D-manifolds and d-orbifolds: a theory of derived differential geometry. III.

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Based on survey paper:

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and preliminary version of book which may be downloaded from

`people.maths.ox.ac.uk/`

`~joyce/dmanifolds.html.`

These slides available at

`people.maths.ox.ac.uk/~joyce/talks.html.`

7. Comparing d-manifolds and d-orbifolds with other spaces

In enumerative invariant problems in differential and symplectic geometry, and algebraic geometry over $\mathbb{C}$, there are several classes of geometric structure one puts on moduli spaces, in order to define virtual cycles/virtual chains, and 'count' the points in the moduli space.

There are truncation functors from essentially all these structures to d-manifolds or d-orbifolds. This includes Kuranishi spaces, polyfolds, and $\mathbb{C}$-schemes or Deligne–Mumford $\mathbb{C}$-stacks with obstruction theories.

7.1. Nonlinear elliptic equations

Theorem 9. *Let V be a Banach manifold, $E \to V$ a Banach vector bundle, and $s : V \to E$ a smooth Fredholm section, with constant Fredholm index $n \in \mathbb{Z}$. Then there is a d-manifold $\boldsymbol{X}$, unique up to equivalence in* $\mathbf{dMan}$, *with topological space $X = s^{-1}(0)$ and* vdim $\boldsymbol{X} = n$.

Nonlinear elliptic equations on compact manifolds induce nonlinear Fredholm maps on Hölder or Sobolev spaces of sections. We deduce:

Corollary. *Let $\mathcal{M}$ be a moduli space of solutions of a nonlinear elliptic equation on a compact manifold, with fixed topological invariants. Then $\mathcal{M}$ extends to a d-manifold.*

7.2. Kuranishi spaces

Kuranishi spaces (both without boundary, and with corners) appear in the work of Fukaya–Oh–Ohta–Ono as the geometric structure on moduli spaces of J-holomorphic curves in symplectic geometry.

They do not define morphisms between Kuranishi spaces, so Kuranishi spaces are not a category. But they do define morphisms $\boldsymbol{f}:\boldsymbol{X}\rightarrow Z$ from Kuranishi spaces $\boldsymbol{X}$ to manifolds or orbifolds Z, and 'fibre products' $\boldsymbol{X}\times_Z\boldsymbol{Y}$ of Kuranishi spaces over manifolds or orbifolds.

I began this project to find a better definition of Kuranishi space, with well-behaved morphisms.

Theorem 10(a) *Suppose $\mathcal{X}$ is a d-orbifold with corners. Then (after many choices) one can construct a Kuranishi space $\mathbf{X}'$ with the same topological space and dimension.*
(b) *Let $\mathbf{X}'$ be a Kuranishi space. Then one can construct a d-orbifold with corners $\mathcal{X}''$, unique up to equivalence in $\mathbf{dOrb^c}$, with the same topological space and dimension.*
(c) *Doing* **(a)** *then* **(b)**, *$\mathcal{X}$ and $\mathcal{X}''$ are equivalent in $\mathbf{dOrb^c}$.*
(d) *The constructions of* **(a)**,**(b)** *identify orientations, morphisms $f : \mathcal{X} \to Y$ to manifolds or orbifolds Y, and fibre products over manifolds and orbifolds, for d-orbifolds with corners and Kuranishi spaces.*

Roughly speaking, Theorem 10 says that d-orbifolds with corners $\mathbf{dOrb^c}$ and Kuranishi spaces are equivalent categories, except that Kuranishi spaces are not a category as morphisms are not defined.

The moral is (I claim): *the 'correct' way to define Kuranishi spaces is as d-orbifolds with corners*.

I prove Theorem 10 using 'good coordinate systems' (families of Kuranishi neighbourhoods with nicely compatible coordinate changes).

Given a good coordinate system $(\mathcal{V}_i, \mathcal{E}_i, s_i, \psi_i)$, $i \in I$ on a Kuranishi space $\boldsymbol{X}'$, make corresponding d-orbifold $\boldsymbol{\mathcal{X}}''$ by gluing 'standard models' $\boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ by equivalences.

7.3. Polyfolds

Polyfolds, due to Hofer, Wysocki and Zehnder, are a rival theory to Kuranishi spaces. They do form a category. Polyfolds remember much more information than Kuranishi spaces or d-orbifolds, so the truncation functor goes only one way.

Theorem 11. *There is a functor* $\Pi_{\mathbf{PolFS}}^{\mathbf{dOrb^c}} : \mathbf{PolFS} \to \mathrm{Ho}(\mathbf{dOrb^c})$, *where* $\mathbf{PolFS}$ *is a category whose objects are triples* $(\mathcal{V}, \mathcal{E}, s)$ *of a polyfold with corners* $\mathcal{V}$, *a fillable strong polyfold bundle* $\mathcal{E}$ *over* V, *and an sc-smooth Fredholm section* s *of* E *with constant Fredholm index.*

Here $\mathrm{Ho}(\mathbf{dOrb^c})$ is the homotopy 1-category of the 2-category $\mathbf{dOrb^c}$.

7.4. $\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theories

Theorem 12. *There is a functor* $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}} : \mathbf{Sch_{\mathbb{C}}Obs} \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathbf{Sch_{\mathbb{C}}Obs}$ *is a category whose objects are triples* $(X, E^\bullet, \phi)$, *for* X *a separated, second countable* $\mathbb{C}$*-scheme and* $\phi : E^\bullet \to \tau_{\geqslant -1}(\mathbb{L}_X)$ *a perfect obstruction theory on* X *with constant virtual dimension. We may define a natural orientation on* $\Pi_{\mathbf{SchObs}}^{\mathbf{dMan}}(X, E^\bullet, \phi)$ *for each* $(X, E^\bullet, \phi)$.

The analogue holds for $\Pi_{\mathbf{StaObs}}^{\mathbf{dOrb}} : \mathbf{Sta_{\mathbb{C}}Obs} \to \mathrm{Ho}(\mathbf{dOrb})$, *replacing* $\mathbb{C}$*-schemes by Deligne–Mumford* $\mathbb{C}$*-stacks, and d-manifolds by d-orbifolds.*

In algebraic geometry, the standard method of forming virtual cycles is to use a proper scheme or Deligne–Mumford stack equipped with a *perfect obstruction theory*, in the sense of Behrend–Fantechi. They are used to define algebraic Gromov–Witten invariants, Donaldson–Thomas invariants of Calabi–Yau 3-folds, ... Note that we can make moduli of J-holomorphic curves in projective complex manifolds into d-orbifolds either symplectically using Kuranishi spaces/polyfolds in Theorems 10, 11, or algebro-geometrically using Theorem 12. So we can compare symplectic and algebraic Gromov–Witten invariants.

7.5. Spivak's derived manifolds

Using Jacob Lurie's derived algebraic geometry, David Spivak (in arXiv:0810.5174) defined an ∞-category of *derived manifolds*.

Theorem 13. (D. Borisov) *Write* $\mathbf{DerMan}$ *for the* ∞*-category of Spivak's derived manifolds of pure dimension, and* $\pi_1(\mathbf{DerMan})$ *for its* 2*-category truncation. There is a* 2*-functor* $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}:\pi_1(\mathbf{DerMan})\rightarrow\mathbf{dMan}$ *which is almost an equivalence of* 2*-categories.*

That is, $\Pi_{\mathbf{DerMan}}^{\mathbf{dMan}}$ *induces bijections on equivalence classes of objects, and on* 2*-isomorphism classes of* 1*-morphisms. On* 2*-morphisms it is surjective, but may not be injective.*

Combining the truncation functors of Theorems 9–13 with results in the literature on existence of geometric structures like Kuranishi spaces, . . . on moduli spaces, proves existence of d-manifold or d-orbifold structures on many important moduli spaces in geometry. So we can apply virtual cycle/virtual chain constructions for d-manifolds and d-orbifolds to get alternative definitions of G–W invariants, D–T invariants, Lagrangian Floer cohomology, etc. We may also be able to define new, finer invariants using d-manifold or d-orbifold bordism.

8. *** Work in progress ***

8.1. D-manifold and d-orbifold homology and cohomology

Based on my old unpublished work arXiv:0707.3572, arXiv:0710.5634 on 'Kuranishi homology', for Y a manifold and R a ring or $\mathbb{Q}$-algebra, I hope to define 'd-manifold homology' $dH_*(Y;R)$ and 'd-manifold cohomology' $dH^*(Y;R)$, which are isomorphic to ordinary (singular) homology $H_*^{\mathrm{sing}}(Y;R)$ and cohomology $H^*(Y;R)$. Here $dH_*(Y;R)$ is the cohomology of a complex of R-modules $\bigl(dC_*(Y;R);\partial\bigr)$.

There will also be orbifold/d-orbifold versions of the theories.

A bit like the definition of d-manifold bordism, chains in $dC_k(Y;R)$ for k in $\mathbb{Z}$ will be R-linear combinations of equivalence classes $[\boldsymbol{X},\boldsymbol{f},\boldsymbol{G}]$, where $\boldsymbol{X}$ is a compact, oriented d-manifold with corners, $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}=F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$ is a 1-morphism in $\mathbf{dMan^c}$, and $\boldsymbol{G}$ is some extra 'gauge-fixing data' associated to $\boldsymbol{X}$, for which there will be many possible choices. If we did not include $\boldsymbol{G}$ then chains $(\boldsymbol{X},\boldsymbol{f})$ might have infinite automorphism groups, leading to bad behaviour ($dH_*(Y;R)=0$). We define $\boldsymbol{G}$ to ensure $\mathrm{Aut}(\boldsymbol{X},\boldsymbol{f},\boldsymbol{G})$ is finite.

The boundary operator $\partial : dC_k(Y;R) \to dC_{k-1}(Y;R)$ maps

$$\partial : [\boldsymbol{X}, \boldsymbol{f}, \boldsymbol{G}] \longmapsto [\partial \boldsymbol{X}, \boldsymbol{f} \circ \boldsymbol{i}_{\boldsymbol{X}}, \boldsymbol{G}|_{\partial \boldsymbol{X}}].$$

Note that $\partial^2 \boldsymbol{X}$ has a free, orientation-reversing involution $\sigma : \partial^2 \boldsymbol{X} \to \partial^2 \boldsymbol{X}$. Using this we show that $\partial^2 = 0 : dC_k(Y;R) \to dC_{k-2}(Y;R)$.
Singular homology $H_*^{\mathrm{sing}}(Y;R)$ may be defined using $\bigl(C_*^{\mathrm{sing}}(Y;R);\partial\bigr)$, where $C_k^{\mathrm{sing}}(Y;R)$ is spanned by *smooth* maps $f : \Delta_k \to Y$, for Δ_k the standard k-simplex, thought of as a manifold with corners.

We define an R-linear map $F_{\mathrm{sing}}^{\mathrm{dMan}}$: $C_k^{\mathrm{sing}}(Y;R) \to dC_k(Y;R)$ by

$$F_{\mathrm{sing}}^{\mathrm{dMan}} : f \longmapsto \big[F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(\Delta_k), F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(f), \boldsymbol{G}_{\Delta_k}\big],$$

for $\boldsymbol{G}_{\Delta_k}$ some standard choice of gauge-fixing data for Δ_k. We can arrange that $F_{\mathrm{sing}}^{\mathrm{dMan}} \circ \partial = \partial \circ F_{\mathrm{sing}}^{\mathrm{dMan}}$, so that $F_{\mathrm{sing}}^{\mathrm{dMan}}$ induces morphisms $F_{\mathrm{sing}}^{\mathrm{dMan}} : H_k^{\mathrm{sing}}(Y;R) \to dH_k(Y;R)$, and we will (I hope) prove these are isomorphisms.

What is the point of d-manifold and d-orbifold (co)homology?

These (co)homology theories have two special features:

(a) they are very well adapted for forming *virtual cycles* and *virtual chains* in moduli problems. They are particularly powerful for moduli spaces 'with corners', as in Lagrangian Floer homology and Symplectic Field Theory.

(b) issues to do with *transversality* — for instance, defining intersection products on transverse chains — often disappear in d-manifold and d-orbifold (co)homology, because of Theorem 2.

Current methods for forming virtual cycles and virtual chains in symplectic geometry (Kuranishi spaces FOOO, polyfolds HWZ) involve making a (multi-valued) perturbation of the moduli space, and then triangulating the perturbed moduli space by simplices to get a singular chain. When the moduli spaces have boundary and corners, one must choose perturbations compatible with other previously-chosen perturbations at the boundary, and insertions of singular chains. This gets very complicated and messy.

In d-orbifold (co)homology, given a moduli space $\bar{\mathcal{M}}$ with evaluation maps ev : $\bar{\mathcal{M}} \rightarrow L$ for L a manifold, we make $\bar{\mathcal{M}}$ into an oriented d-orbifold with corners $\bar{\boldsymbol{\mathcal{M}}}$ with evaluation 1-morphism $\mathbf{ev} : \bar{\boldsymbol{\mathcal{M}}} \rightarrow \boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dOrb^c}}(Y)$. Then we choose some gauge-fixing data $\boldsymbol{G}$ and define the virtual chain to be $[\bar{\boldsymbol{\mathcal{M}}}, \mathbf{ev}, \boldsymbol{G}]$ in $dC_*(Y;\mathbb{Q})$. Thus, *the moduli space is its own virtual chain*. There is *no need for perturbation*. Instead, we need only choose gauge-fixing data, which is easier and can be done compatibly with infinitely many choices. This leads to big simplifications in Fukaya–Oh–Ohta–Ono's Lagrangian Floer cohomology.

8.2. An application: String Topology

(In progress, joint with L. Amorim.)

Let M be an n-manifold. The *loop space* $\mathcal{L}M$ of M is the infinite-dimensional manifold of smooth maps $\gamma : \mathcal{S}^1 \to M$. Can also consider $\mathcal{L}M/\mathcal{S}^1$. *String Topology*, introduced by Chas and Sullivan, studies new algebraic operations on the homology $H_*(\mathcal{L}M;\mathbb{Q})$. They are defined using transversely intersecting families of loops in M.

Now d-manifold homology deals very nicely with issues of transversality. So it may be a good tool for studying String Topology.

I propose to define a chain model $(dC_*(\mathcal{L}M,\mathbb{Q}),\mathrm{d})$ for $H_*(\mathcal{L}M;\mathbb{Q})$ such that the String Topology operations can be defined *at the chain level*, not just at the homology level, and satisfy the expected identities on the nose at the chain level, not just up to homotopy.

The basic idea is this: let $\boldsymbol{X}$ be a d-manifold with corners. Then a smooth map $\boldsymbol{f}:\boldsymbol{X}\rightarrow\mathcal{L}M$ is the same as a 1-morphism $\boldsymbol{f}:\boldsymbol{X}\times\boldsymbol{\mathcal{S}}^{\mathbf{1}}\rightarrow \boldsymbol{M}=F_{\mathbf{Man}}^{\mathbf{dMan^c}}(M)$ in $\mathbf{dMan^c}$. Thus, to deal with loop spaces we don't need to extend theory to infinite dimensions, we can just work in $\mathbf{dMan^c}$.

The first version of $(dC_*(\mathcal{L}M,\mathbb{Q}),\mathrm{d})$ has chains $[\boldsymbol{X},\boldsymbol{f},\boldsymbol{G}]$ where $\boldsymbol{X}$ is a compact, oriented d-manifold with corners, $\boldsymbol{f}:\boldsymbol{X}\times\boldsymbol{\mathcal{S}}^{\mathbf{1}}\to\boldsymbol{M}$ is a 1-morphism in $\mathbf{dMan^c}$, and $\boldsymbol{G}$ is gauge-fixing data for $\boldsymbol{X}$. I claim this complex computes $H_*(\mathcal{L}M;\mathbb{Q})$; the proof should be basically the same as $F_{\mathrm{sing}}^{\mathrm{dMan}}$ an isomorphism in 8.1.
Will need a more complicated definition of $(dC_*(\mathcal{L}M,\mathbb{Q}),\mathrm{d})$ to define String Topology operations on (in progress).
Note: I hope to apply this String Topology model to prove conjectures/partial proofs of Fukaya on topology of Lagrangians.

8.3. Donaldson–Thomas type invariants for Calabi–Yau 4-folds

***** Work in progress *****

Let X be a projective complex manifold of dimension m, and $\mathcal{M}$ be a moduli scheme of stable coherent sheaves on X. Then $\mathcal{M}$ has an *obstruction theory* $\phi : E^\bullet \to \mathbb{L}_M$ which is perfect of amplitude in $[1-m, 0]$, and for each coherent sheaf F in $\mathcal{M}$ encodes the groups $\mathrm{Ext}^i(F,F)$ for $i = 1, \ldots, m$.

If X is a Calabi–Yau m-fold then we can make $E^\bullet, \phi$ perfect of amplitude in $[2-m, 0]$, and get a duality $\theta : E^\bullet \to (E^\bullet)^\vee[m-2]$.

The cases corresponding to d-manifolds (the ‘quasi-smooth’ case) is when $E^\bullet,\phi$ has amplitude in $[-1,0]$, i.e. sheaves on surfaces when $m=2$ (Donaldson theory) or Calabi–Yau 3-folds when $m=3$ (Donaldson–Thomas invariants). In these cases we can make moduli schemes $\mathcal{M}$ into d-manifolds, and define virtual cycles and invariants.

I believe there is a third case: Calabi–Yau 4-folds. Then $E^\bullet,\phi$ has amplitude in $[-2,0]$, and $E^\bullet$ has a duality taking degree i to degree $-2-i$.

I claim that I can define a d-manifold structure on moduli schemes $\mathcal{M}$ of stable coherent sheaves on Calabi–Yau 4-folds. This encodes 'half' of $E^\bullet, \phi$: all of $\mathrm{Ext}^1(F,F)^*$ in degree 0, the 'real part' of the complex vector space $\mathrm{Ext}^2(F,F)^*$ in degree -1, and none of $\mathrm{Ext}^3(F,F)^*$ in degree -2. The *real* virtual dimension of the d-manifold $\boldsymbol{\mathcal{M}}$ is the *complex* virtual dimension (i.e. half of the real virtual dimension) of the obstruction theory $E^\bullet, \phi$. So, for strictly complex-algebraic input, I use d-manifolds to define a virtual cycle which can have odd real dimension. This is very weird. I know of no way to do this using algebraic geometry.

Question for the audience:

Can you think of your own applications for d-manifolds and d-orbifolds?