Axiomatic Set Theory: Problem sheet 1

1. Which of the ZF axioms (A1)–(A2), (A7) and (A8) hold in the structure $\langle \mathbb{Q}, < \rangle$? Also, find an instance of (A5) that is true in $\langle \mathbb{Q}, < \rangle$ and one that is false.

2. Write the following as formulas of LST:
(a) $x = \langle y, z \rangle$;
(b) $x = y \times z$;
(c) $x = y \cup \{y\}$;
(d) "x is a successor set";
(e) $x = \omega$.

3. Assuming ZF*, show that there exists a *transitive* set M such that
(a) $\varnothing \in M$, and
(b) if $x \in M$ and $y \in M$, then $\{x, y\} \in M$, and
(c) every element of M contains at most two elements.
Show further that if σ is an axiom of ZF*+AC other than (A8), (A4) or (A7), then $\langle M, \in \rangle \models \sigma$. (It follows that if ZF* is consistent then so is (ZF*+AC$\setminus\{$(A8), (A4), (A7)$\}$.)

4. Assuming ZF show that if a is a non-empty transitive set then $\varnothing \in a$.

5. Deduce (A3) (pairing) from the other axioms of ZF*.

Axiomatic Set Theory: Problem sheet 2

1. (a) Assuming ZF (ie. ZF^*+Foundation) prove that the following two definitions of "ordinal" are equivalent:

(i) An ordinal is a transitive set well-ordered by $\in$.

(ii) An ordinal is a transitive set totally ordered by $\in$.

(b) Prove theorem 3.10—the principle of induction for On—using only ZF^*.

2. (ZF^*) Define a "natural" ordinal exponentiation using the recursion theorem for ordinals, and show that for all ordinals α, β and γ, $\alpha^{(\beta+\gamma)} = \alpha^\beta \alpha^\gamma$, and $\alpha^{(\beta\cdot\gamma)} = (\alpha^\beta)^\gamma$. Show also that $2^\omega = \omega$.

3. (ZF^*) Suppose $F : On \to On$ is a class term satisfying:

(1) $\alpha < \beta \to F(\alpha) < F(\beta)$ (for $\alpha, \beta \in On$)

(2) $F(\delta) = \bigcup_{\alpha<\delta} F(\alpha)$ (for limit ordinals δ).

Prove that for all $\alpha \in On$ there exists $\beta \in On$ such that $\beta > \alpha$ and $F(\beta) = \beta$ (ie. F has arbitrarily large fixed points). What is the smallest non-zero fixed point of the term $F : On \to On$ defined by $F(x) = \omega.x$ (for $x \in On$)?

4. (ZF) Let H_ω denote the class of *hereditarily finite sets*, ie. $H_\omega = \{x : TC(x)$ is finite$\}$. Prove that $H_\omega = V_\omega$ (and hence that H_ω is a set). Prove that $\langle V_\omega, \in\rangle \models$ the axiom of foundation, and $\langle V_\omega, \in\rangle \models \neg$ the axiom of infinity.

[It is easy, but tedious, to check that $\langle V_\omega, \in\rangle \models$ the other axioms of ZF. This shows that the other axioms of ZF do not imply the axiom of infinity.]

Axiomatic Set Theory: Problem sheet 3

1. (ZF*) Prove that $(V, \in) \vDash$ A4 (Union) and that $(V, \in) \vDash$ A8 (Infinity).

2. (ZF*) Prove that the axiom of foundation is equivalent to $\forall x(x \in V)$.

3. (ZF*) Let $\alpha \in On$ and suppose that $a \in V_\alpha$ and $b \subseteq a$. Prove that $b \in V_\alpha$.

4. (ZF*) Later in the course we shall be concerned with those formulas whose truth does not depend on which transitive class they are interpreted in. More precisely, let A be a transitive class. A formula $\phi(v_1, \ldots, v_n)$ (*without* parameters) of LST is called A-*absolute* if for any $a_1, \ldots, a_n \in A$, $\phi(a_1, \ldots, a_n)$ holds (ie. $(V^*, \in) \vDash \phi(a_1, \ldots, a_n)$) iff $\phi(a_1, \ldots, a_n)$ holds in A (ie. $(A, \in) \vDash \phi(a_1, \ldots, a_n)$). Prove that the following statements (or the natural formulas of LST which these translate) are A-absolute, for any transitive class A:

(i) $v_1 \subseteq v_2$ (ii) $v_1 = \bigcup v_2$ (iii) $v_1 = \{v_2, v_3\}$ (iv) $v_1 = v_2 \cup \{v_2\}$.

Show that "$v_1 = \mathbb{P} v_2$" is not ω-absolute. (Note that ω is a transitive class.)

Axiomatic Set Theory: Problem sheet 4

1. Prove that $\forall\alpha,\beta\in On$, (i) $V_\alpha\cap On=\alpha$, and (ii) if $\alpha\in V_\beta$, then $V_\alpha\in V_\beta$.

2. Complete the proof of Lévy's Reflection Principle.

3. A *club* is, by definition, a closed, unbounded class of ordinals. Prove that if U_1 and U_2 are clubs then so is $U_1\cap U_2$. More generally, suppose that X is a class such that $X\subseteq\omega\times On$. For $i\in\omega$, let $X_i=\{\alpha\in On:\langle i,\alpha\rangle\in X\}$. Suppose that for all $i\in\omega$, X_i is a club. Prove that $\bigcap_{i\in\omega}X_i$ is a club.

4. (i) It is known that there is a formula $\phi(x)$ of LST (without parameters) such that (in ZF one can prove that) for any set a, $\phi(a)$ iff "$\langle a,\in\rangle\models$ ZF and a is transitive". Further, this formula is A-absolute for any transitive class A (see sheet 3, question 4). Show that one cannot prove the sentence $\exists x\phi(x)$ from ZF. [Hint: Consider the least $\alpha\in On$ such that $\exists x\in V_\alpha(\phi(x))$.]

(ii) As formulated in the lectures, ZF is a countably infinite collection of axioms (since there is one separation and replacement axiom for each formula of LST, and there are clearly a countably infinite number of such formulas). Prove that there is no finite subcollection, T, say, of ZF, such that $T\vdash$ ZF.

5. * What is wrong with the following argument:

Let $\{\sigma_i:i\in\omega\}$ be an enumeration of all the axioms of ZF. By Lévy's Reflection Principle, for each $i\in\omega$, the class $\{\alpha\in On:\langle V_\alpha,\in\rangle\models\sigma_i\}$ (call it X_i) is a club (since $(V,\in)\models\sigma_i$). By question (3) above, $\bigcap_{i\in\omega}X_i$ is a club (we are using question (3) by setting $X=\{\langle i,\alpha\rangle:\alpha\in X_i\}$). In particular, $\bigcap_{i\in\omega}X_i$ is non-empty. Let $\beta\in\bigcap_{i\in\omega}X_i$. Then $\beta\in X_i$ for all $i\in\omega$, so $\langle V_\beta,\in\rangle\models\sigma_i$ for all $i\in\omega$, so $\langle V_\beta,\in\rangle\models$ ZF. Hence $\phi(V_\beta)$ holds, so $\exists x\phi(x)$ (where $\phi(x)$ is the formula in (4)(i)). Since $(V,\in)$ is an arbitrary model of ZF, we have ZF $\vdash\exists x\phi(x)$!

Axiomatic Set Theory: Problem sheet 5

1. Assuming (as was shown in the lectures), that $a \in L \to \bigcup a \in L$ and $a \in L \to \mathbb{P}a \cap L \in L$, verify carefully that $\langle L, \in \rangle \models$ union, powerset.

2. The *rank* of a set A, $rk(A)$, is defined to be the least $\alpha \in On$ such that $A \subseteq V_\alpha$. Prove that $\forall \alpha \in On(rk(L_\alpha) = \alpha)$.

3. Suppose $F : V \to V$ is a term definable without parameters (ie. the formula defining "$F(x) = y$" has no parameters). Suppose further that it is an *elementary map*, ie. for any formula $\phi(v_0, \dots, v_{n-1})$ of LST (without parameters), and any $a_0, \dots, a_{n-1} \in V$,

$$\phi(a_0, \dots, a_{n-1}) \Leftrightarrow \phi(F(a_0), \dots, F(a_{n-1})).$$

Prove that F is the identity. [Hint: first show that for all ordinals α, $F(\alpha) = \alpha$, by considering the first β for which $F(\beta) \neq \beta$.]

[Remark: Assuming only ZF, it is not known whether such an elementary map definable *with* parameters can exist other than the identity, although if ZFC is assumed it is known that there is no such.]

4. Let E denote the set of even natural numbers. Prove that $E \in L_{\omega+1}$.

5. For $\phi(\mathbf{v})$ a formula of LST (without parameters) and a any set, let $\phi_a(\mathbf{v})$ denote the formula (with parameter a) obtained by relativizing $\phi(\mathbf{v})$ to the class a. Prove that for any transitive class A and $a, \mathbf{b} \in A$, $(A, \in) \models \phi_a(\mathbf{b})$ iff $\phi_a(\mathbf{b})$ (ie. $\phi_a(\mathbf{v})$ is A-absolute).

Axiomatic Set Theory: Problem sheet 6

1. Prove lemmas 7.2, 7.3 and 7.4.

2. Prove 7.11 (30), ie. that "x is a finite sequence of elements of y" (ie. $x \in {}^{<\omega}y$) is Σ_0^{ZF}, assuming that (1)–(29) of 7.11 are all Σ_0^{ZF}.

3. Prove that "x is a well-ordering of y" is Δ_1^{ZF}.

4. Show that for every Σ_1 formula $\phi(x_1, \ldots, x_n)$, there exists a corresponding Σ_0 formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that

$$ZF \vdash \forall x_1, \ldots x_n(\phi(x_1, \ldots, x_n) \leftrightarrow \exists y_1, \ldots, y_m \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)).$$

5. Prove that ordinal addition, multiplication and exponentiation are Δ_1^{ZF}.

Axiomatic Set Theory: Problem sheet 7

1. Prove that for any infinite cardinal κ, $cf(\kappa)$ is a regular cardinal.

2. Suppose κ, λ are infinite cardinals such that $\kappa \geq \lambda$. Prove that if $\lambda \geq cf(\kappa)$, then $\kappa^\lambda > \kappa$. Suppose now that $\lambda < cf(\kappa)$, and that κ has the property that for any cardinal μ, if $\mu < \kappa$ then $2^\mu \leq \kappa$. Prove that $\kappa^\lambda = \kappa$. Hence show that if GCH is assumed, then for any infinite cardinals κ, λ with $\kappa \geq \lambda$, we have $\kappa^\lambda = \kappa$ or κ^+.

3. Suppose κ is an *uncountable regular* cardinal. Let $g : \kappa \to \kappa$ be any function. Prove that for any $\alpha < \kappa$, there exists $\beta < \kappa$, with $\alpha \leq \beta$, such that β is closed under g (ie. for all $\gamma < \beta$, $g(\gamma) < \beta$).

4. (Optional) Let κ be an uncountable regular cardinal with the property that for any cardinal $\mu < \kappa$, we have $2^\mu < \kappa \ldots (*)$.

Prove that (i) if α is any cardinal and $\alpha < \kappa$, then $|V_\alpha| < \kappa$, (ii) $|V_\kappa| = \kappa$, (iii) if κ is regular, then $\langle V_\kappa, \in\rangle \models$ ZFC.

(For (iii) you need consider only the replacement scheme, since we essentially showed that if α is a limit ordinal and $\alpha > \omega$, then $\langle V_\alpha, \in\rangle$ satisfies all the axioms of ZFC except, possibly, replacement.)

Deduce that in ZFC one cannot prove the existence of a cardinal that satisfies (*).