\alpha(x)$ and $\nabla\alpha(x)$. These coefficients therefore lie in $C^{k-1,\,a}_{\mathrm{loc}}$. We also notice that

$$S_\alpha(\alpha)(x) = -E_F(x, \alpha(x), \nabla\alpha(x)) \in C^{k-2,\,a}_{2\lambda-3}(\Lambda^2 T^*N) \subseteq C^{k-2,\,a}_{\lambda-2}(\Lambda^2 T^*N),$$

since $\lambda < 1$. However, $E_F(x, \alpha(x), \nabla\alpha(x))$ only depends on α and $\nabla\alpha$, and is at worst quadratic in these quantities by Proposition 7.1.9, so it must in fact lie in $C^{k-1,\,a}_{\lambda-2}(\Lambda^2 T^*N)$ since we are given control on the decay of the first k derivatives of α at infinity.

If $\gamma \in C^2_\lambda(\Lambda^2_+ T^*N)$ and $S_\alpha(\gamma) \in C^{k-1,\,a}_{\lambda-2}(\Lambda^2 T^*N)$, applying Theorem 6.4.1(b) implies that $\gamma \in C^{k+1,\,a}_\lambda(\Lambda^2_+ T^*N)$. Since $k \geq 2$, α and $S_\alpha(\alpha)$ satisfy these conditions by the discussion above. We deduce that $\alpha \in C^{k+1,\,a}_{\lambda}(\Lambda^2_+T^*N)$ only knowing a priori that $\alpha \in C^{k,\,a}_{\lambda}(\Lambda^2_+T^*N)$. We proceed by induction to show that $\alpha \in C^{k,\,a}_{\lambda}(\Lambda^2_+T^*N)$ for all $k \geq 2$.

As a consequence of the regularity results above, we deduce the following.

Proposition 7.1.10. *Let $(\alpha,\beta) \in L^p_{k+1,\,\lambda}(U) \times L^p_{k+1,\,\lambda}(\Lambda^4T^*N)$ for some $p > 4$ and $k \geq 2$. If $G(\alpha,\beta) = 0$ and $\|\alpha\|_{C^1_1}$ is sufficiently small, $(\alpha,\beta) \in C^\infty_\lambda(U) \times C^\infty_\lambda(\Lambda^4T^*N)$.*

7.2 Study of the Cokernel

The work in §7.1 leads us to consider

$$G : L^p_{k+1,\,\lambda}(U) \times L^p_{k+1,\,\lambda}(\Lambda^4T^*N) \to L^p_{k,\,\lambda-1}(\Lambda^3T^*N), \tag{7.13}$$

using the notation of Definition 7.1.8 and Proposition 7.1.10 and the results derived in the proof of Proposition 7.1.9. As noted in Definition 7.1.8, the linearisation of (7.13) at $(0,0)$ acts as

$$d + d^* : L^p_{k+1,\,\lambda}(\Lambda^2_+T^*N \oplus \Lambda^4T^*N) \to L^p_{k,\,\lambda-1}(\Lambda^3T^*N). \tag{7.14}$$

In this section we are concerned with the *cokernel* of (7.14).

7.2.1 The image of $d + d^*$

By Theorem 6.3.3, there exists a countable discrete subset $\mathcal{D}'$ of rates λ such that

$$d + d^* : L^p_{k+1,\,\lambda}(\Lambda^{\text{even}}T^*N) \to L^p_{k,\,\lambda-1}(\Lambda^{\text{odd}}T^*N) \tag{7.15}$$

is not Fredholm. Clearly, $\mathcal{D}' \supseteq \mathcal{D}_{\text{AC}}$, where $\mathcal{D}_{\text{AC}}$ is given in Proposition 6.3.4(a). For $\lambda \notin \mathcal{D}'$

$$L^p_{k,\,\lambda-1}(\Lambda^{\text{odd}}T^*N) = (d + d^*)(L^p_{k+1,\,\lambda}(\Lambda^{\text{even}}T^*N)) \oplus \mathcal{C},$$

where $\mathcal{C}$ is a finite-dimensional space which may be taken to consist of forms with compact support, as well as closed under the Hodge star. Alternatively, if $\lambda > -1$ (so that $-\lambda - 3 < \lambda - 1$), $\mathcal{C}$ may be chosen as the kernel $\mathcal{K}$ of the adjoint map

$$d + d^* : L^q_{l+1,\,-\lambda-3}(\Lambda^{\text{odd}}T^*N) \to L^q_{l,\,-\lambda-4}(\Lambda^{\text{even}}T^*N), \tag{7.16}$$

where $1/p + 1/q = 1$ and $l \in \mathbb{N}$, which is graded and closed under the Hodge star.

For any $\lambda \notin \mathcal{D}'$ define an inner product on $L^p_{k,\,\lambda-1}(\Lambda^{\text{odd}}T^*N) \times \mathcal{K}$ by

$$\langle \gamma\,,\,\eta \rangle = \int_N \gamma \cdot \eta. \tag{7.17}$$

We are able to take $\mathcal{C}$ to be any finite-dimensional subspace of $L^p_{k,\,\lambda-1}(\Lambda^{\mathrm{odd}}T^*N)$, with $\dim\mathcal{C} = \dim\mathcal{K}$, such that the inner product on $\mathcal{C}\times\mathcal{K}$ is nondegenerate. In particular, there is a natural isomorphism $\mathcal{C}\cong\mathcal{K}^*$, where the dual space is defined by the inner product (7.17).

Let $\lambda > -1$ and choose $\mathcal{C} = \mathcal{K}$. If $\gamma \in L^p_{k,\,\lambda-1}(\Lambda^3T^*N)$ then $(*\gamma,\gamma)\in L^p_{k,\,\lambda-1}(\Lambda^{\mathrm{odd}}T^*N)$ and hence there exist some $\gamma_m \in L^p_{k+1,\,\lambda}(\Lambda^m T^*N)$, for $m = 0, 2, 4$, and $\eta\in\mathcal{C}$ such that

$$(*\gamma,\gamma) = (d+d^*)(\gamma_0,\gamma_2,\gamma_4)+\eta.$$

By applying the Hodge star,

$$(*\gamma,\gamma) = (d+d^*)(*\gamma_4,*\gamma_2,*\gamma_0)+*\eta.$$

Adding the above formulae and averaging gives:

$$\gamma = d\left(\frac{\gamma_2+*\gamma_2}{2}\right)+d^*\left(\frac{*\gamma_0+\gamma_4}{2}\right)+\tilde{\eta}$$

where $\tilde{\eta}\in\mathcal{C}\cap L^p_{k,\,\lambda-1}(\Lambda^3T^*N)$. We deduce that

$$L^p_{k,\,\lambda-1}(\Lambda^3T^*N) = \Big(d(L^p_{k+1,\,\lambda}(\Lambda^2_+T^*N)) + d^*(L^p_{k+1,\,\lambda}(\Lambda^4T^*N))\Big)\oplus\mathcal{C}^3,$$

where $\mathcal{C}^3 = \mathcal{C}\cap L^p_{k,\,\lambda-1}(\Lambda^3T^*N)$. Moreover, for $\lambda\notin\mathcal{D}'$, $\lambda > -1$,

$$d(L^p_{k+1,\,\lambda}(\Lambda^2T^*N)) = d(L^p_{k+1,\,\lambda}(\Lambda^2_+T^*N)). \tag{7.18}$$

We must surely have that the images are equal for $\lambda\notin\mathcal{D}_{\mathrm{AC}}$, $\lambda > -1$, as well.

The argument above shows that (7.18) will hold for any $\lambda\notin\mathcal{D}_{\mathrm{AC}}$ where the cokernel $\mathcal{C}$ can be taken to be graded and closed under the Hodge star. We shall see in the next subsection that this occurs whenever $\lambda\in(-2,1)\setminus\mathcal{D}_{\mathrm{AC}}$.

7.2.2 Obstructions and the kernel of the adjoint map

We are concerned with solutions $(\alpha,\beta)\in L^p_{k+1,\,\lambda}(\Lambda^2_+T^*N\oplus\Lambda^4T^*N)$ to $G(\alpha,\beta) = F(\alpha)+d^*\beta = 0$. Using the notation of Proposition 7.1.9, $F(\alpha) = d\alpha + P_F(\alpha)$ with $P_F(\alpha)\in L^p_{k,\,\lambda-1}(\Lambda^3T^*N)$. We can thus consider the problem as solving:

$$d\alpha + d^*\beta = -P_F(\alpha) = \gamma.$$

Our deformation theory will be unobstructed if and only if

$$\int_N \gamma\wedge *\eta = 0 \tag{7.19}$$

for all 3-forms η in the kernel $\mathcal{K}^3$ of the adjoint map to (7.14):

$$d^*_+ + d : L^q_{l+1,\,-\lambda-3}(\Lambda^3T^*N)\to L^q_{l,\,-\lambda-4}(\Lambda^2_+T^*N\oplus\Lambda^4T^*N), \tag{7.20}$$

where $1/p + 1/q = 1$ and $l \in \mathbb{N}$.

Suppose $\eta \in \mathcal{K}^3$ and $\lambda > -2$. Then $d^*\eta \in L^q_{l,\,-\lambda-4}(\Lambda^2_- T^*N)$. Note that we may embed $L^q_{l+1,\,-\lambda-3}$ in $C^1_{-\lambda-3}$ for $l \geq 4$ by Theorem 6.2.4, recalling that $p > 4$. Thus,

$$\int_N |d^*\eta|^2 = \int_N -d^*\eta \wedge d^*\eta = \int_N -d{*}\eta \wedge d{*}\eta = \int_N -d({*}\eta \wedge d{*}\eta) = 0,$$

which is valid since $|{*}\eta| = O(\rho^{-\lambda-3})$ and $|d^*\eta| = O(\rho^{-\lambda-4})$ as $\rho \to \infty$, with $\lambda > -2$. We deduce the following result.

Proposition 7.2.1. *If η lies in the kernel of* (7.20) *for $\lambda > -2$ and $l \geq 4$, it satisfies $d\eta = d^*\eta = 0$.*

Under the conditions of Proposition 7.2.1, ${*}\eta \in L^q_{l+1,\,-\lambda-3}(\Lambda^1 T^*N)$ with $d{*}\eta = d^*{*}\eta = 0$. Recall that $N \setminus K \cong (R, \infty) \times \Sigma$ and so, on $N \setminus K$,

$${*}\eta = \chi + f\,dr$$

for a function f and 1-form χ on $(R, \infty) \times \Sigma$, where χ has no dr component. Write

$$d = d_\Sigma + dr \wedge \frac{\partial}{\partial r}$$

on $(R, \infty) \times \Sigma$. The equation $d{*}\eta = 0$ implies that

$$d_\Sigma \chi = 0 \qquad \text{and} \qquad \frac{\partial \chi}{\partial r} - d_\Sigma f = 0. \tag{7.21}$$

Define a function ζ on $(R, \infty) \times \Sigma$ by

$$\zeta(r, \sigma) = -\int_r^\infty f(s, \sigma)\, ds.$$

This is well-defined since the modulus of f is $O(r^{-3-\lambda})$ as $r \to \infty$, where $-3-\lambda < -1$ since $\lambda > -2$. Noting that the modulus of χ with respect to g_Σ is $O(r^{-2-\lambda})$ as $r \to \infty$, with $-2-\lambda < 0$, we calculate using (7.21):

$$\begin{aligned} d\zeta &= -\int_r^\infty d_\Sigma f(s, \sigma)\, ds + f\,dr = -\int_r^\infty \frac{\partial \chi}{\partial r}(s, \sigma)\, ds + f\,dr \\ &= \Big[-\chi(s, \sigma) \Big]_r^\infty + f\,dr = \chi(r, \sigma) + f\,dr = {*}\eta. \end{aligned}$$

If $\{R\} \times \Sigma$ has a tubular neighbourhood in N, which can be ensured by making R larger if necessary, we can extend ζ smoothly to a function on N. Hence $\zeta \in L^q_{l+2,\,-\lambda-2}(\Lambda^0 T^*N)$ with $d\zeta = {*}\eta$ on $N \setminus K$. This leads us to the next proposition.

Proposition 7.2.2. *Let η lie in the kernel $\mathcal{K}^3$ of* (7.20) *for $\lambda > -2$ and $l \geq 4$. There exists a function ζ on N of order $O(\rho^{-2-\lambda})$ as $\rho \to \infty$ such that ${*}\eta - d\zeta = \xi$ is a closed compactly supported 1-form. Moreover, the map $\eta \mapsto [\xi]$ from $\mathcal{K}^3$ to $H^1_{\mathrm{cs}}(N)$ is injective for $\lambda \in (-2, 1)$.*

Proof. Clearly our construction above ensures that ξ is a closed 1-form which is zero outside of the compact subset K of N. Thus $[\xi] \in H^1_{\text{cs}}(N)$. Suppose that $[\xi] = 0$. Then $\xi = d\tilde{\zeta}$ for some function $\tilde{\zeta}$ with compact support. Therefore

$$0 = d^* {*\eta} = d^*(d\zeta + \xi) = d^* d(\zeta + \tilde{\zeta}).$$

Hence $\zeta + \tilde{\zeta}$ is a harmonic function which tends to zero, since $-2 - \lambda < 0$, as $\rho \to \infty$. We may employ Theorem 1.2.5 to deduce that $\zeta + \tilde{\zeta} = 0$ and the result follows. $\square$

Proposition 7.2.5 below shows that the map from $\mathcal{K}^3$ to $H^1_{\text{cs}}(N)$ is an isomorphism when $\lambda \in (-2, 0)$.

Recall that $F(\alpha)$, hence $\gamma = P_F(\alpha)$, is exact. Use the notation of Proposition 7.2.2. Then

$$\int_N \gamma \wedge *\eta = \int_N \gamma \wedge d\zeta + \int_N \gamma \wedge \xi.$$

Here, $[\gamma] \in H^3_{\text{dR}}(N)$, $[\xi] \in H^1_{\text{cs}}(N)$ and the product $H^3_{\text{dR}}(N) \times H^1_{\text{cs}}(N) \to \mathbb{R}$ is well-defined, so

$$\int_N \gamma \wedge \xi = [\gamma] \cdot [\xi] = 0$$

because γ is exact. By integration by parts, using the fact that γ is closed,

$$\int_N \gamma \wedge d\zeta = \int_N d(\gamma \cdot \zeta) = 0.$$

Therefore (7.19) holds for all $\eta \in \mathcal{K}^3$ and $\lambda > -2$. We write this result below.

Proposition 7.2.3. *The deformation theory of AC coassociative 4-folds is unobstructed if the rate λ lies in $(-2, 1)$ but not in the set $\mathcal{D}_{\text{AC}}$ defined in Proposition 6.3.4(a).*

7.2.3 Dimension of the cokernel

We start with a result concerning functions on cones [28, Lemma 2.3].

Proposition 7.2.4. *Suppose that $f : C \to \mathbb{R}$ is a nonzero function such that*

$$f(r, \sigma) = r^\mu f_\Sigma(\sigma)$$

for some $f_\Sigma : \Sigma \to \mathbb{R}$ and $\mu \in \mathbb{R}$. Denoting the Laplacians on C and Σ by Δ_C and Δ_Σ respectively,

$$\Delta_C f = r^{\mu - 2}(\Delta_\Sigma f_\Sigma - \mu(\mu + 2) f_\Sigma).$$

Therefore, since Σ is compact, $\mu(\mu + 2) \geq 0$ and so there exist no nonzero homogeneous harmonic functions of order $O(r^\mu)$ on C with $\mu \in (-2, 0)$.

Use the notation of Proposition 7.2.2 and let $\lambda \in (-2,0)$. Clearly $\Delta\zeta = -d^*\xi$ and hence $d^*\xi$ lies in the image of the map

$$\Delta_{-\lambda-2} = \Delta : L^q_{l+2,\,-\lambda-2}(\Lambda^0 T^*N) \to L^q_{l,\,-\lambda-4}(\Lambda^0 T^*N).$$

Therefore $d^*\xi$ is L^2-orthogonal to the kernel of the L^2 adjoint of $\Delta_{-\lambda-2}$. Let $\mu \in (-2,0)$. Proposition 7.2.4 implies that there are no elements of $\mathcal{D}(\Delta)$, defined in Theorem 6.3.3, between $-\lambda-2$ and $-\mu-2$. Thus $\operatorname{Coker}\Delta_{-\lambda-2} = \operatorname{Coker}\Delta_{-\mu-2}$. Obviously, $d^*\xi$ is then L^2-orthogonal to the kernel of the L^2 adjoint of $\Delta_{-\mu-2}$. Consequently, there exists $\zeta_\mu \in L^q_{l+2,\,-\mu-2}(\Lambda^0T^*N)$ such that $\Delta_{-\mu-2}\zeta_\mu = -d^*\xi$ and hence $\zeta-\zeta_\mu$ is harmonic. Moreover, $\zeta-\zeta_\mu = O(\rho^{-\min(\lambda,\mu)-2})$ as $\rho\to\infty$ and, since $-\min(\lambda,\mu)-2<0$, we use Theorem 1.2.5 to deduce that $\zeta-\zeta_\mu = 0$. Hence $*\eta$ and η lie in $L^q_{l+1,\,-\mu-3}$ for any $\mu\in(-2,0)$. The dimension of the cokernel of (7.14) is therefore constant for $\lambda\in(-2,0)$, $\lambda\notin\mathcal{D}_{\mathrm{AC}}$.

Recall the space $\mathcal{H}^m$ given in Definition 6.5.1. Using (6.5), we notice that

$$\mathcal{H}^3 = \{\gamma \in L^2_{0,\,-2}(\Lambda^3T^*N) \,:\, d\gamma = d^*\gamma = 0\}$$

and hence it is equal to the kernel $\mathcal{K}^3$ of (7.20) for $q=2$, $\lambda=-1$ and $l=0$. By Corollary 6.4.2 applied to the uniformly elliptic operator given by (7.20), $\mathcal{K}^3$ in the case $\lambda=-1$ is isomorphic to $\mathcal{H}^3$. Moreover, the dimension of $\mathcal{K}^3$ is equal to the dimension of the cokernel of (7.14) if $\lambda\notin\mathcal{D}_{\mathrm{AC}}$ and $\mathcal{H}^3\cong H^3_{\mathrm{dR}}(N)$ by Theorem 6.5.2(a).

We put together our recent results and observations.

Proposition 7.2.5. *For all $\lambda\in(-2,0)$, $\lambda\notin\mathcal{D}_{\mathrm{AC}}$, the cokernel of* (7.14) *has dimension equal to $b^3(N)$ and so it is independent of λ.*

Consider the cokernel of (7.15), which has equal dimension to the kernel $\mathcal{K}$ of the adjoint map (7.16) as long as $\lambda\notin\mathcal{D}'$, using the notation of §7.2.1. Let $\lambda>-2$. If $(\eta_1,\eta_3)\in\mathcal{K}$ then $d\eta_1 + d^*\eta_3 = 0$. However, an integration by parts arguments shows, valid since $-4-\lambda<-2$, that $d\eta_1 = d^*\eta_3 = 0$. We may thus use the construction leading to Proposition 7.2.2 to map (η_1,η_3) to a compactly supported 1-form. Then, using the arguments proceeding Proposition 7.2.4, (η_1,η_3) lies in $L^q_{l+1,\,-\mu-3}(\Lambda^{\mathrm{odd}}T^*N)$ for any $\mu\in(-2,0)$. Therefore the dimension of $\mathcal{K}$, and hence of the cokernel of (7.15), is constant for $\lambda\in(-2,0)$, $\lambda\notin\mathcal{D}'$.

We now deduce the result mentioned after (7.18), because we can take the cokernel of (7.15) to be graded for $\lambda>-2$, $\lambda\notin\mathcal{D}_{\mathrm{AC}}$, since we know it is for $\lambda>-1$.

Proposition 7.2.6. *For $\lambda>-2$ and $\lambda\notin\mathcal{D}_{\mathrm{AC}}$,*

$$d(L^p_{k+1,\,\lambda}(\Lambda^2T^*N)) = d(L^p_{k+1,\,\lambda}(\Lambda^2_+T^*N)).$$

7.3 The Deformation Space

7.3.1 The image of F

Let v be the vector field given by dilations, which, in coordinates $(x_1, \ldots, x_7)$ on $\mathbb{R}^7$, is written:

$$v = x_1 \frac{\partial}{\partial x_1} + \ldots + x_7 \frac{\partial}{\partial x_7}\,. \tag{7.22}$$

Then the Lie derivative of φ_0 along v is:

$$\mathcal{L}_v \varphi_0 = d(v \cdot \varphi_0) = 3\varphi_0. \tag{7.23}$$

Therefore, $\psi = \frac{1}{3}\, v \cdot \varphi_0$ is a smooth 2-form such that $d\psi = \varphi_0$. Note that $\psi|_C \equiv 0$ since

$$(v \cdot \varphi_0)|_C = v \cdot (\varphi_0|_C) = 0,$$

as $v \in TC$ and C is coassociative. Define, for $\alpha \in C^1_{\mathrm{loc}}(U)$, $H(\alpha) = f_\alpha^*(\psi|_{N_\alpha})$ so that

$$F(\alpha) = f_\alpha^*(d\psi|_{N_\alpha}) = d(f_\alpha^*(\psi|_{N_\alpha})) = d(H(\alpha)).$$

Recall the diffeomorphism $\Psi_\alpha : (R, \infty) \times \Sigma \to N_\alpha \setminus K_\alpha$, where K_α is compact, introduced before Proposition 7.1.7. The decay of $H(\alpha)$ at infinity is determined by:

$$\Psi_\alpha^*(\psi) = (\Psi_\alpha^* - \iota^*)(\psi) + \iota^*(\psi) = (\Psi_\alpha^* - \iota^*)(\psi)$$

since $\psi|_C \equiv 0$. For $(r, \sigma) \in (R, \infty) \times \Sigma$,

$$(\Psi_\alpha^* - \iota^*)(\psi)|_{(r,\sigma)} = \Big(d\Psi_\alpha|_{(r,\sigma)}^*(\psi|_{\Psi_\alpha(r,\sigma)}) - d\iota|_{(r,\sigma)}^*(\psi|_{\Psi_\alpha(r,\sigma)}) \Big) + d\iota|_{(r,\sigma)}^*(\psi|_{\Psi_\alpha(r,\sigma)} - \psi|_{r\sigma}), \tag{7.24}$$

using the linearity of $d\iota^*$ to derive the last term. Since $|\psi| = O(r)$ and Ψ_α satisfies (1.1) so that $|d\Psi_\alpha^* - d\iota^*| = O(r^{\lambda-1})$ as $r \to \infty$, the expression in brackets in (7.24) is $O(r^\lambda)$. The final term in (7.24) is determined by the behaviour of $d\iota^*$, $\nabla\psi$ and $\Psi_\alpha - \iota$. Hence, as $|d\iota^*|$ and $|\nabla\psi|$ are $O(1)$, using (1.1) again implies that this term is $O(r^\lambda)$. We conclude that if $\alpha \in L^p_{k+1,\,\lambda}(U)$ then $H(\alpha) \in L^p_{k,\,\lambda}(\Lambda^2 T^* N)$. Notice that $H(\alpha)$ has one degree of differentiability less than one would expect since it depends on α and $\nabla\alpha$.

Let $\lambda > -2$ and let $\lambda \notin \mathcal{D}_{\mathrm{AC}}$. By Proposition 7.2.1 the kernel $\mathcal{K}^3$ of (7.20) consists of closed and coclosed 3-forms. The following integration by parts argument is therefore valid for $\alpha \in L^p_{k+1,\,\lambda}(U)$ and $\eta \in \mathcal{K}^3 \subseteq L^q_{l+1,\,-\lambda-3}(\Lambda^3 T^* N)$:

$$\langle F(\alpha)\,,\,\eta\rangle_{L^2} = \langle d(H(\alpha))\,,\,\eta\rangle_{L^2} = \langle H(\alpha)\,,\,d^*\eta\rangle_{L^2} = 0.$$

Hence, $F(\alpha)$ is L^2-orthogonal to $\mathcal{K}^3$. Since $\lambda \notin \mathcal{D}_{\mathrm{AC}}$, the map (7.14) is Fredholm and therefore has closed image. Moreover, $\mathcal{K}^3$ is independent of k, hence $F(\alpha)$ must lie in the image of (7.14). We deduce the following.

Proposition 7.3.1. *For $p > 4$, $k \geq 2$, $\lambda \in (-2, 1)$ and $\lambda \notin \mathcal{D}_{\mathrm{AC}}$,*

$$G : L^p_{k+1,\,\lambda}(U) \times L^p_{k+1,\,\lambda}(\Lambda^4 T^*N) \to d(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*N)) + d^*(L^p_{k+1,\,\lambda}(\Lambda^4 T^*N)) \subseteq L^p_{k,\,\lambda-1}(\Lambda^3 T^*N)$$

and has derivative

$$dG|_{(0,0)} = d + d^* : L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*N \oplus \Lambda^4 T^*N) \to d(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*N)) + d^*(L^p_{k+1,\,\lambda}(\Lambda^4 T^*N)).$$

7.3.2 The moduli space

Let $p > 4$, $k \geq 2$ and let $\lambda \in (-2, 1)$ with $\lambda \notin \mathcal{D}_{\mathrm{AC}}$. By Definition 6.2.1, $X = L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*N \oplus \Lambda^4 T^*N)$ is a Banach space. Clearly $V = L^p_{k+1,\,\lambda}(U) \times L^p_{k+1,\,\lambda}(\Lambda^4 T^*N)$ is an open neighbourhood of $(0, 0)$ in X if $L^p_{k+1,\,\lambda}$ embeds in C^0_λ, since $\lambda < 1$ and U, given by Proposition 7.1.5, grows with order $O(\rho)$ as $\rho \to \infty$. Theorem 6.2.4 gives the condition for this to occur as $k + 1 > \frac{4}{p}$, which is satisfied. Moreover, $Y = d(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*N)) + d^*(L^p_{k+1,\,\lambda}(\Lambda^4 T^*N))$ is a Banach space because it is the image of a Fredholm map and hence a closed subspace of $L^p_{k,\,\lambda-1}(\Lambda^3 T^*N)$.

By Proposition 7.3.1, G maps V to Y and its derivative $dG|_{(0,0)} : X \to Y$ is clearly surjective with finite-dimensional kernel which splits X. Using the Implicit Function Theorem for Banach spaces (Theorem 6.2.5), we deduce that $G^{-1}(0)$ is locally diffeomorphic to the kernel of $dG|_{(0,0)}$. By Proposition 7.1.10, forms (α, β) in $G^{-1}(0)$ such that $\|\alpha\|_{C^1_1}$ is sufficiently small are smooth. Therefore $G^{-1}(0)$ is locally diffeomorphic to the kernel of the map

$$d + d^* : C^\infty_\lambda(\Lambda^2_+ T^*N \oplus \Lambda^4 T^*N) \to C^\infty_{\lambda-1}(\Lambda^3 T^*N).$$

Define a map π_G on $G^{-1}(0)$ by $\pi_G(\alpha, \beta) = \beta$. Then π_G is a smooth map such that $\pi_G^{-1}(0) = F^{-1}(0)$. Let $(\alpha, \beta) \in G^{-1}(0)$ and recall that, by the work in §7.3.1, there exists $H(\alpha) \in C^\infty_\lambda(\Lambda^2 T^*N)$ such that $F(\alpha) = d(H(\alpha))$. Therefore, since $F(\alpha) + d^*\beta = 0$, $d^*\beta \in d(C^\infty_\lambda(\Lambda^2 T^*N))$ and, by Proposition 7.2.6, must then lie in $d(C^\infty_\lambda(\Lambda^2_+ T^*N))$. Hence

$$\pi_G : G^{-1}(0) \to \{\beta \in C^\infty_\lambda(\Lambda^4 T^*N) \,:\, d^*\beta \in d(C^\infty_\lambda(\Lambda^2_+ T^*N))\} = \mathcal{B}_\lambda.$$

Therefore $d\pi_G|_{(0,0)} : \operatorname{Ker} dG|_{(0,0)} \to \mathcal{B}_\lambda$ is surjective and so π_G is locally surjective. Consequently, if $\alpha \in F^{-1}(0)$ and is sufficiently near 0 in C^1_1, it is smooth.

We deduce the following theorem, which is the main result of the chapter.

Theorem 7.3.2. *Let N be a coassociative 4-fold in $\mathbb{R}^7$ which is AC to a cone C in $\mathbb{R}^7$ with rate λ for a generic $\lambda \in (-2, 1)$ (i.e. $\lambda \notin \mathcal{D}_{\mathrm{AC}}$ where $\mathcal{D}_{\mathrm{AC}}$ is given by Proposition 6.3.4(a)). The moduli space $\mathcal{M}(N, \lambda)$ of coassociative deformations of N, which are AC to C with rate λ, is a (smooth) manifold near N with dimension equal to that of the kernel of*

$$d : C^\infty_\lambda(\Lambda^2_+ T^*N) \to C^\infty_{\lambda-1}(\Lambda^3 T^*N).$$

This dimension is also equal to

$$\dim G^{-1}(0) - \dim \mathcal{B}_\lambda,$$

where G *is given in Definition 7.1.8 and*

$$\mathcal{B}_\lambda = \{\beta \in C^\infty_\lambda(\Lambda^4 T^*N) \,:\, d^*\beta \in d(C^\infty_\lambda(\Lambda^2_+ T^*N))\}.$$

We remind the reader that $\mathcal{B}_\lambda = \{0\}$ if $\lambda < 0$ by the comments made at the end of §7.1.2. This fact makes our dimension calculations easier in the next section.

7.4 Dimension of the Moduli Space

We start with some notation.

Definition 7.4.1. Denote the kernel of (7.14) by $\mathcal{K}_\lambda$ and the kernel of (7.20) by $\mathcal{C}_\lambda$. Note that, by the results in §7.3.2,

$$\dim \mathcal{M}(N, \lambda) = \dim \mathcal{K}_\lambda - \dim \mathcal{B}_\lambda$$

for $\lambda \in (-2, 1) \setminus \mathcal{D}_{\mathrm{AC}}$, where $\mathcal{B}_\lambda$ is defined in Theorem 7.3.2.

Using Corollary 6.4.2, since $L^p_{k+1,\,\lambda} \hookrightarrow C^1_\lambda$ by Theorem 6.2.4, and the Maximum Principle (Theorem 1.2.5), $\mathcal{K}_\lambda \cong \{\alpha \in C^\infty_\lambda(\Lambda^2_+ T^*N) \,:\, d\alpha = d^*\alpha = 0\}$ if $\lambda < 0$. From (6.5),

$$\mathcal{H}^2 = \{\alpha \in L^2_{0,\,-2}(\Lambda^2 T^*N) \,:\, d\alpha = d^*\alpha = 0\}$$

where $\mathcal{H}^2$ is given in Definition 6.5.1. Recalling Example 6.5.3, we deduce the following proposition.

Proposition 7.4.2. *In the notation of Example 6.5.3 and Definition 7.4.1,* $\dim \mathcal{K}_{-2} = \dim \mathcal{H}^2_+$.

The next proposition we state follows from standard results in algebraic topology if we consider N as the interior of a manifold which has boundary Σ.

Proposition 7.4.3. *Let the map* $\phi_m : H^m_{\mathrm{cs}}(N) \to H^m_{\mathrm{dR}}(N)$ *be defined by* $\phi_m([\xi]) = [\xi]$. *Let* $r > R$ *and let* $\Psi_r : \Sigma \to N$ *be the embedding given by* $\Psi_r(\sigma) = \Psi(r, \sigma)$. *Define* $p_m : H^m_{\mathrm{dR}}(N) \to H^m_{\mathrm{dR}}(\Sigma)$ *by* $p_m([\xi]) = [\Psi_r^*\xi]$. *Let* $f \in C^\infty(N)$ *be such that* $f = 0$ *on* K *and* $f = 1$ *on* $(R+1, \infty) \times \Sigma$. *If* $\pi_\Sigma : (R, \infty) \times \Sigma \cong N \setminus K \to \Sigma$ *is the projection map, define* $\partial_m : H^m_{\mathrm{dR}}(\Sigma) \to H^{m+1}_{\mathrm{cs}}(N)$ *by* $\partial_m([\xi]) = [d(f\pi_\Sigma^*\xi)]$. *Then the following sequence is exact:*

$$\cdots \longrightarrow H^m_{\mathrm{cs}}(N) \xrightarrow{\phi_m} H^m_{\mathrm{dR}}(N) \xrightarrow{p_m} H^m_{\mathrm{dR}}(\Sigma) \xrightarrow{\partial_m} H^{m+1}_{\mathrm{cs}}(N) \longrightarrow \cdots . \tag{7.25}$$

We note that $H^0_{\mathrm{cs}}(N) = H^4_{\mathrm{dR}}(N) = 0$, which enables us to calculate the dimension of various spaces more easily using the long exact sequence (7.25).

Since $\mathcal{D}_{\mathrm{AC}}$ is discrete, there exists a greatest $\lambda_- \in \mathcal{D}_{\mathrm{AC}}$ with $\lambda_- < -2$ and a least $\lambda_+ \in \mathcal{D}_{\mathrm{AC}}$ with $\lambda_+ > -2$. Recall that $-2 \in \mathcal{D}_{\mathrm{AC}}$ if and only if $b^1(\Sigma) \neq 0$ by the argument preceding Proposition 6.3.4. We are thus able to make the following proposition.

Proposition 7.4.4. *Suppose that $\lambda \in (\lambda_-, 0)$, $\lambda \notin \mathcal{D}_{\mathrm{AC}}$ and $b^1(\Sigma) = 0$, which is equivalent to $-2 \notin \mathcal{D}_{\mathrm{AC}}$. The dimension of $\mathcal{M}(N, \lambda)$, as given in Theorem 7.3.2, is*

$$\dim \mathcal{H}^2_+ + \sum_{\mu \in \mathcal{D}_{\mathrm{AC}} \cap (\lambda_-,\, \lambda)} \mathrm{d}(\mu)$$

where $\mathrm{d}(\mu)$ is given in Proposition 6.3.8 and $\mathcal{H}^2_+$ is defined in Example 6.5.3.

Proof. The index of (7.14) is constant for $\lambda_- < \lambda < \lambda_+$ and hence the dimension of the kernel and cokernel are also constant. Therefore, for $\lambda_- < \lambda < \lambda_+$,

$$\dim\, \mathcal{K}_\lambda = \dim\, \mathcal{K}_{-2} = \dim\, \mathcal{H}^2_+$$

by Proposition 7.4.2. Proposition 7.2.5 implies that the dimension of $\mathcal{C}_\lambda$ is $b^3(N)$ for all $\lambda_- < \lambda < 0$. Applying Theorem 6.3.6 and the results of §7.3.2 completes the proof. □

The next result enables us to calculate the dimension of $\mathcal{M}(N, \lambda)$ when $-2 \in \mathcal{D}_{\mathrm{AC}}$.

Proposition 7.4.5. *For $\lambda_- < \lambda < -2$, $\dim \mathcal{K}_\lambda = \dim \mathcal{K}_{-2}$. If $-2 < \lambda < \lambda_+$,*

$$\dim \mathcal{K}_\lambda = \dim \mathcal{K}_{-2} - b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b^3(N).$$

Proof. For $\lambda < -2$ it is clear, since $L^p_{k,\,\lambda} \subseteq L^p_{k,\,-2}$, that $\mathcal{K}_\lambda \subseteq \mathcal{K}_{-2}$.

Since $-2 \in \mathcal{D}_{\mathrm{AC}}$, there exists a closed and coclosed 2-form η on Σ. By the work in [37, §3 & §4], for η to correspond to a form which adds to the kernel of (7.14) at -2, there must exist a closed self-dual 2-form α on N which is asymptotic to the 2-form $\zeta = \eta + \rho^{-1} d\rho \wedge *\eta$ defined on $N \setminus K \cong (R, \infty) \times \Sigma$. However, ζ grows with order $O(\rho^{-2})$ as $\rho \to \infty$ and hence α must at least grow at this rate. Therefore α does not lie in L^2 and hence $\alpha \notin \mathcal{K}_{-2}$.

Using the notation of Proposition 7.4.3, we see that such a form α will exist if and only if $[\eta]$ lies in $p_2(H^2_{\mathrm{dR}}(N))$, since only then can we consider η as a form on $N \setminus K$. The dimension of this space can be calculated using (7.25):

$$\dim\, p_2(H^2_{\mathrm{dR}}(N)) = -b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b^3(N).$$

By Theorem 6.3.6, there are no changes in $\mathcal{K}_\lambda$ for $\lambda_- < \lambda < -2$ and the argument thus far shows that the kernel forms added at $\lambda = -2$ do not lie in $\mathcal{K}_{-2}$. We conclude that $\mathcal{K}_\lambda = \mathcal{K}_{-2}$ for $\lambda_- < \lambda < -2$. The latter part of the proposition follows from the formula and arguments above, since $\mathcal{K}_\lambda$ does not alter for $-2 < \lambda < \lambda_+$. □

Note that $-b^0(\Sigma)+b^1(\Sigma)+b^0(N)-b^1(N)+b^3(N)$ is zero if $b^1(\Sigma)=0$, which can be easily checked using (7.25). Therefore, Proposition 7.4.5 shows that the function $k(\lambda) = \dim\mathcal{K}_\lambda$ is lower semi-continuous at -2 and is continuous there if and only if $-2 \notin \mathcal{D}_{\mathrm{AC}}$. Our next result shows that the function $c(\lambda) = \dim\mathcal{C}_\lambda$ is upper semi-continuous at -2.

Proposition 7.4.6. *For $-2<\lambda<\lambda_+$, $\dim\mathcal{C}_\lambda = \dim\mathcal{C}_{-2}$. If $\lambda_- < \lambda < -2$,*

$$\dim\mathcal{C}_\lambda = b^0(\Sigma) - b^0(N) + b^1(N).$$

Proof. Since $L^q_{l,\,-\lambda-3} \subseteq L^q_{l,\,-1}$ when $\lambda > -2$, $\mathcal{C}_\lambda \subseteq \mathcal{C}_{-2}$ for $\lambda > -2$.

Since $-2 \in \mathcal{D}_{\mathrm{AC}}$, there exists a closed and coclosed 1-form ζ on Σ. The correspondence of ζ to a cokernel form implies, again by [37, §3 & §4], the existence of a 3-form γ on N with $(d_+^* + d)\gamma = 0$, such that $*\gamma$ is asymptotic to ζ defined on $N \setminus K \cong (R,\infty)\times\Sigma$. Clearly, $|\zeta| = O(\rho^{-1})$ as $\rho\to\infty$. Therefore γ does not lie in $\mathcal{C}_{-2}$, as it must grow with at least order $O(\rho^{-1})$.

Using the notation of Proposition 7.4.3 we see, as in the proof of Proposition 7.4.5, that such a 3-form γ will exist if and only if $[\zeta]$ lies in $p_1(H^1_{\mathrm{dR}}(N))$. We calculate the dimension of this space using (7.25):

$$\dim\, p_1(H^1_{\mathrm{dR}}(N)) = b^0(\Sigma) - b^0(N) + b^1(N) - b^3(N).$$

Theorem 6.3.6 and Proposition 7.2.5 imply the result. $\square$

Theorem 6.3.6, along with Propositions 7.2.5 and 7.4.5, give us the dimension of $\mathcal{K}_\lambda$, hence $\mathcal{M}(N,\lambda)$, for all $\lambda \in (-2,0)$ with $\lambda \notin \mathcal{D}_{\mathrm{AC}}$.

Proposition 7.4.7. *Suppose that $\lambda \in (-2,0)$ and $\lambda\notin\mathcal{D}_{\mathrm{AC}}$. The dimension of $\mathcal{M}(N,\lambda)$, as given in Theorem 7.3.2, is*

$$\dim\mathcal{H}^2_+ - b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b^3(N) + \sum_{\mu\in\mathcal{D}_{\mathrm{AC}}\cap(-2,\,\lambda)} \mathrm{d}(\mu)$$

where $\mathrm{d}(\mu)$ is given in Proposition 6.3.8 and $\mathcal{H}^2_+$ is defined in Example 6.5.3.

We now discuss the case $\lambda > 0$ and begin by studying the point $0 \in \mathcal{D}_{\mathrm{AC}}$. Recall that $\mathrm{d}(0)$, as given by Proposition 6.3.8, is equal to the dimension of

$$D_0 = \{(\alpha,\beta) \in C^\infty(\Lambda^2T^*\Sigma \oplus \Lambda^3T^*\Sigma) \,:\, d\alpha = 0 \text{ and } d{*}\alpha + d^*\beta = 2\alpha\}.$$

It is clear, using integration by parts, that the equations which $(\alpha,\beta)\in D_0$ satisfy are equivalent to

$$d{*}\alpha = 2\alpha \quad\text{and}\quad d^*\beta = 0.$$

The latter equation corresponds to constant 3-forms on Σ, so the solution set has dimension equal to $b^0(\Sigma)$. If we define

$$Z = \{\alpha \in C^\infty(\Lambda^2 T^*\Sigma) \, : \, d{*}\alpha = 2\alpha\}, \tag{7.26}$$

then $\mathrm{d}(0) = b^0(\Sigma) + \dim Z$.

Suppose that $\beta_3 \in C^\infty(\Lambda^3 T^*\Sigma)$ satisfies $d^*\beta_3 = 0$ and corresponds to a form on N which adds to the kernel of (7.14) at 0. Then there exists, by [37, §3 & §4], a 4-form β_4 on N asymptotic to the form $\rho^3 d\rho \wedge \beta_3$ on $N \setminus K \cong (R,\infty) \times \Sigma$ and a self-dual 2-form α_2 of order $o(1)$ as $\rho \to \infty$ such that $d\alpha_2 + d^*\beta_4 = 0$.

Since $d^*\beta_4$ is exact, $*\beta_4$ is a harmonic function which is asymptotic to a function c, constant on each *end* of N, as given in Definition 6.1.1. Applying [28, Theorem 7.10] gives a unique harmonic function f on N which converges to c with order $O(\rho^\mu)$ for all $\mu \in (-2,0)$. The theorem cited is stated for an AC special Lagrangian submanifold L, but only uses the fact that L is an AC Riemannian manifold and hence is applicable here. Therefore, $*\beta_4 - f = o(1)$ as $\rho \to \infty$ and hence, by Theorem 1.2.5, $*\beta_4 = f$.

We deduce that $d^*\beta_4$ and $d\alpha_2$ are $O(\rho^{-3+\epsilon})$ as $\rho \to \infty$ for any $\epsilon > 0$ small, hence they lie in L^2. Integration by parts, now justified, shows that $d^*\beta_4 = 0$ and we conclude that $*\beta_4$ is constant on each component of N. Hence, the piece of $b^0(\Sigma)$ in $\mathrm{d}(0)$ that adds to the dimension of the kernel is equal to $b^0(N)$. Note that the other 3-forms β_3 on Σ satisfying $d^*\beta_3 = 0$ must correspond to cokernel forms and so $b^0(\Sigma) - b^0(N)$ is subtracted from the dimension of the cokernel at 0.

For each end of N, we can define self-dual 2-forms of order $O(1)$ given by translations of it, written as $\frac{\partial}{\partial x_j} \cdot \varphi_0$ for $j = 1, \ldots, 7$. If the end is a flat $\mathbb{R}^4$, we only get three such self-dual 2-forms from it. So, if k' is the number of ends which are not 4-planes,

$$\dim Z \geq 7k' + 3(b^0(\Sigma) - k') = 3b^0(\Sigma) + 4k'.$$

Moreover, the translations of the components of N must correspond to kernel forms, since they are genuine deformations of N. Therefore, if k is the number of components of N which are not 4-planes, at least $3b^0(N) + 4k$ is added to the dimension of the kernel at 0 from $\dim Z$.

We state the following inequalities for the dimension of $\mathcal{M}(N,\lambda)$ for $\lambda \in (0,1)$.

Proposition 7.4.8. *Use the notation of Theorem 7.3.2. If* $\lambda \in (0,1)$,

$$\dim \mathcal{M}(N,\lambda) \leq \dim \mathcal{K}_0 + b^0(N) + \dim Z - \dim \mathcal{B}_\lambda + \sum_{\mu \in \mathcal{D}_{\mathrm{AC}} \cap (0,\,\lambda)} \mathrm{d}(\mu)$$

where $\mathrm{d}(\mu)$ *is given in Proposition 6.3.8,* Z *is defined in* (7.26) *and*

$$\dim \mathcal{K}_0 = \dim \mathcal{H}^2_+ - b^0(\Sigma) + b^1(\Sigma) + b^0(N) - b^1(N) + b^3(N) + \sum_{\mu \in \mathcal{D}_{\mathrm{AC}} \cap (-2,\,0)} \mathrm{d}(\mu).$$

Moreover, if k is the number of components of N which are not a flat $\mathbb{R}^4$,

$$\dim\mathcal{M}(N,\lambda)\geq\dim\mathcal{K}_0+b^0(\Sigma)+3b^0(N)+4k-b^3(N)-\dim\mathcal{B}_\lambda+\sum_{\mu\in\mathcal{D}_{\mathrm{AC}}\cap(0,\,\lambda)}\mathrm{d}(\mu).$$

The first inequality follows from the arguments preceding the proposition. The second is deduced from these same arguments and Proposition 7.2.5, which show, in particular, that the dimension of the cokernel can be reduced by at most $b^3(N)-b^0(\Sigma)+b^0(N)$ as λ increases in $(0,1)$.

7.5 Study of Rates $\lambda<-2$

Suppose for this section that N is a coassociative 4-fold which is AC with rate $\lambda<-2$ and suppose, for convenience, that N is connected.

Consider the deformation $N\mapsto tN$ for $t>0$. Clearly tN is coassociative and AC with rate λ for all $t>0$. If v is the dilation vector field given in (7.22), define a self-dual 2-form α on N as in Proposition 2.3.14 by $\alpha=(v\cdot\varphi_0)|_N$, which corresponds to the deformation above. Using (7.23),

$$d\alpha=d(v\cdot\varphi_0)|_N=3\varphi_0|_N=0$$

since N is coassociative. As $\lambda<-2$, $\alpha\in L^2(\Lambda^2_+T^*N)$ and hence $\alpha\in\mathcal{H}^2_+$, defined in Example 6.5.3. Clearly, if $\dim\mathcal{H}^2_+=0$ then $\alpha=0$, which implies that $tN=N$ for all $t>0$; that is, N is a cone and hence $N\cong\mathbb{R}^4$ as it is nonsingular.

Since α lies in L^2,

$$X(N)=\int_N|\alpha|^2=\|\alpha\|^2_{L^2}.$$

This is a well-defined invariant of N. We define a second invariant as follows. Let Γ be a 2-cycle in N and let D be a 3-cycle in $\mathbb{R}^7$ such that $\partial D=\Gamma$. Define $[Y(N)]\in H^2_{\mathrm{dR}}(N)$ by:

$$[Y(N)]\cdot[\Gamma]=\int_D\varphi_0.$$

We show that this is well-defined. Suppose that D and D' are 3-cycles such that $\partial D=\partial D'=\Gamma$. Then $\partial(D-D')=0$ and so

$$\int_{D-D'}\varphi_0=[\varphi_0]\cdot[D-D']=0$$

since $[\varphi_0]\in H^3_{\mathrm{dR}}(\mathbb{R}^7)$ is zero. Therefore $\int_D\varphi_0=\int_{D'}\varphi_0$. Now suppose that Γ and Γ' are 2-cycles in N such that $\Gamma-\Gamma'=\partial E$ for some 3-cycle $E\subseteq N$. Let D and D' be 3-cycles such that $\partial D=\Gamma$ and $\partial D'=\Gamma'$. Then $\partial D=\partial(E+D')=\Gamma$ and thus

$$[Y(N)]\cdot[\Gamma]=\int_D\varphi_0=\int_{E+D'}\varphi_0=\int_E\varphi_0+\int_{D'}\varphi_0=\int_{D'}\varphi_0=[Y(N)]\cdot[\Gamma'],$$

since $E\subseteq N$ and $\varphi_0|_N=0$. Hence $[Y(N)]$ is well-defined.

We notice that $Y(tN) = t^3 Y(N)$, so

$$\frac{d}{dt}(Y(tN))\bigg|_{t=1} = 3Y(N).$$

Since we may consider α as defined by $\frac{d}{dt}|_{t=1}$, the left-hand side of the above equation is equal to α. Hence $3Y(N) = \alpha$. Recall the definition of $\mathcal{J}$ in Example 6.5.3 as the image of $H^2_{\mathrm{cs}}(N)$ in $H^2_{\mathrm{dR}}(N)$. As $\alpha \in \mathcal{H}^2$ and the map from $\mathcal{H}^2$ to $\mathcal{J}$ given by $\gamma \mapsto [\gamma]$ is an isomorphism, $[Y(N)]$ lies in $\mathcal{J}$. In Example 6.5.3 we showed that (6.13) defines a product on $\mathcal{J} \times \mathcal{J}$ and hence

$$9[Y(N)]^2 = [\alpha] \cup [\alpha] = \int_N \alpha \wedge \alpha = \int_N |\alpha|^2 = X(N).$$

We have used the fact that if $\alpha = \eta + d\xi$, for some compactly supported 2-form η and $\xi \in C^\infty_{\lambda+1}(\Lambda^1 T^* N)$, then an integration by parts argument, valid since $\lambda < -2$, shows that

$$\int_N \alpha \wedge \alpha = \int_N \eta \wedge \eta.$$

We have thus derived a test which determines whether N is a cone.

Proposition 7.5.1. *Let N be a connected coassociative 4-fold in $\mathbb{R}^7$ which is AC with rate $\lambda < -2$. If $\dim \mathcal{H}^2_+ = 0$ then N is a cone and hence a linear $\mathbb{R}^4$ in $\mathbb{R}^7$. Moreover, N is a cone if and only if $X(N) = 0$ or, equivalently, $[Y(N)]^2 = 0$.*

We note, for interest, that a similar argument holds for a connected SL m-fold L which is AC with rate $\lambda < -m/2$, in the following sense. By [45, Theorem 3.6], we may associate to the deformation $L \mapsto tL$ a closed and coclosed 1-form α which lies in $L^2(\Lambda^1 T^* L)$, using (6.5) since $\lambda < -m/2$. By Theorem 6.5.2(a), the closed and coclosed 1-forms in L^2 uniquely represent the cohomology classes in $H^1_{\mathrm{cs}}(L)$ or, equivalently, $H^{m-1}_{\mathrm{dR}}(L)$. Therefore, if $b^{m-1}(L) = 0$ then L is a cone and hence a linear $\mathbb{R}^m$ in $\mathbb{C}^m$. Moreover, we may define the invariant $X(L)$ in an analogous way to $X(N)$ and see that L is a cone if and only if $X(L) = 0$.

We define two invariants, $[Y(L)] \in H^1_{\mathrm{dR}}(L)$ and $[Z(L)] \in H^{m-1}_{\mathrm{dR}}(L)$ by

$$[Y(L)] \cdot [\Gamma] = \int_D \omega_m \quad \text{and} \quad [Z(L)] \cdot [\Gamma'] = \int_{D'} \operatorname{Im} \Omega_m,$$

where Γ is a 1-cycle in L, D is a 2-cycle such that $\partial D = \Gamma$, Γ' is an $(m-1)$-cycle in L, D' is an m-cycle such that $\partial D' = \Gamma'$ and ω_m and Ω_m are given in Definition 3.1.1.

We can also define two further invariants of L, $[Y'(L)]$ and $[Z'(L)]$, for certain rates λ. If $\lambda < -1$, we can map α to an element $[Y'(L)] \in H^1_{\mathrm{cs}}(L)$ using a similar construction to the one leading to Proposition 7.2.2. Similarly, if $\lambda < 1 - m$, we may associate to $*\alpha$, and hence to α, an element $[Z'(L)] \in H^{m-1}_{\mathrm{cs}}(L)$.

For $\lambda < -m/2$, since $m \geq 2$, we calculate:

$$2m[Y'(L)] \cdot [Z(L)] = [\alpha] \cup [*\alpha] = \int_L \alpha \wedge *\alpha = \int_L |\alpha|^2 = X(L).$$

Hence $X(L) = 0$ if and only if $[Y'(L)] \cdot [Z(L)] = 0$. If $\lambda < 1 - m$, $X(L) = 0$ if and only if $[Y(L)] \cdot [Z'(L)] = 0$. We write these observations as a proposition.

Proposition 7.5.2. *Let L be a connected SL m-fold in $\mathbb{C}^m$ which is AC with rate $\lambda < -m/2$. If $b^{m-1}(L) = 0$ then L is a cone and hence a linear $\mathbb{R}^m$ in $\mathbb{C}^m$. Moreover, L is a cone if and only if $X(L) = 0$ or, equivalently, $[Y'(L)] \cdot [Z(L)] = 0$. If $\lambda < 1 - m$, L is a cone if and only if $[Y(L)] \cdot [Z'(L)] = 0$.*

7.6 An Example

We return to the SU(2)-invariant coassociative 4-folds M_c given by Theorem 5.2.4. These can formulated as in [17, Theorem IV.3.2] in terms of the *quaternions*, described in §1.3, and the now familiar octonions.

Let $e \in \operatorname{Im}\mathbb{H}$ be a fixed unit vector and let $c \in \mathbb{R}$. Let $\operatorname{Im}\mathbb{O} \cong \operatorname{Im}\mathbb{H} \oplus \langle f \rangle_{\mathbb{H}}$, where $f \in (\operatorname{Im}\mathbb{H})^{\perp} \subseteq \operatorname{Im}\mathbb{O}$ such that $|f| = 1$. Then

$$M_c = \{sqe\bar{q} + r\bar{q}f \,:\, q \in \mathbb{H},\, |q| = 1 \text{ and } s(4s^2 - 5r^2)^2 = c \text{ for } r \geq 0,\, s \in \mathbb{R}\}.$$

Suppose $c \neq 0$ and take $c > 0$ without loss of generality. This forces $s > 0$ and $4s^2 - 5r^2 \neq 0$. It is then clear that M_c has two components, M_c^+ and M_c^-, corresponding to $4s^2 - 5r^2 > 0$ and $4s^2 - 5r^2 < 0$ respectively.

Considering M_c^+, we get one end which is AC to $(0,\infty) \times \mathcal{S}^3$. We calculate the rate as follows. For large r, s is approximately equal to $\frac{\sqrt{5}}{2}r$ and hence $4s^2 - 5r^2 = O(r^{-\frac{1}{2}})$. Therefore $s = \frac{\sqrt{5}}{2}r + O(r^{-\frac{3}{2}})$ and thus M_c^+ converges with rate $-3/2$ to $(0,\infty) \times \mathcal{S}^3$. For each $r \neq 0$ we have an $\mathcal{S}^3$ orbit in M_c^+, but when $r = 0$ there is an $\mathcal{S}^2$ orbit. Therefore, topologically, M_c^+ is an $\mathbb{R}^2$ bundle over $\mathcal{S}^2$. Hence $H^2(M_c^+) = \mathbb{R}$. Suppose, for a contradiction, that $\dim \mathcal{H}^2_+ = 1$. Therefore, there exists a smooth, closed, self-dual 2-form in L^2, which corresponds to a coassociative deformation of M_c^+ that is AC to the SU(2)-invariant cone with rate at most -2. However, this deformation must itself be invariant under SU(2), and thus is AC with rate $-3/2$ as it lies in the family given by Theorem 5.2.4. Hence $\dim \mathcal{H}^2_+ = 0$ and, since the SU(2) action has generic orbit $\mathcal{S}^3$, we conclude that M_c^+ is isomorphic to the bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$.

We now turn to M_c^-. Here there are two ends, one of which has the same behaviour as the end of M_c^+ and the other is where $s \to 0$. For the latter case, we quickly see that $s = O(r^{-4})$ and so

M_c^- converges at rate -4 to $\mathbb{R}^4$. As the case $r = 0$ is excluded here, M_c^- is topologically $\mathbb{R} \times \mathcal{S}^3$ and converges with rate $-3/2$ to $(0, \infty) \times \mathcal{S}^3$ and rate -4 to $\mathbb{R}^4$ at its two ends. Hence $H^2(M_c^-) = 0$.

Consequently, for $c \neq 0$, $\dim \mathcal{H}_+^2 = 0$ for M_c. However, the rate $-3/2$ is greater than -2, so M_c has a nontrivial deformation space.

For $c = 0$, M_0 is a cone with three ends, two of which are diffeomorphic to $(0, \infty) \times \mathcal{S}^3$ and the third, corresponding to $s \equiv 0$, is a flat $\mathbb{R}^4$.

Chapter 8

Deformation Theory of Coassociative 4-folds with Conical Singularities

This final chapter is dedicated to the study of *deformations* of coassociative 4-folds in a G_2 manifold which have *conical singularities*. In Section 8.2, we stratify the types of deformations allowed into three problems, each with an associated nonlinear first order differential operator whose kernel gives a local description of the moduli space. The main result for each problem, given in §8.3, states that the moduli space is locally homeomorphic to the kernel of a smooth map between smooth manifolds. Furthermore, using the material in Chapter 6 helps to provide a lower bound on the expected dimension of these moduli spaces. The last section shows that, in weakening the condition on the G_2 structure of the ambient 7-manifold, there is a generic smoothness result for the moduli spaces of deformations corresponding to our second and third problems. The study of the deformation theory of special Lagrangian m-folds with conical singularities by Joyce in the series of papers [28], [29], [30], [31] and [32] motivates the research detailed here.

8.1 Basic Theory

Let $B(0;\eta)$ denote the open ball about 0 in $\mathbb{R}^7$ with radius $\eta > 0$, i.e. $B(0;\eta) = \{\mathbf{v} \in \mathbb{R}^7 : |\mathbf{v}| < \eta\}$. We define a preferred choice of local coordinates on a G_2 manifold near a finite set of points.

Definition 8.1.1. Let (M, φ, g) be a G_2 manifold as in Definition 2.3.11 and let $z_1, \ldots, z_s$ be points in M. There exist a constant $\eta > 0$, an open set $V_i \ni z_i$ in M with $V_i \cap V_j = \emptyset$ for

$j \neq i$ and a diffeomorphism $\chi_i : B(0;\eta) \subseteq \mathbb{R}^7 \to V_i$ with $\chi_i(0) = z_i$, for $i = 1, \dots, s$, such that $\zeta_i = d\chi_i|_0 : \mathbb{R}^7 \to T_{z_i}M$ is an isomorphism identifying the standard G_2 structure (φ_0, g_0) on $\mathbb{R}^7$ with the pair $(\varphi|_{T_{z_i}M}, g|_{T_{z_i}M})$. We call the set $\{\chi_i : B(0;\eta) \to V_i : i = 1, \dots, s\}$ a G_2 *coordinate system near* $z_1, \dots, z_s$.

We say that two G_2 coordinate systems near $z_1, \dots, z_s$, with maps χ_i and $\tilde{\chi}_i$ for $i = 1, \dots, s$ respectively, are *equivalent* if $d\tilde{\chi}_i|_0 = d\chi_i|_0 = \zeta_i$ for all i.

The definition above is an analogue of the local coordinate system for almost Calabi–Yau manifolds used by Joyce [20, Definition 3.6]. Although the family of G_2 coordinate systems near $z_1, \dots, z_s$ is clearly infinite-dimensional, there are only finitely many equivalence classes, given by the number of possible sets $\{\zeta_1, \dots, \zeta_s\}$. Moreover, the family of choices for each ζ_i is isomorphic to G_2. Note also that this definition does not require the G_2 structure (φ, g) to be *torsion-free*, in the sense of Definition 2.3.9.

Definition 8.1.2. Let (M, φ, g) be a G_2 manifold, let $N \subseteq M$ be compact and let $z_1, \dots, z_s \in N$. We say that N is a 4-fold in M with *conical singularities at* $z_1, \dots, z_s$ *with rate* λ, denoted a *CS 4-fold*, if $\hat{N} = N \setminus \{z_1, \dots, z_s\}$ is a (nonsingular) 4-dimensional submanifold of M and there exist constants $0 < \epsilon < \eta$ and $\lambda > 1$, a compact 3-dimensional Riemannian submanifold (Σ_i, h_i) of $\mathcal{S}^6 \subseteq \mathbb{R}^7$, where h_i is the restriction of the round metric on $\mathcal{S}^6$ to Σ_i, an open set $U_i \ni z_i$ in N with $U_i \subseteq V_i$ and a smooth map $\Phi_i : (0, \epsilon) \times \Sigma_i \to B(0;\eta) \subseteq \mathbb{R}^7$, for $i = 1, \dots, s$, such that $\Psi_i = \chi_i \circ \Phi_i : (0, \epsilon) \times \Sigma_i \to U_i \setminus \{z_i\}$ is a diffeomorphism and Φ_i satisfies

$$|\nabla_i^j (\Phi_i(r_i, \sigma_i) - \iota_i(r_i, \sigma_i))| = O(r_i^{\lambda - j}) \qquad \text{for } j \in \mathbb{N} \text{ as } r_i \to 0, \tag{8.1}$$

where $\iota_i(r_i, \sigma_i) = r_i \sigma_i \in B(0;\eta)$, ∇_i is the Levi–Civita connection of the cone metric $g_i = dr_i^2 + r_i^2 h_i$ on $C_i = (0, \infty) \times \Sigma_i$ coupled with partial differentiation on $\mathbb{R}^7$, $|.|$ is calculated with respect to g_i and $\{\chi_i : B(0;\eta) \to V_i : i = 1, \dots, s\}$ is a G_2 coordinate system near $z_1, \dots, z_s$.

We call C_i the *cone* at the singularity z_i and Σ_i the *link* of the cone C_i. We may write N as the disjoint union

$$N = K \sqcup \bigsqcup_{i=1}^{s} U_i,$$

where K is compact.

If $\hat{N}$ is coassociative in M, we say that N is a *CS coassociative 4-fold*.

Suppose N is a CS 4-fold at $z_1, \dots, z_s$ with rate λ in (M, φ, g) and use the notation of Definition 8.1.2. The induced metric on $\hat{N}$, $g|_{\hat{N}}$, makes $\hat{N}$ into a Riemannian manifold. Moreover, it is clear from (8.1) that the maps Ψ_i satisfy (6.2) in Definition 6.1.3 with the same constant λ. Thus, N may be considered as a CS manifold with rate λ.

It is important to note that, if $\lambda \in (1,2)$, Definition 8.1.2 is independent of the choice of G_2 coordinate system near the singularities, up to equivalence. Suppose we have two equivalent coordinate systems defined using maps χ_i and $\tilde{\chi}_i$. These maps must agree up to second order since the zero and first order behaviour of each is prescribed, as stated in Definition 8.1.1. Therefore, the transformed maps $\tilde{\Phi}_i$ corresponding to $\tilde{\chi}_i$ such that $\tilde{\Psi}_i = \tilde{\chi}_i \circ \tilde{\Phi}_i = \chi_i \circ \Phi_i = \Psi_i$ are defined by:

$$\tilde{\Phi}_i = (\tilde{\chi}_i^{-1} \circ \chi_i) \circ \Phi_i.$$

Hence

$$|\nabla_i^j(\tilde{\Phi}_i(r_i,\sigma_i) - \Phi_i(r_i,\sigma_i))| = O(r_i^{2-j}) \qquad \text{for } j \in \mathbb{N} \text{ as } r_i \to 0,$$

where ∇_i and $|.|$ are calculated as in Definition 8.1.2. Thus, in order that the terms generated by the transformation of the G_2 coordinate system neither dominate nor be of equal magnitude to the $O(r_i^{\lambda-j})$ terms given in (8.1), we need $\lambda < 2$.

We now make a definition which also depends only on equivalence classes of G_2 coordinate systems near the singularities.

Definition 8.1.3. Let N be a CS 4-fold at $z_1,\ldots,z_s$ in a G_2 manifold (M,φ,g). Use the notation of Definitions 8.1.1 and 8.1.2. For $i = 1,\ldots,s$ define a cone $\hat{C}_i$ in $T_{z_i}M$ by $\hat{C}_i = (\zeta_i \circ \iota_i)(C_i)$. We call $\hat{C}_i$ the *tangent cone* at z_i .

One can show that $\hat{C}_i$ is a tangent cone to N at z_i in the sense of *geometric measure theory* (see, for example, [13, p. 233]). We also have a straightforward result relating to the tangent cones at singular points of CS coassociative 4-folds.

Proposition 8.1.4. *Let N be a CS coassociative 4-fold at $z_1,\ldots,z_s$ in a G_2 manifold (M,φ,g). The tangent cones at $z_1,\ldots,z_s$ are coassociative.*

Proof. Use the notation of Definitions 8.1.1 and 8.1.2.

It is enough to show that $\iota_i(C_i)$ is coassociative in $\mathbb{R}^7$ for all i, since $\zeta_i : \mathbb{R}^7 \to T_{z_i}M$ is an isomorphism identifying (φ_0, g_0) with $(\varphi|_{T_{z_i}M}, g|_{T_{z_i}M})$. This is equivalent to the condition $\iota_i^*(\varphi_0) \equiv 0$ for $i = 1,\ldots,s$.

Note that $\varphi|_{\hat{N}} \equiv 0$ implies that, for all i, $\varphi|_{U_i \setminus \{z_i\}} \equiv 0$. Hence, $\Psi_i^*(\varphi) = \Phi_i^*(\chi_i^*(\varphi))$ vanishes on C_i for all i. Using (8.1),

$$|\Phi_i^*(\chi_i^*(\varphi)) - \iota_i^*(\chi_i^*(\varphi))| = O(r_i^{\lambda-1}) \qquad \text{as } r_i \to 0$$

for all i. Moreover,

$$|\iota_i^*(\chi_i^*(\varphi)) - \iota_i^*(\varphi_0)| = O(r_i) \qquad \text{as } r_i \to 0$$

since

$$\chi_i^*(\varphi) = \varphi_0 + O(r_i) \quad \text{and} \quad |\nabla\iota_i| = O(1) \qquad \text{as } r_i \to 0.$$

Therefore, because $\lambda > 1$,

$$|\iota_i^*(\varphi_0)| \to 0 \qquad \text{as } r_i \to 0$$

for all i. As $T_{r_i\sigma_i}\iota_i(C_i) = T_{\sigma_i}\iota_i(C_i)$ for all $(r_i, \sigma_i) \in C_i$, $|\iota_i^*(\varphi_0)|$ is independent of r_i and thus vanishes for all i as required. □

8.2 The Deformation Problems

We have a common notation for the next three sections. Let N be a CS coassociative 4-fold at $z_1, \ldots, z_s$ with rate λ in a G_2 manifold (M, φ, g). Suppose $\lambda \in (1, 2) \setminus \mathcal{D}_{\mathrm{CS}}$, where $\mathcal{D}_{\mathrm{CS}}$ is defined in Proposition 6.3.4(b), and the cone at z_i is C_i with link Σ_i. We shall then use the notation of Definitions 8.1.2 and 8.1.3. In particular, we let $\{\chi_i : B(0;\eta) \to V_i \, : \, i = 1, \ldots, s\}$, with $d\chi_i|_0 = \zeta_i$ for all i, be the G_2 coordinate system near $z_1, \ldots, z_s$ used to define N and let $\hat{C}_i$ be the tangent cone at z_i. Recalling that N is a CS manifold, in the sense of Definition 6.1.3, we therefore have a radius function $\rho : \hat{N} \to (0, 1]$ on $\hat{N}$ as in Definition 6.1.4.

We consider deformations of N which are CS coassociative 4-folds at s points with rate λ in (M, φ, g) with the same cones at the singularities as N, but the singularities need not be at the same points, nor have identical tangent cone. We also, eventually, consider deforming the G_2 structure on the ambient 7-manifold M.

8.2.1 Problem 1: fixed singularities and G_2 structure

The first deformation problem we consider is where the deformations of N have identical singular points to N with the same rate, cones and tangent cones, and the G_2 structure of M is fixed.

Definition 8.2.1. The *moduli space of deformations* $\mathcal{M}_1(N, \lambda)$ for Problem 1 is the set of N' in (M, φ, g) which are CS coassociative 4-folds at $z_1, \ldots, z_s$ with rate λ, having cone C_i and tangent cone $\hat{C}_i$ at z_i for all i, such that there exists a homeomorphism $h : N \to N'$, isotopic to the identity, with $h(z_i) = z_i$ for $i = 1, \ldots, s$ and such that $h|_{\hat{N}} : \hat{N} \to N' \setminus \{z_1, \ldots, z_s\}$ is a diffeomorphism.

We begin our formulation of a local description of $\mathcal{M}_1(N, \lambda)$ with a corollary to Theorem 7.1.2 which is an analogue to Corollary 7.1.3.

Corollary 8.2.2. *For $i = 1, \ldots, s$ choose $\Phi_i : (0, \epsilon) \times \Sigma_i \to B(0;\eta) \subseteq \mathbb{R}^7$ uniquely by imposing the condition that*

$$\Phi_i(r_i, \sigma_i) - \iota_i(r_i, \sigma_i) \in (T_{r_i\sigma_i}\iota_i(C_i))^{\perp}$$

for all $(r_i, \sigma_i) \in (0,\epsilon) \times \Sigma_i$, which can be achieved by making ϵ smaller and K larger if necessary. Let $P_i = \iota_i((0,\epsilon) \times \Sigma_i)$, $Q_i = \Phi_i((0,\epsilon) \times \Sigma_i)$ and define $n_i : \nu(P_i) \to \mathbb{R}^7$ by $n_i(r_i\sigma_i, v) = v + \Phi_i(r_i, \sigma_i)$. For all i, there exist an open subset $\hat{V}_i$ of $\nu(P_i)$ in $\mathbb{R}^7$, containing the zero section, and an open set $\hat{S}_i$ in $B(0;\eta) \subseteq \mathbb{R}^7$ containing Q_i such that $n_i|_{\hat{V}_i} : \hat{V}_i \to \hat{S}_i$ is a diffeomorphism. Moreover, $\hat{V}_i$ and $\hat{S}_i$ can be chosen to grow like r_i on $(0,\epsilon) \times \Sigma_i$, for all i, and such that $P_i \subseteq \hat{S}_i$.

The proof of this result is almost identical to that of Corollary 7.1.3 so we omit it. Corollary 8.2.2 provides us with an analogue to Proposition 7.1.5.

Proposition 8.2.3. *There exist an open set $\hat{U} \subseteq \Lambda^2_+ T^*\hat{N}$ containing the zero section, an open set $\hat{T} \subseteq M$ containing $\hat{N}$ and a diffeomorphism $\delta : \hat{U} \to \hat{T}$ which takes the zero section to $\hat{N}$. Moreover, $\hat{U}$ and $\hat{T}$ can be chosen to grow with order $O(\rho)$ as $\rho \to 0$ and δ is compatible with the identifications $U_i \setminus \{z_i\} \cong (0,\epsilon) \times \Sigma_i$ for all i and the isomorphism $\jmath : \nu(\hat{N}) \to \Lambda^2_+ T^*\hat{N}$ given in Proposition 2.3.14.*

Proof. Use the notation of Corollary 8.2.2 and define $\hat{T}_i = \chi_i(\hat{S}_i)$. Then $\hat{T}_i$ is an open set in M such that $U_i \setminus \{z_i\} \subseteq \hat{T}_i \subseteq V_i$, since $\chi_i(Q_i) = U_i \setminus \{z_i\}$, and which grows with order $O(\rho)$ as $\rho \to 0$.

Consider the bundle $(\Lambda^2_+)_{\chi_i^*(g)} T^*((0,\epsilon) \times \Sigma_i)$, where the notation $(\Lambda^2_+)_h$ indicates that the Hodge star is calculated using the metric h and we consider $(0,\epsilon) \times \Sigma_i \cong P_i \subseteq \mathbb{R}^7$. Then

$$\begin{aligned} \jmath_i : \nu(P_i) &\longrightarrow (\Lambda^2_+)_{\chi_i^*(g)} T^* P_i \\ v|_{r_i\sigma_i} &\longmapsto \left(v|_{r_i\sigma_i} \cdot \chi_i^*(\varphi)|_{\Phi_i(r_i,\sigma_i)} \right) |_{T_{r_i\sigma_i} P_i} \end{aligned}$$

is an isomorphism because $U_i \setminus \{z_i\}$ is coassociative and thus P_i is, with respect to the metric $\chi_i^*(g)$ and 3-form $\chi_i^*(\varphi)$, and hence we may apply Proposition 2.3.14. Note also that

$$\Psi_i^* : (\Lambda^2_+)_g T^*(U_i \setminus \{z_i\}) \longrightarrow (\Lambda^2_+)_{\chi_i^*(g)} T^*((0,\epsilon) \times \Sigma_i)$$

is clearly a diffeomorphism. Therefore, let $\hat{U}_i \subseteq (\Lambda^2_+)_g T^*(U_i \setminus \{z_i\})$ be such that $\Psi_i^*(\hat{U}_i) = \jmath_i(\hat{V}_i)$. Note, by construction, that $\hat{U}_i$ grows with order $O(\rho)$ as $\rho \to 0$.

Define a diffeomorphism $\delta_i : \hat{U}_i \to \hat{T}_i$ such that the following diagram commutes:

$$\begin{tikzcd} \hat{U}_i \arrow[r, "\Psi_i^*"] \arrow[dd, "\delta_i"'] & \jmath_i(\hat{V}_i) \arrow[d, "\jmath_i^{-1}"] \\ & \hat{V}_i \arrow[d, "n_i"] \\ \hat{T}_i & \hat{S}_i. \arrow[l, "\chi_i"'] \end{tikzcd} \tag{8.2}$$

Interpolating smoothly over K, we extend $\bigcup_{i=1}^s \hat{U}_i$ and $\bigcup_{i=1}^s \hat{T}_i$ to $\hat{U}$ and $\hat{T}$ as required and extend the diffeomorphisms δ_i smoothly to a diffeomorphism $\delta : \hat{U} \to \hat{T}$ such that δ acts as the identity on $\hat{N}$, which is identified with the zero section in $\Lambda^2_+ T^*\hat{N}$.

Note that we have a splitting $T\hat{U}|_{(x,0)} = T_x\hat{N} \oplus \Lambda^2_+ T^*_x\hat{N}$ for all $x \in \hat{N}$. Thus we can consider $d\delta$ at $\hat{N}$ as a map from $T\hat{N} \oplus \Lambda^2_+ T^*\hat{N}$ to $T\hat{N} \oplus \nu(\hat{N}) \cong TM|_{\hat{N}}$. Hence, we require in our extension of δ from δ_i to ensure that, in matrix notation,

$$d\delta|_{\hat{N}} = \begin{pmatrix} I & A \\ 0 & \jmath^{-1} \end{pmatrix}, \tag{8.3}$$

where I is the identity and A is arbitrary. This can be achieved because of the definition of δ_i.

The compatibility of δ with $\jmath$ and Ψ_i for all i, mentioned in the statement of the proposition, is given by (8.2) and the behaviour of $d\delta|_{\hat{N}}$ stipulated in (8.3). □

We now define our deformation map for Problem 1. Let $C^k_{\mathrm{loc}}(\hat{U}) = \{\alpha \in C^k_{\mathrm{loc}}(\Lambda^2_+ T^*\hat{N}) : \alpha \in \hat{U}\}$, where $\hat{U}$ is given in Proposition 8.2.3, and adopt similar notation, as in §7.1.2, to define subsets of the spaces of forms described in §6.2.

Definition 8.2.4. Use the notation of Proposition 8.2.3. Let Γ_α be the graph of $\alpha \in C^1_{\mathrm{loc}}(\hat{U})$ and let $\pi_\alpha : \hat{N} \to \Gamma_\alpha$ be given by $\pi_\alpha(x) = (x, \alpha(x))$. Let $f_\alpha = \delta \circ \pi_\alpha$ and let $\hat{N}_\alpha = f_\alpha(\hat{N}) \subseteq \hat{T}$. Define a map F_1 from $C^1_{\mathrm{loc}}(\hat{U})$ to $C^0_{\mathrm{loc}}(\Lambda^3 T^*\hat{N})$ by:

$$F_1(\alpha) = f^*_\alpha(\varphi|_{\hat{N}_\alpha}).$$

By [45, p. 731], which we are allowed to use by our choice of δ, the linearisation of F_1 at 0 is

$$dF_1|_0(\alpha) = L_1(\alpha) = d\alpha$$

for all $\alpha \in C^1_{\mathrm{loc}}(\Lambda^2_+ T^*\hat{N})$.

By Proposition 2.3.13, $\operatorname{Ker} F_1$ is the set of $\alpha \in C^1_{\mathrm{loc}}(\hat{U})$ such that $\hat{N}_\alpha$ is coassociative.

However, we want CS coassociative deformations with singularities at the same points with the same tangent cones. Suppose $\alpha \in C^1_{\mathrm{loc}}(\hat{U})$ and $N_\alpha = \hat{N}_\alpha \cup \{z_1, \ldots, z_s\}$ is such a deformation. Then there exist smooth maps $(\Phi_\alpha)_i : (0, \epsilon) \times \Sigma_i \to B(0; \eta)$ satisfying (8.1) such that $(\Psi_\alpha)_i = \chi_i \circ (\Phi_\alpha)_i$ is a diffeomorphism onto an open subset of $\hat{N}_\alpha$ for all i as in Definition 8.1.2. Note that we are free to use χ_i because the tangent cones at the singularities of N_α must be the same as for N, so any G_2 coordinate system near the singularities used to define N_α must be equivalent to the one given by χ_i for $i = 1, \ldots, s$. Choose $(\Phi_\alpha)_i$ uniquely such that

$$(\Phi_\alpha)_i(r_i, \sigma_i) - \iota_i(r_i, \sigma_i) \in (T_{r_i\sigma_i}\iota_i(C_i))^\perp$$

for all $(r_i, \sigma_i) \in (0, \epsilon) \times \Sigma_i$.

Use the notation of Corollary 8.2.2 and the proof of Proposition 8.2.3. Since

$$\Phi_i(r_i, \sigma_i) - \iota_i(r_i, \sigma_i) \in (T_{r_i\sigma_i}P_i)^\perp \cong \nu_{r_i\sigma_i}(P_i),$$

$\Phi_i - \iota_i$ can be identified using $\jmath_i$ with the graph of $\beta_i \in (\Lambda^2_+)_{\chi_i^*(g)} T^*((0,\epsilon) \times \Sigma_i)$. Thus,

$$|\nabla_i^j \beta_i| = O(r_i^{\lambda - j}) \qquad \text{for } j \in \mathbb{N} \text{ as } r_i \to 0$$

by (8.1) and therefore $\beta_i \in C^\infty_\lambda((\Lambda^2_+)_{\chi_i^*(g)} T^*((0,\epsilon) \times \Sigma_i))$.

We may similarly deduce, by the definition of δ, Φ_i and $(\Phi_\alpha)_i$, that $(\Phi_\alpha)_i - \iota_i = ((\Phi_\alpha)_i - \Phi_i) + (\Phi_i - \iota_i)$ corresponds to the graph of $\Psi_i^*(\alpha) + \beta_i$ on $(0,\epsilon) \times \Sigma_i$, recalling that

$$\Psi_i^* : \Lambda^2_+ T^*(U_i \setminus \{z_i\}) \to (\Lambda^2_+)_{\chi_i^*(g)} T^*((0,\epsilon) \times \Sigma_i)$$

is a diffeomorphism for all i. Since N_α has the same types of singularities as N, both β_i and $\Psi_i^*(\alpha) + \beta_i$ lie in $C^\infty_\lambda((\Lambda^2_+)_{\chi_i^*(g)} T^*((0,\epsilon) \times \Sigma_i))$ for each i. Thus α must lie in $C^\infty_\lambda(\Lambda^2_+ T^*\hat{N})$.

We conclude that $\hat{N}_\alpha$ is a sufficiently nearby deformation of $\hat{N}$ with the same conical singularities if and only if $\alpha \in C^\infty_\lambda(\hat{U}) \subseteq C^\infty_\lambda(\Lambda^2_+ T^*\hat{N})$. We state this as a proposition.

Proposition 8.2.5. *The moduli space of deformations for Problem 1 is locally homeomorphic to* $\operatorname{Ker} F_1 = \{\alpha \in C^\infty_\lambda(\hat{U}) : F_1(\alpha) = 0\}$.

We define an associated map G_1 to F_1 in a similar manner to Definition 7.1.8.

Definition 8.2.6. Define $G_1 : C^1_{\mathrm{loc}}(\hat{U}) \times C^1_{\mathrm{loc}}(\Lambda^4 T^*\hat{N}) \to C^0_{\mathrm{loc}}(\Lambda^3 T^*\hat{N})$ by:

$$G_1(\alpha, \beta) = F_1(\alpha) + d^*\beta.$$

Then G_1 is a first order elliptic operator at $(0,0)$ since

$$dG_1|_{(0,0)} = d + d^* : C^1_{\mathrm{loc}}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \longrightarrow C^0_{\mathrm{loc}}(\Lambda^3 T^*\hat{N}).$$

If $G_1(\alpha,\beta) = 0$ and $\beta \in C^\infty_\lambda(\Lambda^4 T^*\hat{N})$, $*\beta$ is a harmonic function which decays with order $O(\rho^\lambda)$ as $\rho \to 0$. Since $\lambda > 1$, $*\beta \to 0$ as $\rho \to 0$ and hence, by the Maximum Principle (Theorem 1.2.5), it must be 0. We therefore deduce the following.

Proposition 8.2.7. $\operatorname{Ker} F_1 \cong \{(\alpha,\beta) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^*\hat{N}) : G_1(\alpha,\beta) = 0\}$.

The work on regularity in §7.1.3 for the AC deformation problem applies, with minor modification, to Problem 1. We may thus state the analogous result to Proposition 7.1.10.

Proposition 8.2.8. *Let* $(\alpha,\beta) \in L^p_{k+1,\lambda}(\hat{U}) \times L^p_{k+1,\lambda}(\Lambda^4 T^*\hat{N})$ *for some* $p > 4$ *and* $k \geq 2$. *If* $G_1(\alpha,\beta) = 0$ *and* $\|\alpha\|_{C^1_1}$ *is sufficiently small,* $(\alpha,\beta) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^*\hat{N})$.

8.2.2 Problem 2: moving singularities and fixed G_2 structure

For this problem we again consider deformations of N in (M, φ, g) which are CS coassociative 4-folds at s points with the same rate and cones at the singularities, but now we allow the singular points and tangent cones at those points to differ from those of N. However, we still assume that the G_2 structure on M is fixed.

Definition 8.2.9. The *moduli space of deformations* $\mathcal{M}_2(N, \lambda)$ for Problem 2 is the set of N' in (M, φ, g) which are CS coassociative 4-folds at $z_1', \ldots, z_s'$ with rate λ, having cone C_i and tangent cone $\hat{C}_i'$ at z_i' for all i, such that there exists a homeomorphism $h : N \to N'$, isotopic to the identity, with $h(z_i) = z_i'$ for $i = 1, \ldots, s$ and such that $h|_{\hat{N}} : \hat{N} \to N' \setminus \{z_1', \ldots, z_s'\}$ is a diffeomorphism.

Here it is more difficult to create a local description of the moduli space which is compatible with the analytic framework in which our study is made. What one would consider more 'intuitive' approaches do not, as far as the author is aware, bear fruit. We therefore follow what is, at first sight, a slightly indirect route.

For each $i = 1, \ldots, s$ let B_i be an open set in M containing z_i such that $B_i \cap B_j = \emptyset$ for $i \neq j$. Let $B = \prod_{i=1}^{s} B_i$. For each $\mathbf{z}' = (z_1', \ldots, z_s') \in B$, we have a family $I(\mathbf{z}')$ of choices of s-tuples $\boldsymbol{\zeta}' = (\zeta_1', \ldots, \zeta_s')$ of isomorphisms $\zeta_i' : \mathbb{R}^7 \to T_{z_i'}M$ identifying (φ_0, g_0) with $(\varphi|_{T_{z_i'}M}, g|_{T_{z_i'}M})$. Clearly, for each $\mathbf{z}' \in B$, $I(\mathbf{z}') \cong G_2$. We thus make the following definition.

Definition 8.2.10. The *translation space* is

$$\mathcal{T} = \{(\mathbf{z}', \boldsymbol{\zeta}') \,:\, \mathbf{z}' \in B,\, \boldsymbol{\zeta}' \in I(\mathbf{z}')\}.$$

Note that $\mathcal{T}$ is a principal G_2^s bundle over B and hence is a smooth manifold.

Let H_i denote the Lie subgroup of G_2 preserving $\iota_i(C_i)$ in $\mathbb{R}^7$ for $i = 1, \ldots, s$ and let $H = \prod_{i=1}^{s} H_i \subseteq G_2^s$. Then H acts freely on $\mathcal{T}$ by

$$(\mathbf{z}', \boldsymbol{\zeta}') \longmapsto (\mathbf{z}', (\zeta_1' \circ A_1^{-1}, \ldots, \zeta_s' \circ A_s^{-1})),$$

where $(A_1, \ldots, A_s) \in H$. Thus there exists an H-orbit through $(\mathbf{z}, \boldsymbol{\zeta})$ in $\mathcal{T}$, where

$$\mathbf{z} = (z_1, \ldots, z_s) \quad \text{and} \quad \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_s).$$

Define $\hat{\mathcal{T}}$ to be a small open ball in $\mathbb{R}^n$ containing 0, where $n = \dim \mathcal{T} - \dim H$, and let $h_{\hat{\mathcal{T}}} : \hat{\mathcal{T}} \to \mathcal{T}$ be an embedding with $h_{\hat{\mathcal{T}}}(0) = (\mathbf{z}, \boldsymbol{\zeta})$ such that $h_{\hat{\mathcal{T}}}(\hat{\mathcal{T}})$ is transverse to the H-orbit through $(\mathbf{z}, \boldsymbol{\zeta})$. Hence, $\dim \mathcal{T} = \dim \hat{\mathcal{T}} + \dim H$. Write $h_{\hat{\mathcal{T}}}(t) = (\mathbf{z}(t), \boldsymbol{\zeta}(t))$ for $t \in \hat{\mathcal{T}}$, with $\mathbf{z}(0) = \mathbf{z}$ and $\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}$.

Our choice of $\hat{\mathcal{T}}$ ensures that if $t, t' \in \hat{\mathcal{T}}$, with $t \neq t'$, are such that $\mathbf{z}(t) = \mathbf{z}(t')$, the s-tuples of tangent cones, $\{\hat{C}_1(t), \ldots, \hat{C}_s(t)\}$ and $\{\hat{C}_1(t'), \ldots, \hat{C}_s(t')\}$, are distinct. Furthermore, $\hat{\mathcal{T}}$ is an open ball in $\mathbb{R}^n \cong T_0\hat{\mathcal{T}}$ and hence can be considered as an open subset of $T_0\hat{\mathcal{T}}$.

We use $\hat{\mathcal{T}}$ to extend N to a family of nearby CS 4-folds and provide an analogue to Proposition 8.2.3 for Problem 2. In defining N we chose a G_2 coordinate system $\{\chi_i : B(0;\eta) \to V_i : i = 1, \ldots, s\}$ with $d\chi_i|_0 = \zeta_i$ for $i = 1, \ldots, s$. Extend this to a smooth family of G_2 coordinate systems

$$\Big\{\{\chi_i(t) : B(0;\eta) \to V_i(t) : i = 1, \ldots, s\} : t \in \hat{\mathcal{T}}\Big\},$$

where $V_i(t)$ is an open set in M containing $z_i(t)$, $\chi_i(t)(0) = z_i(t)$, $d\chi_i(t)|_0 = \zeta_i(t)$, $\chi_i(0) = \chi_i$ and $V_i(0) = V_i$ for $i = 1, \ldots, s$.

Proposition 8.2.11. *Use the notation of Proposition 8.2.3 and Definition 8.2.10.*

(a) There exists a family $\mathcal{N} = \{N(t) : t \in \hat{\mathcal{T}}\}$ of CS 4-folds in M, with $N(0) = N$, such that $N(t)$ has singularities at $z_1(t), \ldots, z_s(t)$ with rate λ, cones $C_1, \ldots, C_s$ and tangent cones $\hat{C}_1(t), \ldots, \hat{C}_s(t)$ defined by $\hat{C}_i(t) = (\zeta_i(t) \circ \iota_i)(C_i)$.

(b) Let $\hat{N}(t) = N(t) \setminus \{z_1(t), \ldots, z_s(t)\}$ and write

$$N(t) = K(t) \sqcup \bigsqcup_{i=1}^{s} U_i(t)$$

where $K(t)$ is compact and $U_i(t) \setminus \{z_i(t)\} \cong (0, \epsilon) \times \Sigma_i$ for all i, in the obvious way, ensuring that $K(0) = K$ and $U_i(0) = U_i$. For $t \in \hat{\mathcal{T}}$, there exist open sets $\hat{T}(t) \subseteq M$ containing $\hat{N}(t)$ and diffeomorphisms $\delta(t) : \hat{U} \to \hat{T}(t)$ taking the zero section to $\hat{N}(t)$, varying smoothly in t, with $\hat{T}(0) = \hat{T}$ and $\delta(0) = \delta$. Moreover, $\hat{T}(t)$ can be chosen to grow with order $O(\rho)$ as $\rho \to 0$ and $\delta(t)$ is compatible with the identifications $U_i(t) \setminus \{z_i(t)\} \cong (0, \epsilon) \times \Sigma_i$ for all i.

Note that $\mathcal{N}$ does *not* necessarily consist of CS *coassociative* 4-folds and that $\delta(t)$ is not required to be compatible with the isomorphism $\nu(\hat{N}) \cong \Lambda^2_+ T^*\hat{N}$ for $t \neq 0$.

Proof. Use the notation from the proof of Proposition 8.2.3. For $t \in \hat{\mathcal{T}}$, define $\hat{T}_i(t) = \chi_i(t)(\hat{S}_i)$ and

$$U_i(t) = \Big(\chi_i(t) \circ \Phi_i((0, \epsilon) \times \Sigma_i)\Big) \cup \{z_i(t)\}$$

for $i = 1, \ldots, s$. Then $\hat{T}_i(t)$ contains $U_i(t) \setminus \{z_i(t)\}$. Define a diffeomorphism $\delta_i(t)$ such that the following diagram commutes:

$$\begin{tikzcd}
\hat{U}_i \arrow[r, "\Psi_i^*"] \arrow[dd, "\delta_i(t)"'] & \jmath_i(\hat{V}_i) \arrow[d, "\jmath_i^{-1}"] \\
 & \hat{V}_i \arrow[d, "n_i"] \\
\hat{T}_i(t) & \hat{S}_i. \arrow[l, "\chi_i(t)"']
\end{tikzcd} \tag{8.4}$$

We then interpolate smoothly over K to extend $\bigcup_{i=1}^{s} \hat{T}_i(t)$ to $\hat{T}(t)$ and $\delta_i(t)$ to $\delta(t)$ as required. Note by construction that $\hat{T}(t)$ grows with order $O(\rho)$ as $\rho \to 0$.

Let $e(t) = \delta(t)|_{\hat{N}}$ and define $\hat{N}(t) = e(t)(\hat{N})$. Then $e(t) : \hat{N} \to \hat{N}(t)$ is a diffeomorphism for all $t \in \hat{\mathcal{T}}$ and $e(0)$ is the identity. Let $N(t) = \hat{N}(t) \cup \{z_1(t), \ldots, z_s(t)\}$. We then have a family $\mathcal{N} = \{N(t) : t \in \hat{\mathcal{T}}\}$ as claimed. Note that $K(t) = e(t)(K)$.

By the construction of $\delta(t)$ and the family $\mathcal{N}$, it is clear that the proposition is proved, where the compatibility conditions on $\delta(t)$ are given by (8.4). □

The next definition is analogous to Definition 8.2.4.

Definition 8.2.12. Use the notation of Proposition 8.2.11. Let Γ_α be the graph of $\alpha \in C^1_{\text{loc}}(\hat{U})$ and let $\pi_\alpha : \hat{N} \to \Gamma_\alpha$ be given by $\pi_\alpha(x) = (x, \alpha(x))$. For $t \in \hat{\mathcal{T}}$, let $f_\alpha(t) = \delta(t) \circ \pi_\alpha$ and let $\hat{N}_\alpha(t) = f_\alpha(t)(\hat{N})$. Define F_2 from $C^1_{\text{loc}}(\hat{U}) \times \hat{\mathcal{T}}$ to $C^0_{\text{loc}}(\Lambda^3 T^*\hat{N})$ by:

$$F_2(\alpha, t) = f_\alpha(t)^* \left(\varphi|_{\hat{N}_\alpha(t)}\right).$$

The linearisation of F_2 at $(0, 0)$ acts as

$$dF_2|_{(0,0)} : (\alpha, t) \longmapsto d\alpha + L_2(t),$$

where $\alpha \in C^1_{\text{loc}}(\Lambda^2_+ T^*\hat{N})$, $t \in T_0\hat{\mathcal{T}}$ and L_2 is a linear map into the space of smooth exact 3-forms on $\hat{N}$ since φ is exact near $\hat{N}$.

Note that $F_2(\alpha, 0) = F_1(\alpha)$ as given in Definition 8.2.4.

Clearly, $\operatorname{Ker} F_2$ is the set of $\alpha \in C^1_{\text{loc}}(\hat{U})$ and $t \in \hat{\mathcal{T}}$ such that $\hat{N}_\alpha(t)$ is coassociative. However, we have not yet encoded the information that $N_\alpha(t)$ is CS with *rate* λ. This is the subject of the next proposition.

Proposition 8.2.13. *The moduli space of deformations for Problem 2 is locally homeomorphic to* $\operatorname{Ker} F_2 = \{(\alpha, t) \in C^\infty_\lambda(\hat{U}) \times \hat{\mathcal{T}} \,:\, F_2(\alpha, t) = 0\}$.

Proof. For each $t \in \hat{\mathcal{T}}$, we are in the situation of Problem 1 in the sense that we want coassociative deformations $\hat{N}_\alpha(t)$ of $\hat{N}(t)$, defined by a self-dual 2-form α, which have the same singular points, cones and tangent cones as $\hat{N}(t)$. It is thus clear that $\alpha \in C^\infty_\lambda(\hat{U})$ by Proposition 8.2.5. □

We now introduce an associated map G_2 to F_2.

Definition 8.2.14. Define $G_2 : C^1_{\text{loc}}(\hat{U}) \times C^1_{\text{loc}}(\Lambda^4 T^*\hat{N}) \times \hat{\mathcal{T}} \to C^0_{\text{loc}}(\Lambda^3 T^*\hat{N})$ by:

$$G_2(\alpha, \beta, t) = F_2(\alpha, t) + d^*\beta.$$

Then $dG_2|_{(0,0,0)} : (\alpha, \beta, t) \longmapsto d\alpha + d^*\beta + L_2(t)$, in the notation of Definition 8.2.12.

We then have an analogous result to Proposition 8.2.7, which follows in exactly the same fashion because $F_2(\alpha, t)$ is exact.

Proposition 8.2.15. $\operatorname{Ker} F_2 \cong \{(\alpha, \beta, t) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^*\hat{N}) \times \hat{\mathcal{T}} \,:\, G_2(\alpha, \beta, t) = 0\}$.

The next result studies the regularity of the kernel of G_2, recalling that the work in §7.1.3 is applicable to the CS case, with appropriate slight modifications.

Proposition 8.2.16. *Let $(\alpha, \beta, t) \in L^p_{k+1,\,\lambda}(\hat{U}) \times L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N}) \times \hat{\mathcal{T}}$, where $p > 4$ and $k \geq 2$. If $G_2(\alpha, \beta, t) = 0$ and $\|\alpha\|_{C^1_1}$ and t are sufficiently small, $(\alpha, \beta) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^*\hat{N})$.*

Proof. Note that $dG_2(\alpha, \beta, t) = \Delta\beta = 0$ implies that $\beta = 0$ by Theorem 1.2.5 and $d^*G_2(\alpha, \beta, t) = d^*F_2(\alpha, t) = 0$ is an elliptic equation at 0 on α. Using similar notation to §7.1.3,

$$d^*F_2(\alpha, t)(x) = R_t(x, \alpha(x), \nabla\alpha(x))\nabla^2\alpha(x) + E_t(x, \alpha(x), \nabla\alpha(x)),$$

where R_t and E_t are smooth functions of their arguments. If we define

$$S_{(\alpha,\,t)}(\gamma)(x) = R_t(x, \alpha(x), \nabla\alpha(x))\nabla^2\gamma(x),$$

then $S_{(\alpha,\,t)}$ is a smooth linear differential operator. The ellipticity of $S_\alpha = S_{(\alpha,0)}$ results from the coassociativity of $\hat{N}$. Ellipticity is an *open* condition so, although $\hat{N}(t)$ is not necessarily coassociative, the fact that it is 'close' to being coassociative means that $S_{(\alpha,\,t)}$ is elliptic, as long as we shrink $\hat{\mathcal{T}}$ as necessary to make t sufficiently small.

The regularity results for $S_{(\alpha,\,t)}$ follow in the same way as in the proof of Proposition 7.1.10 since $F_2(\alpha, t)$ depends smoothly on t and $\hat{N}(t)$ is asymptotically coassociative near the singular points, which validates the use of the theory from §6.4. Recall that $L^p_{k+1,\,\lambda} \hookrightarrow C^{k,\,a}_\lambda$ where $a = 1 - 4/p$. Thus, if $S_{(\alpha,\,t)}(\gamma) \in C^{k-1,\,a}_{\lambda-2}$ and $\gamma \in C^2_\lambda(\hat{U})$, then $\gamma \in C^{k+1,\,a}_\lambda(\hat{U})$.

Since E_0 maps into $C^{k-1,\,a}_{\lambda-2}$, as argued in §7.1.3, and F_2 depends smoothly on t, E_t maps into $C^{k-1,\,a}_{\lambda-2}$ for t sufficiently small. Hence,

$$S_{(\alpha,\,t)}(\alpha)(x) = -E_t(x, \alpha(x), \nabla\alpha(x)) \in C^{k-1,\,a}_{\lambda-2}.$$

We deduce that $\alpha \in C^{k+1,\,a}_\lambda$, given only that $\alpha \in C^{k,\,a}_\lambda$. Induction gives the result. $\square$

8.2.3 Problem 3: moving singularities and varying G_2 structure

For our final problem we consider CS deformations N' of N with the same rate and cones at s singularities, but with possibly different singular points and tangent cones there, such that N' is coassociative under a deformation of the G_2 structure on M.

We begin with the following.

Proposition 8.2.17. *Use the notation of Proposition 8.2.3. Let*

$$T = \hat{T} \cup \bigcup_{i=1}^{s} V_i \supseteq N.$$

By making $\hat{T}$ and V_i, for $i = 1, \ldots, s$, smaller if necessary, T retracts onto N. There exists an isomorphism $\Xi : H^3_{\mathrm{dR}}(T) \to H^3_{\mathrm{cs}}(\hat{N})$.

Proof. Let $[\xi] \in H^3_{\mathrm{dR}}(T)$. Since the sets V_i retract onto $\{z_i\}$ for $i = 1, \ldots, s$, ξ can be chosen such that $\xi|_{V_i} = 0$. Therefore, $\xi|_{U_i \setminus \{z_i\}} = 0$ which implies that the support of $\xi|_{\hat{N}}$ is contained in K, which is compact. Hence $[\xi|_{\hat{N}}]$ is a well-defined element of $H^3_{\mathrm{cs}}(\hat{N})$. Define Ξ by $[\xi] \mapsto [\xi|_{\hat{N}}]$. We show that Ξ is well-defined. Suppose that $\xi' = \xi + dv$, for $v \in C^\infty(\Lambda^2 T^* T)$, such that $\xi'|_{V_i} = 0$ for all i. Then $dv|_{V_i} = 0$ for all i. Since V_i retracts onto $\{z_i\}$ we can choose v such that $v|_{V_i} = 0$ without affecting dv by smoothly interpolating over $\hat{T}$. Thus $v|_{\hat{N}}$ is compactly supported on $\hat{N}$ and $\xi|_{\hat{N}} + d(v|_{\hat{N}}) = \xi'|_{\hat{N}}$. Hence Ξ is well-defined and injective.

Any closed form on $\hat{N}$ with support in K can be extended smoothly to a closed form on T which vanishes on V_i for all i. Thus, any cohomology class in $H^3_{\mathrm{cs}}(\hat{N})$ has a representative γ that can be lifted to a form ξ on T such that $\Xi([\xi]) = [\gamma]$, which implies that Ξ is surjective. □

The reason for this result is two-fold. Firstly, the condition that $\Xi([\varphi|_T]) = 0$ in $H^3_{\mathrm{cs}}(\hat{N})$ is implied by the coassociativity of $\hat{N}$ and it forces $[\varphi|_{\hat{N}}] = 0$ in $H^3_{\mathrm{cs}}(\hat{N})$. This is stronger than the seemingly more natural condition of $[\varphi|_{\hat{N}}] = 0$ in $H^3_{\mathrm{dR}}(\hat{N})$, which would be the correct requirement if $\hat{N}$ were compact by the work of McLean [45]. Secondly, if a G_2 structure (φ', g') on M is such that $\Xi([\varphi'|_T]) \neq 0$ then $\varphi'|_{\hat{N}'} \neq 0$ for any nearby deformation $\hat{N}'$ of $\hat{N}$, so there are *no* coassociative deformations.

Proposition 8.2.17 allows us to define a distinguished family of 'nearby' G_2 structures to (φ, g).

Definition 8.2.18. Let $\hat{\mathcal{F}}$ be a small open ball about 0 in $\mathbb{R}^m$ for some m. Let

$$\mathcal{F} = \{(\varphi^f, g^f) \,:\, f \in \hat{\mathcal{F}}\}$$

be a family of torsion-free G_2 structures, with $(\varphi^0, g^0) = (\varphi, g)$, such that $\Xi([\varphi^f|_T]) = 0$ in $H^3_{\mathrm{cs}}(\hat{N})$ and the map $h_{\hat{\mathcal{F}}} : \hat{\mathcal{F}} \to \mathcal{F}$ given by $h_{\hat{\mathcal{F}}}(f) = (\varphi^f, g^f)$ is an embedding.

Note that $\hat{\mathcal{F}}$ can be considered as an open subset of $T_0\hat{\mathcal{F}}$.

We now describe the moduli space for Problem 3.

Definition 8.2.19. The *moduli space of deformations* $\mathcal{M}_3(N, \lambda)$ for Problem 3 is the set of pairs (N', f) of $f \in \hat{\mathcal{F}}$ and N' in (M, φ^f, g^f) which are CS coassociative 4-folds at $z'_1, \ldots, z'_s$ with rate λ, having cone C_i and tangent cone $\hat{C}'_i$ at z'_i for all i, such that there exists a homeomorphism $h : N \to$

N', isotopic to the identity, with $h(z_i) = z'_i$ for $i = 1, \ldots, s$ and such that $h|_{\hat{N}} : \hat{N} \to N' \setminus \{z'_1, \ldots, z'_s\}$ is a diffeomorphism.

Note that we have a projection map $\pi_{\hat{\mathcal{F}}} : \mathcal{M}_3(N, \lambda) \to \hat{\mathcal{F}}$, with $\pi_{\hat{\mathcal{F}}}(N', f) = f$, whose fibres $\pi_{\hat{\mathcal{F}}}^{-1}(f)$ are equal to the moduli space for Problem 2 defined using the G_2 structure (φ^f, g^f).

We must adapt our translation space from Problem 2 to incorporate the varying G_2 structure.

Definition 8.2.20. Use the notation of Definitions 8.2.10 and 8.2.18. For $f \in \hat{\mathcal{F}}$ and $\mathbf{z}' \in B$ let $I^f(\mathbf{z}')$ denote the family of choices of s-tuples $\boldsymbol{\zeta}' = (\zeta'_1, \ldots, \zeta'_s)$ of isomorphisms $\zeta'_i : \mathbb{R}^7 \to T_{z'_i}M$ identifying (φ_0, g_0) with $(\varphi^f|_{T_{z'_i}M}, g^f|_{T_{z'_i}M})$.

The *translation space* corresponding to $\hat{\mathcal{F}}$ is

$$\mathcal{T}^{\hat{\mathcal{F}}} = \{(\mathbf{z}', \boldsymbol{\zeta}', f) \,:\, \mathbf{z}' \in B,\, f \in \hat{\mathcal{F}},\, \boldsymbol{\zeta}' \in I^f(\mathbf{z}')\}.$$

Note that it is a principal G_2^s bundle over $B \times \hat{\mathcal{F}}$.

There is a natural free action of H on $\mathcal{T}^{\hat{\mathcal{F}}}$ and hence an H-orbit through $(\mathbf{z}, \boldsymbol{\zeta}, 0)$. Therefore, we may embed $\hat{\mathcal{T}} \times \hat{\mathcal{F}}$ into $\mathcal{T}^{\hat{\mathcal{F}}}$ by $h_{\hat{\mathcal{T}} \times \hat{\mathcal{F}}} : (t, f) \mapsto (\mathbf{z}(t, f), \boldsymbol{\zeta}(t, f), f)$ such that $h_{\hat{\mathcal{T}} \times \hat{\mathcal{F}}}(\hat{\mathcal{T}} \times \hat{\mathcal{F}})$ is transverse to this H-orbit and $\mathbf{z}(0, f) = \mathbf{z}$ for all f.

Use the notation introduced before Proposition 8.2.11. Extend the G_2 coordinate system near $z_1, \ldots, z_s$ used to define N to a smooth family of G_2 coordinate systems

$$\Big\{\{\chi_i(t, f) : B(0; \eta) \to V_i(t, f) \,:\, i = 1, \ldots, s\} \,:\, (t, f) \in \hat{\mathcal{T}} \times \hat{\mathcal{F}}\Big\}$$

such that $V_i(t, f)$ is an open set in M containing $z_i(t, f)$, $\chi_i(t, f)(0) = z_i(t, f)$, $d\chi_i(t, f)|_0 = \zeta_i(t, f)$, $\chi_i(t, 0) = \chi_i(t)$, $V_i(0, f) = V_i$ and $V_i(t, 0) = V_i(t)$ for $i = 1, \ldots, s$. We state the analogue of Proposition 8.2.11.

Proposition 8.2.21. *Use the notation of Propositions 8.2.3 and 8.2.11 and Definition 8.2.20.*

(a) There exists a family $\mathcal{N}^{\hat{\mathcal{F}}} = \{N(t, f) : (t, f) \in \hat{\mathcal{T}} \times \hat{\mathcal{F}}\}$ of CS 4-folds in M, with $N(0, f) = N$ and $N(t, 0) = N(t)$, such that $N(t, f)$ has singularities at $z_1(t, f), \ldots, z_s(t, f)$ with rate λ, cones $C_1, \ldots, C_s$ and tangent cones $\hat{C}_1(t, f), \ldots, \hat{C}_s(t, f)$ defined by $\hat{C}_i(t, f) = (\zeta_i(t, f) \circ \iota_i)(C_i)$.

(b) Let $\hat{N}(t, f) = N(t, f) \setminus \{z_1(t, f), \ldots, z_s(t, f)\}$ and write

$$N(t, f) = K(t, f) \sqcup \bigsqcup_{i=1}^{s} U_i(t, f)$$

where $K(t, f)$ is compact and $U_i(t, f) \setminus \{z_i(t, f)\} \cong (0, \epsilon) \times \Sigma_i$ for all i, in the obvious way, ensuring that $K(0, f) = K$, $K(t, 0) = K(t)$, $U_i(0, f) = U_i$ and $U_i(t, 0) = U_i(t)$. For $(t, f) \in \hat{\mathcal{T}} \times \hat{\mathcal{F}}$, there exist open sets $\hat{T}(t, f) \subseteq M$ containing $\hat{N}(t, f)$ and diffeomorphisms $\delta(t, f)$:

$\hat{U} \to \hat{T}(t,f)$ taking the zero section to $\hat{N}(t,f)$, varying smoothly in t and f, with $\hat{T}(0,f) = \hat{T}$, $\hat{T}(t,0) = \hat{T}(t)$ and $\delta(t,0) = \delta(t)$. Moreover, $\hat{T}(t,f)$ can be chosen to grow with order $O(\rho)$ as $\rho \to 0$ and $\delta(t,f)$ is compatible with the identifications $U_i(t,f) \setminus \{z_i(t,f)\} \cong (0,\epsilon) \times \Sigma_i$ for $i = 1,\ldots,s$.

The proof is almost identical to that of Proposition 8.2.11 and so we omit it. The compatibility conditions on $\delta(t,f)$ are given by similar commutative diagrams to (8.4). Note that $\delta(t,f)$ is not required to be compatible with the isomorphism $\nu(\hat{N}) \cong \Lambda^2_+ T^*\hat{N}$ for $(t,f) \neq (0,0)$.

We proceed by defining our final deformation map.

Definition 8.2.22. Use the notation of Proposition 8.2.21. Let Γ_α be the graph of $\alpha \in C^1_{\text{loc}}(\hat{U})$ and let $\pi_\alpha : \hat{N} \to \Gamma_\alpha$ be given by $\pi_\alpha(x) = (x, \alpha(x))$. For $(t,f) \in \hat{\mathcal{T}} \times \hat{\mathcal{F}}$, let $f_\alpha(t,f) = \delta(t,f) \circ \pi_\alpha$ and let $\hat{N}_\alpha(t,f) = f_\alpha(t,f)(\hat{N})$. Define F_3 from $C^1_{\text{loc}}(\hat{U}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}}$ to $C^0_{\text{loc}}(\Lambda^3 T^*\hat{N})$ by:

$$F_3(\alpha,t,f) = f_\alpha(t,f)^* \left(\varphi^f|_{\hat{N}_\alpha(t,f)}\right).$$

The linearisation of F_3 at $(0,0,0)$ acts as

$$dF_3|_{(0,0,0)} : (\alpha,t,f) \longmapsto d\alpha + L_2(t) + L_3(f),$$

where $\alpha \in C^1_{\text{loc}}(\Lambda^2_+ T^*\hat{N})$, $(t,f) \in T_0\hat{\mathcal{T}} \oplus T_0\hat{\mathcal{F}}$, L_2 is given in Definition 8.2.12 and L_3 is a linear map into the space of smooth exact 3-forms on $\hat{N}$ by the condition imposed on φ^f in Definition 8.2.18.

Note that $F_3(\alpha,t,0) = F_2(\alpha,t)$ as given in Definition 8.2.12.

Now, $\operatorname{Ker} F_3$ corresponds to choices of $\hat{N}_\alpha(t,f)$ which are coassociative with respect to (φ^f, g^f). The next result is then clear from considering the proof of Proposition 8.2.13.

Proposition 8.2.23. *The moduli space of deformations for Problem 3 is locally homeomorphic to* $\operatorname{Ker} F_3 = \{(\alpha,t,f) \in C^\infty_\lambda(\hat{U}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}} : F_3(\alpha,t,f) = 0\}$.

We again have an associated map to our deformation map.

Definition 8.2.24. Define $G_3 : C^1_{\text{loc}}(\hat{U}) \times C^1_{\text{loc}}(\Lambda^4 T^*\hat{N}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}} \to C^0_{\text{loc}}(\Lambda^3 T^*\hat{N})$ by:

$$G_3(\alpha,\beta,t,f) = F_3(\alpha,t,f) + d^*\beta.$$

Then $dG_3|_{(0,0,0,0)} : (\alpha,\beta,t,f) \longmapsto d\alpha + d^*\beta + L_2(t) + L_3(f)$, in the notation of Definition 8.2.22.

The next result is analogous to Propositions 8.2.7 and 8.2.15 and may be immediately deduced from the exactness of $F_3(\alpha,t,f)$, which follows from the condition imposed on φ^f in Definition 8.2.18.

Proposition 8.2.25. $\operatorname{Ker} F_3 \cong \{(\alpha,\beta,t,f) \in C^\infty_\lambda(\hat{U}) \times C^\infty_\lambda(\Lambda^4 T^*\hat{N}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}} : G_3(\alpha,\beta,t,f) = 0\}$.

The argument used to prove the regularity result Proposition 8.2.16 is easily generalised to the map G_3, so we end the section with the following.

Proposition 8.2.26. *Let $(\alpha,\beta,t,f)\in L^p_{k+1,\lambda}(\hat{U})\times L^p_{k+1,\lambda}(\Lambda^4T^*\hat{N})\times\hat{\mathcal{T}}\times\hat{\mathcal{F}}$, where $p>4$ and $k\geq 2$. If $G_3(\alpha,\beta,t,f)=0$ and $\|\alpha\|_{C^1_1}$, t and f are sufficiently small, $(\alpha,\beta)\in C^\infty_\lambda(\hat{U})\times C^\infty_\lambda(\Lambda^4T^*\hat{N})$.*

8.3 The Deformation and Obstruction Spaces

In Proposition 7.2.3 we found that the deformation theory of AC coassociative 4-folds was unobstructed for generic rates in a given range. We used this fact to conclude in Theorem 7.3.2 that the moduli space was locally smooth and of dimension equal to that of the deformation space.

In the CS case we are not so fortunate. We therefore describe the deformation and obstruction spaces for each of our Problems and show in each scenario that, if the obstruction space is zero, we have an analogous result to Theorem 7.3.2. We recollect the common notation introduced at the start of §8.2. In addition, fix some $p>4$ and integer $k\geq 2$.

8.3.1 Problem 1

Recall the maps F_1 and G_1 given in Definitions 8.2.4 and 8.2.6 respectively. Their kernels give a local description for the moduli space $\mathcal{M}_1(N,\lambda)$ by Propositions 8.2.5 and 8.2.7. Therefore the kernels of $dF_1|_0$ and $dG_1|_{(0,0)}$ describe the *infinitesimal deformations*.

Definition 8.3.1. The *infinitesimal deformation space* for Problem 1 is

$$\mathcal{I}_1(N,\lambda)=\{\alpha\in C^\infty_\lambda(\Lambda^2_+T^*\hat{N})\,:\,d\alpha=0\}\cong\{(\alpha,\beta)\in C^\infty_\lambda(\Lambda^2_+T^*\hat{N}\oplus\Lambda^4T^*\hat{N})\,:\,d\alpha+d^*\beta=0\}.$$

The equivalence of the spaces follows by Proposition 8.2.7 or, more simply, by the Maximum Principle for harmonic functions (Theorem 1.2.5).

Using Proposition 8.2.8 or, since $L^p_{k+1,\lambda}\hookrightarrow C^1_\lambda$ by Theorem 6.2.4, Corollary 6.4.2 implies that

$$\mathcal{I}_1(N,\lambda)\cong\{(\alpha,\beta)\in L^p_{k+1,\lambda}(\Lambda^2_+T^*\hat{N}\oplus\Lambda^4T^*\hat{N})\,:\,d\alpha+d^*\beta=0\}.$$

Therefore, $\mathcal{I}_1(N,\lambda)$ is finite-dimensional.

We turn to possible *obstructions* to the deformation theory and start with the following.

Proposition 8.3.2. *The map F_1 takes $L^p_{k+1,\lambda}(\hat{U})$ into $d(L^p_{k+1,\lambda}(\Lambda^2T^*\hat{N}))$.*

Proof. Let $\alpha\in L^p_{k+1,\lambda}(\hat{U})$ and let T be as in Proposition 8.2.17. As noted after that proposition, $[\varphi|_T]=0$ in $H^3_{\mathrm{dR}}(T)$ and hence $\varphi|_T$ is exact. Thus, $\varphi|_T=d\psi$ for some $\psi\in C^\infty(\Lambda^2T^*T)$. However, we want to select ψ in a particular way near the singularities. On $B(0;\eta) \subseteq \mathbb{R}^7$, for each $i = 1, \ldots, s$,

$$\chi_i^*(\varphi) = \varphi_0 + O(r_i).$$

If v is the dilation vector field on $\mathbb{R}^7$ given in (7.22), we can choose ψ to satisfy

$$\chi_i^*(\psi) = \frac{1}{3}(v \cdot \varphi_0) + O(r_i^2)$$

on V_i, since $d(v \cdot \varphi_0) = 3\varphi_0$, then extend ψ smoothly to a form on T such that $d\psi = \varphi|_T$. Note that

$$(v \cdot \varphi_0)|_{\iota_i(C_i)} = v \cdot (\varphi_0|_{\iota_i(C_i)}) = 0$$

as $v \in T(\iota_i(C_i))$. Hence $\chi_i^*(\psi) = O(r_i^2)$ on $\iota_i(C_i)$, for all i, and similar results hold for the derivatives of ψ. Define

$$H_1(\alpha) = f_\alpha^*(\psi|_{\hat{N}_\alpha})$$

so that $F_1(\alpha) = d(H_1(\alpha))$. Note that $\chi_i^*(\psi)|_{\iota_i(C_i)} = O(r_i^2)$ is dominated by $O(r_i^\lambda)$ terms as $r_i \to 0$ since $\lambda < 2$. Further, $f_\alpha^*(\psi|_{\hat{N}_\alpha})$ has the same growth as $\chi_i^*(\psi)|_{(\Phi_\alpha)_i((0,\epsilon)\times\Sigma_i)}$ as $r_i \to 0$, using the notation preceding Proposition 8.2.5. However,

$$\chi_i^*(\psi)|_{(\Phi_\alpha)_i((0,\epsilon)\times\Sigma_i)} = \chi_i^*(\psi)|_{((\Phi_\alpha)_i-\iota_i)((0,\epsilon)\times\Sigma_i)} + \chi_i^*(\psi)|_{\iota_i((0,\epsilon)\times\Sigma_i)}.$$

The first term depends on $|(\Phi_\alpha)_i - \iota_i|$ and hence is $O(r_i^\lambda)$ as $r_i \to 0$. This dominates the second term by our observation above. Hence, $H_1(\alpha) \in L^p_{k,\,\lambda}$ because H_1 depends on α and $\nabla\alpha$. Note that $H_1(\alpha)$ has one degree of differentiability less than expected.

From this, recalling that $\lambda \notin \mathcal{D}_{\mathrm{CS}}$, we deduce that $F_1(\alpha)$ lies in $d(L^p_{k,\,\lambda}(\Lambda^2 T^*\hat{N}))$ and hence is L^2-orthogonal to elements of the kernel of

$$d + d^* : L^q_{l+1,\,-3-\lambda}(\Lambda^3 T^*\hat{N}) \to L^q_{l,\,-4-\lambda}(\Lambda^2 T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}),$$

where $q > 1$ such that $1/p + 1/q = 1$. We show that

$$d(L^p_{k,\,\lambda}(\Lambda^2 T^*\hat{N})) \oplus d^*(L^p_{k,\,\lambda}(\Lambda^4 T^*\hat{N})) \subseteq L^p_{k-1,\,\lambda-1}(\Lambda^3 T^*\hat{N})$$

is *characterised* as the subspace which is L^2-orthogonal to this kernel.

Consider

$$d + d^* : L^p_{k,\,\lambda}(\Lambda^{\mathrm{even}} T^*\hat{N}) \to L^p_{k-1,\,\lambda-1}(\Lambda^{\mathrm{odd}} T^*\hat{N}).$$

This elliptic map has image which comprises precisely of those elements of $L^p_{k-1,\,\lambda-1}(\Lambda^{\mathrm{odd}} T^*\hat{N})$ which are L^2-orthogonal to the kernel $\mathcal{K}$ of

$$d + d^* : L^q_{l+1,\,-3-\lambda}(\Lambda^{\mathrm{odd}} T^*\hat{N}) \to L^q_{l,\,-4-\lambda}(\Lambda^{\mathrm{even}} T^*\hat{N}).$$

The space $\mathcal{K}$ can be written as the direct sum $\mathcal{K} = \mathcal{K}^1 \oplus \mathcal{K}^3 \oplus \mathcal{K}^m$, where

$$\mathcal{K}^j = \mathcal{K} \cap L^q_{l+1,\,-3-\lambda}(\Lambda^j T^*\hat{N})$$

for $j = 1$ and 3 and $\mathcal{K}^m$ is some transverse subspace. Then

$$d(L^p_{k,\,\lambda}(\Lambda^2 T^*\hat{N})) \oplus d^*(L^p_{k,\,\lambda}(\Lambda^4 T^*\hat{N})) = \{\alpha_3 \,:\, \exists\, \alpha_1 \text{ such that } (\alpha_1, \alpha_3) \in \mathcal{K}^\perp\},$$

where we take the orthogonal complement in $L^p_{k-1,\,\lambda-1}$. Note that the projection $\pi_1(\mathcal{K}^m)$ of $\mathcal{K}^m$ onto the space of 1-forms must meet $\mathcal{K}^1$ in the zero form since, if $(\alpha_1, \alpha_3) \in \mathcal{K}^m$ and $\alpha_1 \in \mathcal{K}^1$ then $\alpha_3 \in \mathcal{K}^3$, which contradicts the direct sum decomposition of $\mathcal{K}$. Therefore, $\pi_1(\mathcal{K}^m)$ and $\mathcal{K}^1$ are transverse finite-dimensional subspaces of $L^q_{l+1,\,-3-\lambda}(\Lambda^1 T^*\hat{N})$. Hence, there exists a space $\mathcal{A}$ of smooth compactly supported 1-forms on $\hat{N}$ which is L^2-orthogonal to $\mathcal{K}^1$ and such that $\mathcal{A} \times \mathcal{K}^m \to \mathbb{R}$ given by $(\gamma, \xi) \mapsto (\gamma, 0) \cdot \xi$ is a dual pairing. If $\alpha_3 \in L^p_{k-1,\,\lambda-1}(\Lambda^3 T^*\hat{N})$ such that $\alpha_3 \in (\mathcal{K}^3)^\perp$, there exists a unique $\alpha_1 \in \mathcal{A}$ such that $(\alpha_1, 0) \cdot \xi = -(0, \alpha_3) \cdot \xi$ for all $\xi \in \mathcal{K}^m$, which implies that $(\alpha_1, \alpha_3) \in (\mathcal{K}^m)^\perp$. We conclude that

$$\begin{aligned}(\mathcal{K}^3)^\perp &= \{\alpha_3 \in (\mathcal{K}^3)^\perp \,:\, \exists\, \alpha_1 \in (\mathcal{K}^1)^\perp \text{ such that } (\alpha_1, \alpha_3) \in (\mathcal{K}^m)^\perp\} \\ &= \{\alpha_3 \,:\, \exists\, \alpha_1 \text{ such that } (\alpha_1, \alpha_3) \in \mathcal{K}^\perp\} \\ &= d(L^p_{k,\,\lambda}(\Lambda^2 T^*\hat{N})) \oplus d^*(L^p_{k,\,\lambda}(\Lambda^4 T^*\hat{N})) \subseteq L^p_{k-1,\,\lambda-1}(\Lambda^3 T^*\hat{N}).\end{aligned}$$

However, $\mathcal{K}^3$ is independent of k, and hence $F_1(\alpha)$ must lie in the image of $d + d^*$ from $L^p_{k+1,\,\lambda}$, since $F_1(\alpha)$ lies in $L^p_{k,\,\lambda-1}$. We may thus write $F_1(\alpha) = d\gamma + d^*\beta$ for some $\gamma \in L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N})$ and $\beta \in L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N})$. Moreover, $dd^*\beta = 0$ and so β is harmonic and $O(\rho^\lambda)$ as $\rho \to 0$. By Theorem 1.2.5, (noting that $*\beta$ is a harmonic function on $\hat{N}$), $\beta = 0$. The proposition is thus proved. $\square$

We deduce from Propositions 8.2.5, 8.2.7, 8.2.8 and 8.3.2 that $\mathcal{M}_1(N, \lambda)$ is locally homeomorphic to the kernel of

$$G_1 : L^p_{k+1,\,\lambda}(\hat{U}) \times L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N}) \to d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N})) \oplus d^*(L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N})).$$

Therefore, our deformation theory will be obstructed if and only if the map

$$d : L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N}) \to d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N}))$$

is *not* surjective. This leads us to the next result and definition.

Proposition 8.3.3. *There exists a finite-dimensional subspace $\mathcal{O}_1(N, \lambda)$ of $L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$ such that*

$$d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N})) = d(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N})) \oplus \mathcal{O}_1(N, \lambda).$$

Proof. The Fredholmness of $d + d^*$ implies that the images of $L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N}) \oplus L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N})$ and $L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N}) \oplus L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N})$ under $d + d^*$ are both closed and have finite codimension in $L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$. Since

$$\{0\} = d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N})) \cap d^*(L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N})) = d(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N})) \cap d^*(L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N}))$$

by the Maximum Principle (Theorem 1.2.5), we deduce that

$$d(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N})) \quad \text{and} \quad d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N}))$$

are both closed and that the former has finite codimension in the latter. Thus, $\mathcal{O}_1(N,\lambda)$ can be chosen as stated. $\square$

Definition 8.3.4. The *obstruction space* for Problem 1 is

$$\mathcal{O}_1(N,\lambda) \cong \frac{d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N}))}{d(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N}))}.$$

We proceed as follows. Define

$$U_1 = L^p_{k+1,\,\lambda}(\hat{U}) \times L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N}), \qquad X_1 = L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}),$$
$$Y_1 = \mathcal{O}_1(N,\lambda) \subseteq L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N}) \text{ and } \qquad Z_1 = d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N})) \oplus d^*(L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N})).$$

Then X_1, Y_1 and Z_1 are Banach spaces and U_1 is an open neighbourhood of $(0,0)$ in X_1 because $L^p_{k+1,\,\lambda} \hookrightarrow C^0_\lambda$ by Theorem 6.2.4 and $\hat{U}$ grows with order $O(\rho)$ as $\rho \to 0$ by Proposition 8.2.3. Thus, $W_1 = U_1 \times Y_1$ is an open neighbourhood of $(0,0,0)$ in $X_1 \times Y_1$. Define $\mathcal{G}_1 : W_1 \to Z_1$ by:

$$\mathcal{G}_1(\alpha,\beta,\gamma) = G_1(\alpha,\beta) + \gamma.$$

Then $\mathcal{G}_1$ is well-defined by Propositions 8.3.2 and 8.3.3 and its derivative at $(0,0,0)$ acts from $X_1 \times Y_1$ to Z_1 as

$$d\mathcal{G}_1|_{(0,0,0)} : (\alpha,\beta,\gamma) \longmapsto d\alpha + d^*\beta + \gamma.$$

Clearly, $d\mathcal{G}_1|_{(0,0,0)}$ is surjective by construction and its kernel, using the fact that $(d+d^*)(X_1) \cap Y_1 = \{0\}$, is given by:

$$\begin{aligned} \operatorname{Ker} d\mathcal{G}_1|_{(0,0,0)} &= \{(\alpha,\beta,\gamma) \in X_1 \times Y_1 : d\alpha + d^*\beta + \gamma = 0\} \\ &\cong \{(\alpha,\beta) \in X_1 : d\alpha + d^*\beta = 0\} \cong \mathcal{I}_1(N,\lambda). \end{aligned}$$

The conclusion, by implementing the Implicit Function Theorem for Banach spaces (Theorem 6.2.5), is that $\operatorname{Ker} \mathcal{G}_1$ is a smooth manifold near zero which may be identified with an open neighbourhood

$\hat{\mathcal{M}}_1(N,\lambda)$ of 0 in $\mathcal{I}_1(N,\lambda)$. Formally, if we write $X_1 = \mathcal{I}_1(N,\lambda) \oplus A$ for some closed subspace A of X_1, there exist open sets $\hat{\mathcal{M}}_1(N,\lambda) \subseteq \mathcal{I}_1(N,\lambda)$, $V_A \subseteq A$, $V_Y \subseteq Y_1$, all containing 0, with $\hat{\mathcal{M}}_1(N,\lambda) \times V_A \subseteq U_1$, and smooth maps $\mathcal{V}_A : \hat{\mathcal{M}}_1(N,\lambda) \to V_A$ and $\mathcal{V}_Y : \hat{\mathcal{M}}_1(N,\lambda) \to V_Y$ such that

$$\operatorname{Ker} \mathcal{G}_1 \cap (\hat{\mathcal{M}}_1(N,\lambda) \times V_A \times V_Y) = \{(x, \mathcal{V}_A(x), \mathcal{V}_Y(x)) \,:\, x \in \hat{\mathcal{M}}_1(N,\lambda)\}.$$

If we define a smooth map $\pi_1 : \hat{\mathcal{M}}_1(N,\lambda) \to \mathcal{O}_1(N,\lambda)$ by $\pi_1(x) = \mathcal{V}_Y(x)$, the moduli space $\mathcal{M}_1(N,\lambda)$ near N is locally homeomorphic to the kernel of π_1 near 0. We can think of π_1 as a map on an open neighbourhood of $(0,0,0)$ in $\operatorname{Ker} \mathcal{G}_1$ which projects onto the obstruction space. We write these results as a theorem.

Theorem 8.3.5. *Use the notation of Definitions 8.2.1, 8.3.1 and 8.3.4. There exists a smooth manifold $\hat{\mathcal{M}}_1(N,\lambda)$, which is an open neighbourhood of 0 in $\mathcal{I}_1(N,\lambda)$, and a smooth map $\pi_1 : \hat{\mathcal{M}}_1(N,\lambda) \to \mathcal{O}_1(N,\lambda)$, with $\pi_1(0) = 0$, such that an open neighbourhood of 0 in $\operatorname{Ker} \pi_1$ is homeomorphic to an open neighbourhood of N in $\mathcal{M}_1(N,\lambda)$.*

We deduce from this theorem that, if the obstruction space is zero, the moduli space is a smooth manifold near N of dimension equal to that of the infinitesimal deformation space. We expect the obstruction space to be zero for generic choices of N and the G_2 structure on M.

8.3.2 Problem 2

Recall the notation introduced in Definitions 8.2.10, 8.2.12 and 8.2.14. We begin by defining the infinitesimal deformation space for this problem.

Definition 8.3.6. The *infinitesimal deformation space* for Problem 2 is

$$\begin{aligned}\mathcal{I}_2(N,\lambda) &= \{(\alpha, t) \in C^\infty_\lambda(\Lambda^2_+ T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \,:\, d\alpha + L_2(t) = 0\} \\ &\cong \{(\alpha, \beta, t) \in C^\infty_\lambda(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \,:\, d\alpha + d^*\beta + L_2(t) = 0\}.\end{aligned}$$

The equivalence in the definition follows from Proposition 8.2.15 or from the observation that $d\alpha + L_2(t)$ is exact and so $\beta = 0$ by the Maximum Principle (Theorem 1.2.5). Note that there is a subspace of $\mathcal{I}_2(N,\lambda)$ which is isomorphic to $\mathcal{I}_1(N,\lambda)$.

By Proposition 8.2.16,

$$\mathcal{I}_2(N,\lambda) \cong \{(\alpha, \beta, t) \in L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \,:\, d\alpha + d^*\beta + L_2(t) = 0\}.$$

Therefore, $\mathcal{I}_2(N,\lambda)$ is finite-dimensional.

To start our consideration of obstructions, we have the generalisation of Proposition 8.3.2.

Proposition 8.3.7. *The map F_2 takes $L^p_{k+1,\,\lambda}(\hat{U}) \times \hat{\mathcal{T}}$ into $d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N}))$.*

Proof. Use the notation from Proposition 8.2.11 and its proof and from the proof of Proposition 8.3.2. Recall that we have an open set $T \supseteq \hat{T}$ in M containing N, which retracts onto N, and $\psi \in C^\infty(\Lambda^2 T^*T)$ such that $d\psi = \varphi|_T$. We may similarly construct open sets $T(t) \supseteq \hat{T}(t)$ in M, with $T(0) = T$, which contain $N(t)$ and retract onto it, varying smoothly with t. We also have $\psi(t) \in C^\infty(\Lambda^2 T^*T(t))$, with $\psi(0) = \psi$, such that $d\psi(t) = \varphi|_{T(t)}$, using the fact that φ is exact on $N(t)$. Again, the $\psi(t)$ vary smoothly with t. Formally, let

$$T(t) = \hat{T}(t) \cup \bigcup_{i=1}^{s} V_i(t).$$

By making $\hat{T}(t)$ and $V_i(t)$ smaller if necessary, $T(t)$ will be an open set as stated. We may choose $\psi(t)$ such that

$$\chi_i(t)^*(\psi(t)) = \frac{1}{3}\,(v \cdot \varphi_0) + O(r_i^2)$$

on $V_i(t)$ and then extend smoothly to a form $\psi(t)$ on $T(t)$ as required. Define

$$H_2(\alpha, t) = f_\alpha(t)^* \left(\psi(t)|_{\hat{N}_\alpha(t)}\right).$$

Then $d(H_2(\alpha,t)) = F_2(\alpha,t)$. Moreover, by the same reasoning that $H_1(\alpha) \in L^p_{k,\,\lambda}$ in the proof of Proposition 8.3.2, $H_2(\alpha,t)$ lies in $L^p_{k,\,\lambda}$. Therefore, $F_2(\alpha,t)$ lies in $d(L^p_{k,\,\lambda}(\Lambda^2 T^*\hat{N}))$. However, because $F_2(\alpha,t) \in L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$, the argument at the end of the proof of Proposition 8.3.2 implies that $F_2(\alpha,t) \in d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N}))$ as required. □

We now define the obstruction space.

Definition 8.3.8. From Propositions 8.3.3 and 8.3.7, since L_2 is a linear map on a finite-dimensional vector space, there exists a finite-dimensional subspace $\mathcal{O}_2(N,\lambda)$ of $L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$ such that

$$d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N})) = (d(L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N})) + L_2(T_0\hat{\mathcal{T}})) \oplus \mathcal{O}_2(N,\lambda).$$

We define $\mathcal{O}_2(N,\lambda)$ to be the *obstruction space* for Problem 2.

Note that $\mathcal{O}_2(N,\lambda)$ may be chosen to be contained in $\mathcal{O}_1(N,\lambda)$.

Following the scheme for Problem 1, we let

$$\begin{aligned} U_2 &= L^p_{k+1,\,\lambda}(\hat{U}) \times L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N}) \times \hat{\mathcal{T}}, \qquad & X_2 &= L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \oplus T_0\hat{\mathcal{T}}, \\ Y_2 &= \mathcal{O}_2(N,\lambda) \subseteq L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N}) \text{ and} & Z_2 &= d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N})) \oplus d^*(L^p_{k+1,\,\lambda}(\Lambda^4 T^*\hat{N})). \end{aligned}$$

Recall that $\hat{\mathcal{T}} \subseteq \mathbb{R}^n \cong T_0\hat{\mathcal{T}}$ is open. Then X_2, Y_2 and Z_2 are Banach spaces, U_2 is an open neighbourhood of $(0,0,0)$ in X_2 and hence $W_2 = U_2 \times Y_2$ is an open neighbourhood of $(0,0,0,0)$ in

$X_2 \times Y_2$. Define $\mathcal{G}_2 : W_2 \to Z_2$ by:

$$\mathcal{G}_2(\alpha, \beta, t, \gamma) = G_2(\alpha, \beta, t) + \gamma.$$

Then $d\mathcal{G}_2|_{(0,0,0,0)} : X_2 \times Y_2 \to Z_2$ acts as

$$(\alpha, \beta, t, \gamma) \longmapsto d\alpha + d^*\beta + L_2(t) + \gamma.$$

By construction, $d\mathcal{G}_2|_{(0,0,0,0)}$ is surjective and, using the fact that the image of $dG_2|_{(0,0,0)}$ meets Y_2 at 0 only,

$$\begin{aligned}\operatorname{Ker} d\mathcal{G}_2|_{(0,0,0,0)} &= \{(\alpha, \beta, t, \gamma) \in X_2 \times Y_2 \,:\, d\alpha + d^*\beta + L_2(t) + \gamma = 0\} \\ &\cong \{(\alpha, \beta, t) \in X_2 \,:\, d\alpha + d^*\beta + L_2(t) = 0\} \cong \mathcal{I}_2(N, \lambda).\end{aligned}$$

As for Problem 1, Theorem 6.2.5 gives us that $\operatorname{Ker} \mathcal{G}_2$ is a smooth manifold near zero which may be identified with an open neighbourhood $\hat{\mathcal{M}}_2(N, \lambda)$ of $(0, 0)$ in $\mathcal{I}_2(N, \lambda)$. We can again define a smooth map $\pi_2 : \hat{\mathcal{M}}_2(N, \lambda) \to \mathcal{O}_2(N, \lambda)$ such that $\operatorname{Ker} \pi_2$ is locally homeomorphic near $(0, 0)$ to an open neighbourhood of N in $\mathcal{M}_2(N, \lambda)$. We thus have the following theorem.

Theorem 8.3.9. *Use the notation of Definitions 8.2.9, 8.3.6 and 8.3.8. There exists a smooth manifold* $\hat{\mathcal{M}}_2(N, \lambda)$*, which is an open neighbourhood of* $(0, 0)$ *in* $\mathcal{I}_2(N, \lambda)$*, and a smooth map* $\pi_2 : \hat{\mathcal{M}}_2(N, \lambda) \to \mathcal{O}_2(N, \lambda)$*, with* $\pi_2(0, 0) = 0$*, such that an open neighbourhood of zero in* $\operatorname{Ker} \pi_2$ *is homeomorphic to an open neighbourhood of* N *in* $\mathcal{M}_2(N, \lambda)$*.*

We deduce that, if $\mathcal{O}_2(N, \lambda) = \{0\}$, the moduli space for Problem 2 is a smooth manifold near N of dimension $\dim \mathcal{I}_2(N, \lambda) = \dim \mathcal{I}_1(N, \lambda) + \dim \hat{\mathcal{T}}$, which we expect to occur for generic choices of N and the torsion-free G_2 structure on M. We shall see, in §8.5, that if we choose a suitable generic *closed* G_2 structure on M we may drop the assumption that N is generic and still obtain a smooth moduli space.

8.3.3 Problem 3

We presume in this subsection that the reader is sufficiently familiar with the schemata we have used in the previous two subsections to be able to generalise them to Problem 3. This allows us to present a tidier treatment of the problem.

Recall the notation of Definitions 8.2.18, 8.2.22 and 8.2.24.

Definition 8.3.10. The *infinitesimal deformation space* $\mathcal{I}_3(N, \lambda)$ for Problem 3 is

$$\begin{aligned}\mathcal{I}_3(N, \lambda) &= \{(\alpha, t, f) \in C^\infty_\lambda(\Lambda^2_+ T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \oplus T_0\hat{\mathcal{F}} \,:\, d\alpha + L_2(t) + L_3(f) = 0\} \\ &\cong \{(\alpha, \beta, t, f) \in C^\infty_\lambda(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \oplus T_0\hat{\mathcal{F}} \,:\, d\alpha + d^*\beta + L_2(t) + L_3(f) = 0\}.\end{aligned}$$

By Proposition 8.2.26,

$$\mathcal{I}_3(N,\lambda) \cong \{(\alpha,\beta,t,f) \in L^p_{k+1,\,\lambda}(\Lambda^2_+T^*\hat{N} \oplus \Lambda^4T^*\hat{N}) \oplus T_0\hat{\mathcal{T}} \oplus T_0\hat{\mathcal{F}} \,:\, d\alpha + d^*\beta + L_2(t) + L_3(f) = 0\}.$$

In considering obstructions, we first have the generalisation of Propositions 8.3.2 and 8.3.7.

Proposition 8.3.11. *The map F_3 takes $L^p_{k+1,\,\lambda}(\hat{U}) \times \hat{\mathcal{T}} \times \hat{\mathcal{F}}$ into $d(L^p_{k+1,\,\lambda}(\Lambda^2T^*\hat{N}))$.*

The proposition is proved in a similar way to Proposition 8.3.7 and so we omit the details. The result leads us to define our final obstruction space.

Definition 8.3.12. From Propositions 8.3.3 and 8.3.11, since L_2 and L_3 are linear maps on finite-dimensional vector spaces, there exists a finite-dimensional subspace $\mathcal{O}_3(N,\lambda)$ of $L^p_{k,\,\lambda-1}(\Lambda^3T^*\hat{N})$ such that

$$d(L^p_{k+1,\,\lambda}(\Lambda^2T^*\hat{N})) = (d(L^p_{k+1,\,\lambda}(\Lambda^2_+T^*\hat{N})) + L_2(T_0\hat{\mathcal{T}}) + L_3(T_0\hat{\mathcal{F}})) \oplus \mathcal{O}_3(N,\lambda).$$

We define $\mathcal{O}_3(N,\lambda)$ to be the *obstruction space* for Problem 3.

Note that we may choose our obstruction spaces such that $\mathcal{O}_3(N,\lambda) \subseteq \mathcal{O}_2(N,\lambda) \subseteq \mathcal{O}_1(N,\lambda)$.

The use of the Implicit Function Theorem (Theorem 6.2.5) in the derivation of Theorems 8.3.5 and 8.3.9 can be easily generalised to give the following.

Theorem 8.3.13. *Use the notation of Definitions 8.2.19, 8.3.10 and 8.3.12. There exists a smooth manifold $\hat{\mathcal{M}}_3(N,\lambda)$, which is an open neighbourhood of $(0,0,0)$ in $\mathcal{I}_3(N,\lambda)$, and a smooth map $\pi_3 : \hat{\mathcal{M}}_3(N,\lambda) \to \mathcal{O}_3(N,\lambda)$, with $\pi_3(0,0,0) = 0$, such that an open neighbourhood of zero in $\operatorname{Ker}\pi_3$ is homeomorphic to an open neighbourhood of $(N,0)$ in $\mathcal{M}_3(N,\lambda)$.*

We deduce that, if $\mathcal{O}_3(N,\lambda) = \{0\}$, $\mathcal{M}_3(N,\lambda)$ is a smooth manifold near $(N,0)$ of dimension $\dim\mathcal{I}_3(N,\lambda) = \dim\mathcal{I}_2(N,\lambda) + \dim\hat{\mathcal{F}}$. Moreover, the projection map $\pi_{\hat{\mathcal{F}}} : \mathcal{M}_3(N,\lambda) \to \hat{\mathcal{F}}$ is smooth near $(N,0)$. We expect this to occur for generic choices of N and the torsion-free G_2 structure on M. If we allow ourselves to work with *closed* G_2 structures on M, we shall show in §8.5 that we may drop our genericity assumptions for N and (φ,g) and still get a smooth moduli space.

8.4 Dimension Calculations

We shall relate the expected dimension of the moduli space for Problem 1 to the index of a first order uniformly elliptic operator on $\hat{N}$. The theory of §6.3.2 will then allow us to calculate this index. Recall that $p > 4$, $k \geq 2$ and $\lambda \in (1,2) \setminus \mathcal{D}_{\mathrm{CS}}$. Recollect also the spaces $\mathcal{H}^m$ and $\mathcal{H}^2_+$ given in Definition 6.5.1 and Example 6.5.3 respectively.

Definition 8.4.1. Let

$$(d_+ + d^*)_\lambda = d + d^* : L^p_{k+1,\,\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \to L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N}).$$

By Definition 8.3.1, $\mathcal{I}_1(N, \lambda)$ is isomorphic to the kernel of this map. Define the adjoint map by

$$(d^*_+ + d)_{-3-\lambda} = d^*_+ + d : L^q_{l+1,\,-3-\lambda}(\Lambda^3 T^*\hat{N}) \longrightarrow L^q_{l,\,-4-\lambda}(\Lambda^2_+ T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}),$$

where $q > 1$ such that $1/p + 1/q = 1$ and $l \geq 4$. The cokernel of $(d_+ + d^*)_\lambda$ is then isomorphic to the kernel of $(d^*_+ + d)_{-3-\lambda}$.

We now study the dimension of the kernel and cokernel of $(d_+ + d^*)_\mu$.

Proposition 8.4.2. *The kernel of $(d_+ + d^*)_{-2}$ is isomorphic to $\mathcal{H}^2_+$. Furthermore, if $\mu > -2$ is such that $(-2, \mu] \cap \mathcal{D}_{\mathrm{CS}} = \emptyset$, $\dim \operatorname{Ker} (d_+ + d^*)_\mu = \dim \mathcal{H}^2_+$.*

Proof. Using Theorem 1.2.5, (6.5) and the regularity result Corollary 6.4.2 as in the deduction of Proposition 7.4.2 gives the first part of the result. By a similar argument to Proposition 7.4.5, the function $k(\mu) = \dim\, \operatorname{Ker} (d_+ + d^*)_\mu$ is upper semi-continuous at -2, noting that the direction of semi-continuity is reversed from the AC case. The second part follows from the observation that the choice of μ ensures that there are no changes in the kernel in $(-2, \mu]$ by Theorem 6.3.6. □

Proposition 8.4.3. *If $\mu < -1$ is such that $[\mu, -1) \cap \mathcal{D}_{\mathrm{CS}} = \emptyset$, the cokernel of $(d_+ + d^*)_\mu$ is isomorphic to $H^1_{\mathrm{dR}}(\hat{N})$.*

Proof. Since $\mu < -1$, the argument in §7.2.1 (noting the reversal of the inequality) generalises to show that

$$d(L^p_{k+1,\,\mu}(\Lambda^2 T^*\hat{N})) = d(L^p_{k+1,\,\mu}(\Lambda^2_+ T^*\hat{N})).$$

Thus, the cokernel of $(d_+ + d^*)_\mu$ is isomorphic to the kernel of

$$(d^* + d)_{-3-\mu} = d^* + d : L^q_{l+1,\,-3-\mu}(\Lambda^3 T^*\hat{N}) \longrightarrow L^q_{l,\,-4-\mu}(\Lambda^2 T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}). \quad (8.5)$$

Using (6.5) and Corollary 6.4.2, since $L^q_{l+1,\,\lambda} \hookrightarrow C^1_\lambda$ by Theorem 6.2.4, the kernel of $(d^* + d)_{-3-(-1)} = (d^* + d)_{-2}$ is isomorphic to $\mathcal{H}^3$. By Theorem 6.5.2(b), $\mathcal{H}^3 \cong H^1_{\mathrm{dR}}(\hat{N})$, where the isomorphism is given by $\gamma \mapsto [*\gamma]$. Since $[\mu, -1) \cap \mathcal{D}_{\mathrm{CS}} = \emptyset$, there are no changes in the cokernel in $[\mu, -1)$ by Theorem 6.3.6. Moreover, the dimension of the cokernel is lower semi-continuous in μ, which can be shown as in the proof of Proposition 7.4.6, noting again that the direction of semi-continuity is reversed from the AC case. The result follows. □

Although the proposition gives us the dimension of the cokernel near -1, we would like to know its dimension just above -2 so that we may calculate the index of $(d_+ + d^*)_\mu$ using Proposition 8.4.2. This is achieved through the next result.

Proposition 8.4.4. *The dimension of the cokernel of* $(d_+ + d^*)_\mu$ *is constant for* $\mu \in (-2,0) \setminus \mathcal{D}_{\mathrm{CS}}$ *and equal to* $b^1(\hat{N})$.

Proof. The constancy of the dimension of the cokernel in $(-2,0)$ is deduced in precisely the same manner as Proposition 7.2.5 because this only uses the fact, given in Proposition 7.2.4, that there are no homogeneous harmonic functions on a 4-dimensional cone of order $O(r^\mu)$ for $\mu \in (-2,0)$. The result follows from Proposition 8.4.3. $\square$

We may now calculate the index of $(d_+ + d^*)_\mu$ for all growth rates using Propositions 8.4.2 and 8.4.4 and Theorem 6.3.6.

Proposition 8.4.5. *Use the notation of Propositions 6.3.4(b) and 6.3.8. If* $\lambda \in (1,2)$, $\lambda \notin \mathcal{D}_{\mathrm{CS}}$, *the index of* $(d_+ + d^*)_\lambda$ *is given by:*

$$\operatorname{ind}(d_+ + d^*)_\lambda = \dim \mathcal{H}^2_+ - b^1(\hat{N}) \; - \sum_{\mu \in (-2,\lambda) \cap \mathcal{D}_{\mathrm{CS}}} \mathrm{d}(\mu).$$

However, the obstruction space $\mathcal{O}_1(N,\lambda)$ given in Definition 8.3.4 is a subspace of the cokernel of $(d_+ + d^*)_\lambda$, so we must relate its dimension to that of the cokernel.

Proposition 8.4.6. *The following inequality holds:* $\dim \mathcal{O}_1(N,\lambda) \leq \dim \operatorname{Coker}(d_+ + d^*)_\lambda - b^1(\hat{N})$.

Proof. From the proof of Proposition 8.3.2, the image of

$$(d + d^*)_\lambda = d + d^* : L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N} \oplus \Lambda^4 T^*\hat{N}) \to L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$$

is characterised as the subspace of $L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$ which is L^2-orthogonal to the kernel $\mathcal{K}$ of $(d^* + d)_{-3-\lambda}$ defined by (8.5). Furthermore, as noticed in the proof of Proposition 8.3.3, $\operatorname{Image}(d + d^*)_\lambda$ has finite codimension in $L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$. Therefore, we may choose a finite-dimensional space $\mathcal{C}$ of smooth compactly supported 3-forms on $\hat{N}$ such that

$$L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N}) = \operatorname{Image}(d + d^*)_\lambda \oplus \mathcal{C}$$

and so that the product $\mathcal{C} \times \mathcal{K} \to \mathbb{R}$ given by $(\gamma,\eta) \mapsto \langle \gamma,\eta\rangle_{L^2}$ is nondegenerate.

We may similarly deduce that $\operatorname{Image}(d_+ + d^*)_\lambda$ is the subspace of $L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N})$ which is L^2-orthogonal to the kernel $\mathcal{K}'$ of $(d^*_+ + d)_{-3-\lambda}$. Then $\mathcal{K}' \supseteq \mathcal{K}$ and $\mathcal{K}$ consists of closed and coclosed 3-forms, whereas $\mathcal{K}'$ consists of 3-forms η such that $d\eta = d^*_+\eta = 0$. Hence, we may choose a subspace $\mathcal{K}''$ of $\mathcal{K}'$, transverse to $\mathcal{K}$, comprising 3-forms which are *not* coclosed and such that $\mathcal{K}' = \mathcal{K} \oplus \mathcal{K}''$.

The next stage is to extend $\mathcal{C}$ to a space $\mathcal{C}' = \mathcal{C} \oplus \mathcal{C}''$, where $\mathcal{C}''$ consists of smooth *exact* compactly supported 3-forms on $\hat{N}$, such that

$$L^p_{k,\,\lambda-1}(\Lambda^3 T^*\hat{N}) = \operatorname{Image}(d_+ + d^*)_\lambda \oplus \mathcal{C}'$$

and such that the product $\mathcal{C}'' \times \mathcal{K}'' \to \mathbb{R}$ given by $(\gamma, \eta) \mapsto \langle \gamma, \eta \rangle_{L^2}$ is nondegenerate, which is possible as $\mathcal{K}''$ comprises forms which are not coclosed. By construction, $\mathcal{C}''$ is a valid choice for $\mathcal{O}_1(N, \lambda)$ by Proposition 8.3.3. Therefore,

$$\dim \mathcal{O}_1(N, \lambda) = \dim \mathcal{C}' - \dim \mathcal{C} = \dim \operatorname{Coker}(d_+ + d^*)_\lambda - \dim \mathcal{K}.$$

If γ lies in the kernel of (8.5) for rate $\mu = -1$ then $\gamma \in \mathcal{K}$ for $\lambda \in (1, 2)$ by Theorem 6.2.4. Thus, the map from $\mathcal{K}$ to $H^1_{\mathrm{dR}}(\hat{N})$ given by $\gamma \mapsto [*\gamma]$ is surjective. This gives the result. $\square$

We may now calculate a lower bound for the expected dimension of $\mathcal{M}_1(N, \lambda)$ using Propositions 8.4.5 and 8.4.6.

Proposition 8.4.7. *Using the notation of Propositions 6.3.4(b) and 6.3.8,*

$$\dim \mathcal{I}_1(N, \lambda) - \dim \mathcal{O}_1(N, \lambda) \geq \dim \mathcal{H}^2_+ - \sum_{\mu \in (-2, \lambda) \cap \mathcal{D}_{\mathrm{CS}}} \mathrm{d}(\mu).$$

Recalling that the dimension of $\mathcal{T}$ given in Definition 8.2.10 is $21s$, we derive analogous results for our other problems.

Proposition 8.4.8. *Using the notation of Definitions 8.2.10 and 8.2.18 and Propositions 6.3.4(b) and 6.3.8,*

$$\dim \mathcal{I}_2(N, \lambda) - \dim \mathcal{O}_2(N, \lambda) \geq \dim \mathcal{H}^2_+ + 21s - \dim \mathrm{H} - \sum_{\mu \in (-2, \lambda) \cap \mathcal{D}_{\mathrm{CS}}} \mathrm{d}(\mu).$$

and

$$\dim \mathcal{I}_3(N, \lambda) - \dim \mathcal{O}_3(N, \lambda) \geq \dim \mathcal{H}^2_+ + 21s - \dim \mathrm{H} + \dim \hat{\mathcal{F}} - \sum_{\mu \in (-2, \lambda) \cap \mathcal{D}_{\mathrm{CS}}} \mathrm{d}(\mu).$$

We note that Propositions 8.4.4, 8.4.6 and 8.4.7 imply the following bound on $\dim \mathcal{O}_1(N, \lambda)$.

Proposition 8.4.9. *In the notation of Propositions 6.3.4(b) and 6.3.8,*

$$\dim \mathcal{O}_1(N, \lambda) \leq \sum_{\mu \in [0, \lambda) \cap \mathcal{D}_{\mathrm{CS}}} \mathrm{d}(\mu)$$

We also know that, in Problem 2, we remove the obstructions which correspond to translations of the singularities and G_2 transformations of the tangent cones. These obstructions occur, respectively, at rates 0 and 1. Hence, $\mathrm{d}(0) \geq 7s$, $\mathrm{d}(1) \geq 14s - \dim \mathrm{H}$ and we have the following stronger bound on the dimension of $\mathcal{O}_2(N, \lambda)$.

Proposition 8.4.10. *In the notation of Definition 8.2.10 and Propositions 6.3.4(b) and 6.3.8,*

$$\dim \mathcal{O}_2(N, \lambda) \leq -21s + \dim \mathrm{H} + \sum_{\mu \in [0, \lambda) \cap \mathcal{D}_{\mathrm{CS}}} \mathrm{d}(\mu).$$

8.5 φ-Closed 7-Manifolds

For our deformation problems we have assumed the ambient manifold (M, φ, g) is a G_2 manifold; that is, M is endowed with a G_2 structure such that $d\varphi = d^*\varphi = 0$. However, the results of McLean [45] we have used, which are based upon the linearisation of the map we denoted F_1 in Definition 8.2.4, still hold if this condition on φ is relaxed to just $d\varphi = 0$. Thus, our deformation theory results hold if (M, φ, g) is a φ-closed 7-manifold in the sense of Definition 2.3.11. The effect of $*\varphi$ not being closed on M means that coassociative 4-folds in M are no longer necessarily volume minimizing in their homology class. This does not, however, affect our discussion.

The use of φ-closed 7-manifolds (M, φ, g) is that closed G_2 structures occur in infinite-dimensional families, since the set of closed definite 3-forms on M, in the sense of Definition 2.3.8, is open. We show that we can choose a family $\mathcal{F}$, in a similar fashion to Definition 8.2.18 of Problem 3, of closed G_2 structures on M such that $\dim \mathcal{F} = \dim \mathcal{O}_1(N, \lambda)$ and, further, such that $\mathcal{O}_3(N, \lambda) = \{0\}$. In other words, we have enough freedom in our choice of $\mathcal{F}$ to ensure that $dF_3|_{(0,0,0)}$, as given in Definition 8.2.22, maps onto $d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N}))$. Then $\mathcal{M}_3(N, \lambda)$ is a smooth manifold near $(N, 0)$ by Theorem 8.3.13 and $\pi_{\hat{\mathcal{F}}} : \mathcal{M}_3(N, \lambda) \to \hat{\mathcal{F}}$ is a smooth map near $(N, 0)$. By *Sard's Theorem*, which is stated in the argument leading to Theorem 4.2.6, $\pi_{\hat{\mathcal{F}}}^{-1}(f)$ is a smooth manifold near (N, f) for almost all $f \in \hat{\mathcal{F}}$. As observed in Definition 8.2.19, $\pi_{\hat{\mathcal{F}}}^{-1}(f)$ corresponds to the moduli space of deformations for Problem 2 defined using the G_2 structure (φ^f, g^f). Thus, for any given N, a generic perturbation of the closed G_2 structure within $\mathcal{F}$ ensures that $\mathcal{M}_2(N, \lambda)$ is smooth near N.

We thus prove the following, which is similar to the result [29, Theorem 9.1].

Theorem 8.5.1. *Let (M, φ, g) be a φ-closed 7-manifold in the sense of Definition 2.3.11 and let N in (M, φ, g) be a CS coassociative 4-fold at $z_1, \ldots, z_s$ with rate $\lambda \in (1, 2) \setminus \mathcal{D}_{\mathrm{CS}}$, where $\mathcal{D}_{\mathrm{CS}}$ is defined in Proposition 6.3.4(b). Use the notation of Definitions 8.2.10, 8.3.4 and 8.3.12 and Proposition 6.3.8. Let $m = \dim \mathcal{O}_1(N, \lambda)$ and let $\hat{\mathcal{F}}$ be an open ball about 0 in $\mathbb{R}^m$. There exists a smooth family $\mathcal{F} = \{(\varphi^f, g^f) : f \in \hat{\mathcal{F}}\}$ of closed G_2 structures on M such that $\mathcal{O}_3(N, \lambda) = \{0\}$. Hence, the moduli space of deformations for Problem 3 is a smooth manifold near $(N, 0)$ of dimension greater than or equal to*

$$\dim \mathcal{H}^2_+ + 21s - \dim \mathrm{H} + \dim \mathcal{O}_1(N, \lambda) - \sum_{\mu \in (-2,\lambda) \cap \mathcal{D}_{\mathrm{CS}}} \mathrm{d}(\mu).$$

Moreover, for generic $f \in \hat{\mathcal{F}}$, the moduli space of deformations in (M, φ^f, g^f) for Problem 2 is a smooth manifold near N of dimension greater than or equal to

$$\dim \mathcal{H}^2_+ + 21s - \dim \mathrm{H} - \sum_{\mu \in (-2,\lambda) \cap \mathcal{D}_{\mathrm{CS}}} \mathrm{d}(\mu).$$

Proof. Use the notation in the proof of Proposition 8.4.6. Recall that we have a subspace $\mathcal{K}''$ of $L^q_{l+1,\,-3-\lambda}(\Lambda^3 T^*\hat{N})$ consisting of forms η such that $d\eta = d^*_+\eta = 0$ but $d^*\eta \neq 0$. Moreover, $\mathcal{O}_1(N,\lambda)$ can be chosen to be a space of smooth compactly supported exact 3-forms γ such that $\langle\gamma,\eta\rangle_{L^2} = 0$ for all $\eta \in \mathcal{K}'' \setminus \{0\}$ implies that $\gamma = 0$. Therefore $\mathcal{K}'' \cong (\mathcal{O}_1(N,\lambda))^*$ and hence has dimension m.

Let $\{\eta_1,\ldots,\eta_m\}$ be a basis for $\mathcal{K}''$ and choose a basis $\{d\upsilon_1,\ldots,d\upsilon_m\}$ for $\mathcal{O}_1(N,\lambda)$, where υ_j is a smooth compactly supported 2-form for all j, such that $\langle d\upsilon_i,\eta_j\rangle_{L^2} = \delta_{ij}$. This is possible because the L^2 product on $\mathcal{O}_1(N,\lambda)\times\mathcal{K}''$ is nondegenerate. For $f = (f_1,\ldots,f_m) \in \mathbb{R}^m$ define

$$\upsilon_f = \sum_{j=1}^m f_j\upsilon_j.$$

Using the notation of Proposition 8.2.17, define (φ^f, g^f), for f in a sufficiently small open ball $\hat{\mathcal{F}}$ about 0 in $\mathbb{R}^m$, to be a closed G_2 structure on M such that $\Xi([\varphi^f|_T]) = 0$ in $H^3_{\mathrm{cs}}(\hat{N})$ and $\varphi^f|_{\hat{N}} = d\upsilon_f$. Recall from Definitions 8.2.22 and 8.3.12 that we have a linear map $L_3 : T_0\hat{\mathcal{F}} \cong \mathbb{R}^m \to d(L^p_{k+1,\,\lambda}(\Lambda^2 T^*\hat{N}))$ arising from $dF_3|_{(0,0,0)}$. By construction, $L_3(f) = d\upsilon_f$ for $f \in \mathbb{R}^m$ and hence L_3 maps onto $\mathcal{O}_1(N,\lambda)$. Proposition 8.3.3 and Definition 8.3.12 imply that $\mathcal{O}_3(N,\lambda) = \{0\}$ as required.

The latter parts of the theorem follow from the discussion preceding it and Proposition 8.4.8. $\square$

Afterword: Further Research

There are no solved problems; there are only problems that are more or less solved.
– Henri Poincaré

It is unsurprising and, perhaps, only right that a thesis such as this should end not with a sense of finality but detailing possibilities for the continuation of the study described within it.

The work in Part I generates a number of obvious problems to tackle. The systems of differential equations given in Sections 4.2 and 4.4 defining associative 3-folds are either unsolved or are only solved in certain circumstances. There is similar unfinished work on coassociative and Cayley 4-folds with symmetries in §5.2 and §5.3. It is the author's hope that these systems may be fully solved. In particular, if solutions are found which provide examples of cones, the material in §4.5 and §5.4 will help produce further calibrated submanifolds in $\mathbb{R}^7$ and $\mathbb{R}^8$.

The only notable deficiency in the results of Part II is the precise determination of the expected dimension of the moduli spaces for all relevant growth rates. However, the theory developed in Chapters 7 and 8 does inspire a number of possible research topics. The long-term project involving the 7-dimensional analogue of the SYZ conjecture was mentioned in the preface. As a step towards this goal, it will be necessary to have the ability to *resolve* the singularities of CS coassociative 4-folds. To achieve this using AC coassociative 4-folds, as is likely to be the case, will require the deformation theories described in the last two chapters.

One could also consider deriving analogous theory to Part II for associative and Cayley submanifolds. The results of McLean for compact deformations, described at the end of Sections 2.3 and 2.4, suggest that the investigation for Cayley 4-folds will bear the most fruit. Moreover, one could attempt to prove an 8-dimensional version of the SYZ conjecture. However, since the nature of the deformation theory of Cayley 4-folds is significantly different from the coassociative scenario, the author expects these generalisations to be far from immediate.

We conclude with one final thought.

The moving power of mathematical invention is not reasoning but imagination.
– Augustus de Morgan

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Calibrated Submanifolds and the Exceptional Geometries

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The research presented here lies within the broader subject of *calibrated geometry*. In Chapter 1 the basic theory underlying this topic is given. Number systems and their relationship with group theory are also discussed at an elementary level, providing background material before the exposition of the *octonions*, or *Cayley numbers*, in Chapter 2.

The focus of Chapter 2 is on the exceptional Lie groups G_2 and Spin(7) and the formulation of calibrations on $\mathbb{R}^7$ and $\mathbb{R}^8$ which are associated to these groups. This allows the definition of *associative 3-folds* and *coassociative 4-folds* in $\mathbb{R}^7$ and *Cayley 4-folds* in $\mathbb{R}^8$. Moreover, it is demonstrated that there is a generalisation of these calibrations and calibrated submanifolds for particular 7- and 8-dimensional Riemannian manifolds, known as G_2 and Spin(7) manifolds respectively.

Following these two preliminary chapters, the rest of the dissertation is split into two parts. Part I begins, in Chapter 3, with a review of the theory and constructions of *special Lagrangian m-folds* in $\mathbb{C}^m$ which shall be pertinent in the sequel. Chapter 4 gives construction methods and examples for associative 3-folds in $\mathbb{R}^7$. The final chapter in Part I is a similar presentation for coassociative 4-folds in $\mathbb{R}^7$ and Cayley 4-folds in $\mathbb{R}^8$.

Chapter 6, the first chapter in Part II, reviews various definitions and results from the study of analysis on *asymptotically conical* (AC) manifolds and manifolds with *conical singularities* (CS). The final two chapters are dedicated to the study of *deformations* of AC coassociative 4-folds in $\mathbb{R}^7$ and CS coassociative 4-folds in a G_2 manifold. In Chapter 7 it is proved that an AC coassociative 4-fold, which converges with generic rate in a specified range to a cone at infinity, has a locally smooth moduli space of deformations of known dimension. In Chapter 8, three different deformation problems for CS coassociative 4-folds are studied. For each case there is a weaker result: the moduli space is locally homeomorphic to the kernel of a smooth map between smooth manifolds. However, if the obstructions in the problem are known to be zero, the moduli space is locally smooth and a lower bound is given on its dimension.