# Chapter 2

# Basics

See D. Goldrei *Classic Set Theory*, Chapman and Hall 1996, or H.B. Enderton *Elements of Set Theory*, Academic Press, 1977.

The material for this course is contained in K. Kunen *Set Theory*, North-Holland 1980, or K. Devlin *Constructibility*.

The *language of set theory*, LST, is first-order predicate calculus with equality having the membership relation $\in$ (which is binary) as its only non-logical symbol.

Thus the basic symbols of LST are: $=$, $\in$, $\vee$, $\neg$, $\forall$, ( and ), and an infinite list $v_0, v_1, \ldots, v_n, \ldots$ of variables (although for clarity we shall often use $x, y, z, t, \ldots, u, v, \ldots$ etc. as variables).

The *well-formed formulas*, or just *formulas*, of LST are those expressions that can be built up from the *atomic formulas*: $v_i = v_j$, $v_i \in v_j$, using the rules: (1) if $\phi$ is a formula, so is $\neg\phi$, (2) if $\phi$ and $\psi$ are formulas, so is $(\phi \vee \psi)$, and (3) if $\phi$ is a formula, so is $\forall v_i \phi$.

## 2.1 Some standard abbreviations

We write $(\phi \wedge \psi)$ for $\neg(\neg\phi \vee \neg\psi)$; $(\phi \to \psi)$ for $(\neg\phi \vee \psi)$; $(\phi \leftrightarrow \psi)$ for $((\phi \to \psi) \wedge (\psi \to \phi))$; $\exists x\phi$ for $\neg\forall x\neg\phi$; $\exists! x\phi$ for $\exists x\forall y(\phi \leftrightarrow x = y)$; $\exists x \in y\,\phi$ for $\exists x(x \in y \wedge \phi)$; $\forall x \in y\,\phi$ for $\forall x(x \in y \to \phi)$; $\forall x, y\,\phi$ (etc.) for $\forall x\forall y\phi$; $x \notin y$ for $\neg x \in y$.

We shall also often write $\phi$ as $\phi(x)$ to indicate free occurrences of a variable $x$ in $\phi$. The formula $\phi(z)$ (say) then denotes the result of substituting every free occurrence of $x$ in $\phi$ by $z$. Similarly for $\phi(x, y)$, $\phi(x, y, z)$,$\ldots$, etc.

## 2.2 The Axioms

**(A1.)** *Extensionality*

$$\forall x, y(x = y \leftrightarrow \forall t(t \in x \leftrightarrow t \in y))$$

Two sets are equal iff they have the same members.

**(A2.)** *Empty set*

$$\exists x \forall y y \notin x$$

There is a set with no members, *the empty set*, denoted $\varnothing$.

**(A3.)** *Pairing*

$$\forall x, y \exists z \forall t (t \in z \leftrightarrow (t = x \vee t = y))$$

For any sets $x, y$ there is a set, denoted $\{x, y\}$, whose only elements are $x$ and $y$.

**(A4.)** *Union*

$$\forall x \exists y \forall t (t \in y \leftrightarrow \exists w (w \in x \wedge t \in w))$$

For any set $x$, there is a set, denoted $\bigcup x$, whose members are the members of the members of $x$.

**(A5.)** *Separation Scheme* If $\phi(\mathbf{x}, \mathbf{y})$ is a formula of LST, the following is an axiom:

$$\forall \mathbf{x} \forall u \exists z \forall y (y \in z \leftrightarrow (y \in u \wedge \phi(\mathbf{x}, y))$$

For given sets $\mathbf{x}, u$ there is a set, denoted $\{y \in u : \phi(\mathbf{x}, y)\}$, whose elements are those elements $y$ of $u$ which satisfy the formula $\phi(\mathbf{x}, y)$.

**(A6.)** *Replacement Scheme* If $\phi(x, y)$ is a formula of LST (possibly with other free variables $\mathbf{u}$, say) then the following is an axiom:

$$\forall \mathbf{u} [\forall x, y, y' ((\phi(x, y) \wedge \phi(x, y')) \to y = y') \to \forall s \exists z \forall y (y \in z \leftrightarrow \exists x \in s \, \phi(x, y))]$$

The set $z$ is denoted $\{y : \exists x \phi(x, y) \wedge x \in s\}$.

**(A7.)** *Power Set*

$$\forall x \exists y \forall t (t \in y \leftrightarrow \forall z (z \in t \to z \in x))$$

For any set $x$ there is a set, denoted $\mathbb{P}(x)$, whose members are exactly the subsets of $x$.

**(A8.)** *Infinity*

$$\exists x [\exists y (y \in x \wedge \forall z (z \notin y) \wedge \forall y (y \in x \to \exists z (z \in x \wedge \forall t (t \in z \leftrightarrow (t \in y \vee t = y))))]$$

There is a set $x$ such that $\varnothing \in x$ and whenever $y \in x$, they $y \cup \{y\} \in x$. (Such a set is called a *successor set*. The set $\omega$ of natural numbers is a successor set.)

**(A9.)** *Foundation*

$$\forall x (\exists z (z \in x) \to \exists z (z \in x \wedge \forall y \in z \, y \notin x))$$

If the set $x$ is non-empty, then for some $z \in x$, $z$ has no members in common with $x$.

**(A10.)** *Axiom of Choice*

$$\forall u[[\forall x \in u \exists y(y \in x) \wedge \forall x, y((x \in u \wedge y \in u \wedge x \neq y) \rightarrow \forall z(z \notin x \vee \notin y))] \\ \rightarrow \exists v \forall x \in u \exists! y(y \in x \wedge y \in v)]$$

If $u$ is a non-empty set of non-empty sets, then there is a set $v$ which contains exactly one element of each set in $u$.

We write ZF* for the collection of axioms A1–A8; ZF for A1–A9; ZFC for A1–A10.

## 2.3 Proofs in principle and proofs in practice

Suppose that $T$ is one of the above collections of axioms. If $\sigma$ is a sentence of LST (ie. a formula without free variables), we say that *$\sigma$ is a theorem of $T$*, or that *$\sigma$ can be proved from $T$*, and write $T \vdash \sigma$, if there is a finite sequence $\sigma_1, \ldots, \sigma_n$ of LST formulas such that $\sigma_n$ is $\sigma$, and each $\sigma_i$ is either in $T$ or else follows from earlier formulas in the sequence by a rule of logic. Clearly every theorem of ZF is a theorem of ZFC and every theorem of ZF* is a theorem of ZF. To say that *$T$ is consistent* means that for no sentence $\phi$ of LST is $(\phi \wedge \neg\phi)$ a theorem of $T$ (which is in fact equivalent to saying that there is some sentence which is not provable from $T$). This now makes theorem 1.1 precise: we must show that if ZF is consistent, then so is ZFC.

Now in proving this theorem we shall need to build up a large stock of theorems of ZF (and we shall discuss some theorems of ZFC as well) but to give formal proofs of these would not only be tedious but also infeasible. We shall therefore employ the standard short-cut of *adopting a Platonic viewpoint.* That is, we shall think of the collection of all sets as being a clearly defined notion and whenever we want to show a sentence, $\sigma$, say, of LST has a formal proof (from ZF say) we shall simply give an informal argument that the proposition asserted by $\sigma$ about this collection is true. Indeed, we shall often not bother to write out $\sigma$ as a formula of LST at all; we shall simply write down (in English plus a few logical and mathematical symbols) "what it is saying". Of course we shall take care that, in our informal argument, we only use those propositions about the collection of all sets asserted by the axioms of ZF. Thus, for example, if I write:

**Theorem 2.3.1** *(ZF*) There is no set containing every set.*

then I mean that from the axioms of ZF* there is a formal proof of the LST sentence

$$\forall x \exists y y \notin x.$$

Actually, it probably wouldn't be too difficult to give a formal proof of this, but we shall supply the following as a proof:

*Proof.* Suppose $A$ were a set containing every set. By A5 $\{x \in A : x \notin x\}$ is a set, call it $B$. Then $B \in B$ iff $B \in A$ and $B \notin B$. But $B \in A$ is true (as $A$ contains every set), so $B \in B$ iff $B \notin B$—a contradiction. □

Of course in all such cases, the reader should convince him- or herself that (a) the informal statement we are proving can be written as a sentence of LST, and (b) the given proof can be converted, at least in principle, to a formal proof from the specified collection of axioms.

## 2.4 Interpretations

The Completeness Theorem for first-order predicate calculus (also due to Gödel) states that a sentence $\sigma$ (of any first-order language) is provable from a collection of sentences $S$ (in the same language) if and only if every model of $S$ is a model of $\sigma$. (Equivalently, $S$ is consistent if and only if $S$ has a model—just let $\sigma$ be a contradiction, which by the Soundness Theorem is provable from $S$ iff $S$ is inconsistent, and is logically implied by $S$ only if $S$ has no models.) Let us examine this in our present context. Firstly, a *structure for LST* is specified by a domain of discourse $M$ over which the quantifiers $\forall x \ldots$ and $\exists x \ldots$ range, and a binary relation $E$ on $M$ to interpret the membership relation $\in$. If $\sigma$ is a sentence of LST which is true under this interpretation we say that $\sigma$ *is true in* $\langle M, E\rangle$ or $\langle M, E\rangle$ *is a model of* $\sigma$, and write $\langle M, E\rangle \models \sigma$. If $T$ is a collection of sentences of LST we also write $\langle M, E\rangle \models T$ iff $\langle M, E\rangle \models \sigma$ for each sentence $\sigma$ in $T$. (If $\phi(x_1, \ldots, x_n)$ is a formula of LST with free variables among $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ are in the domain $M$, we also write $\langle M, E\rangle \models \phi(a_1, \ldots, a_n)$ to mean $\phi(x_1, \ldots, x_n)$ is true of $a_1, \ldots, a_n$ in the interpretation $\langle M, E\rangle$.)

For example, suppose $M$ contains just the two distinct elements $a$ and $b$, and $E$ is specified by $a \to b$, ie. $E(a, b)$, not $E(b, a)$, not $E(a, a)$, not $E(b, b)$. Then $\langle M, E\rangle \models$ A2, ie. $M \models \exists x \forall y y \notin x$, since it is true that there is an $x$ in $M$ (namely a) such that for all $y \in M$, not $E(y, x)$. It is also easy to see that $\langle M, E\rangle \models$ A1 and $\langle M, E\rangle \models \neg$A3. Notice that, by the completeness theorem, this implies that A3 is not provable from the axioms A1, A2 since we have found a model of the latter two axioms which is not a model of the former.

**Exercise 2.4.1** *Let $\mathbb{Q}$ be the set of rational numbers and $<$ the usual ordering of $\mathbb{Q}$. Which axioms of ZF are true in $\langle \mathbb{Q}, < \rangle$?*

Note that the Platonic viewpoint adopted here amounts to regarding a sentence, $\sigma$, say, of LST as true, if and only if $\langle V^*, \in\rangle \models \sigma$, where $V^*$ is the collection of all sets, and $\in$ is the usual membership relation.

The completeness theorem provides a method for establishing theorem 1.1. For we can rephrase that theorem as: If ZF has a model then so does ZFC. Indeed we shall construct a subcollection $L$ of $V^*$ such that if we assume $\langle V^*, \in\rangle \models$ ZF, then $\langle L, \in\rangle \models$ ZFC. (Actually our proof will yield somewhat more which ought to be enough to satisfy any purist. Namely, it will produce an effective procedure for converting any proof of a contradiction (ie. a sentence of the form $(\phi \wedge \neg\phi)$) from ZFC to a proof of a contradiction from ZF.)

We now turn to the development of some basic set theory from the axioms $ZF^*$.

## 2.5 New sets from old

The axioms of ZF are of three types: (a) those that assert that all sets have a certain property (A1, A9), (b) those that sets with certain properties exist (A2, A8), and (c) those that tell us how we may construct new sets out of given sets (A3–A7). Our aim here is to combine the operations implicit in the axioms of type (c) to obtain more ways of constructing sets and to introduce notations for these constructions (just as, for example, we introduced the notation $\bigcup x$ for the set $y$ given by A4). It will be convenient to use the class notation $\{x : \phi(x)\}$ for the collection (or *class*) of sets $x$ satisfying the LST formula $\phi(x)$.[1] As we have seen, such a class need not be a set. However, in the following definitions it can be shown (from the axioms $ZF^*$) that we always do get a set. This amounts to showing that for some set $a$, if $b$ is any set such that $\phi(b)$ holds (ie. $V^* \models \phi(b)$) then $b \in a$, so that $\{x : \phi(x)\} = \{x \in a : \phi(x)\}$ which is a set by A5. I leave all the required proofs as exercises—they can also be found in the books.

In the following, $A, B, \ldots, a, b, c, \ldots, f, g, a_1, a_2, \ldots, a_n, \ldots$ etc. all denote sets.

1. $\{a_1, \ldots, a_n\} \ := \{x : x = a_1 \vee \ldots \vee x = a_n\}$.
2. $a \cup b \ := \bigcup\{a, b\} = \{x : x = a \vee x = b\}$.
3. $a \cap b \ := \{x : x = a \wedge x = b\}$.
4. $a \setminus b \ := \{x : x \in a \wedge x \notin b\}$.
5. $\bigcap a \ := \begin{cases} \{x : \forall y \in a x \in y\} \text{ if } a \neq \varnothing \\ \quad \text{undefined if } a = \varnothing \end{cases}$.
6. $\langle a, b\rangle \ := \{\{a\}, \{a, b\}\}$. (**Lemma.** $\langle a, b\rangle = \langle c, d\rangle \leftrightarrow (a = c \wedge b = d)$.)
7. $a \times b \ := \{x : \exists c \in a \, \exists d \in b \, x = \langle c, d\rangle\}$. (**Remark:** Of course the proof that $a \times b$ is a set requires not only "bounding the $x$'s", but also showing that the expression "$\exists c \in a \exists d \in b x = \langle c, d\rangle$" can be written as a formula of LST (with parameters $a, b$).)
8. $a \times b \times c \ := a \times (b \times c), \ldots$, etc.
9. $a^2 \ := a \times a$, $a^3 \ := a \times a \times a, \ldots$, etc.
10. We write $a \subseteq b$ for $\forall x \in a (x \in b)$.

[1] Actually, $\phi(x)$ will be allowed to have parameters (ie. names for given sets), so is not strictly a formula of LST. Notice, however, that parameters are allowed in A5 and A6 (the "**x**" and "**u**").

11. $c$ is a *binary relation* on $a$ we take to mean $c \subseteq a^2$. (Similarly for ternary,..., $n$-ary, ...relations.)

12. If $A$ is a binary relation on $a$ we usually write $xAy$ for $\langle x, y\rangle \in A$.

    $A$ is called a *(strict) partial order* on $a$ iff

    (a) $\forall x, y \in a(xAy \to \neg yAx)$,

    (b) $\forall x, y, z \in a((xAy \wedge yAx) \to xAz)$.

    If in addition we have (3) $\forall x, y \in a(x = y \vee xAy \vee yAx)$, then $A$ is called a *(strict) total* (or *linear) order* of $a$.

13. Write $f : a \to b$ ($f$ is a *function* with *domain* $a$ and *codomain* $b$, or simply $f$ is a *function from $a$ to $b$*) if $f \subseteq a \times b$ and $\forall c \in a \exists! d \in b \langle c, d\rangle \in f$. Write $f(c)$ for this unique $d$.

14. If $f : a \to b$, $f$ is called *injective* (or *one-to-one*) if $\forall c, d \in a(c \neq d \to f(c) \neq f(d))$, *surjective* (or *onto*) if $\forall d \in b \exists c \in a f(c) = d$, and *bijective* if it is both injective and surjective.

15. We write $a \sim b$ if $\exists f(f : a \to b \wedge f$ bijective).

16. ${}^ab := \{f : f : a \to b\}$.

17. A set $a$ is called *a successor set* if

    (a) $\varnothing \in a$ and

    (b) $\forall b(b \in a \to b \cup \{b\} \in a)$.

    Axiom A8 implies a successor set exists and it can be further shown that a unique such set, denoted $\omega$, exists with the property that $\omega \subseteq a$ for every successor set $a$. The set $\omega$ is called the set of natural numbers. If $n, m \in \omega$ we often write $n + 1$ for $n \cup \{n\}$ and $n < m$ for $n \in m$ and 0 for $\varnothing$ (in this context). The relation $\in$ (ie. $<$) is a total order of $\omega$ (more precisely $\{\langle x, y\rangle : x \in \omega, y \in \omega \wedge x \in y\}$ is a total order of $\omega$).

18. The set $\omega$ satisfies the *principle of mathematical induction*, ie. if $\psi(x)$ is any formula of LST such that $\psi(0) \wedge \forall n \in \omega(\psi(n) \to \psi(n+1))$ holds, then $\forall n \in \omega \psi(n)$ holds.

19. The set $\omega$ also satisfies the *well-ordering principle*, ie. for any set $a$, if $a \subseteq \omega$ and $a \neq \varnothing$ then $\exists b \in a \forall c \in a(c > b \vee c = b)$.

20. *Definition by recursion*

    Suppose that $f : A \to A$ is a function and $a \in A$. Then there is a unique function $g : \omega \to A$ such that:

    (a) $g(0) = a$, and

    (b) $\forall n \in \omega g(n + 1) = f(g(n))$.

(Thus, $g(n) = \underbrace{f(f \cdots (f}_{n \text{ times}} a)) \cdots)).$)

More generally, if $f : B \times \omega \times A \to A$ and $h : B \to A$ are functions, then there is a unique function $g : B \times \omega \to A$ such that

(a) $\forall b \in B\, g(b, 0) = h(b)$, and

(b) $\forall b \in B\, \forall n \in \omega\, g(b, n+1) = f(b, n, g(b, n))$.

Using this result one can define the addition, multiplication and exponentiation functions.

(**Remark** I have adopted here the usual convention of writing $g(b, n+1)$ for $g(\langle b, n+1 \rangle)$. Similarly for $f$.)

21. A set $a$ is called *finite* iff $\exists n \in \omega a \sim n$.

22. A set $a$ is called *countably infinite* iff $a \sim \omega$.

23. A set $a$ is called *countable* iff $a$ is finite or countably infinite. (Equivalently: iff $\exists f (f : a \to \omega \wedge f$ injective).)

    (**Theorem** $\mathbb{P}\omega$ is not countable. In fact, for no set $A$ do we have $A \sim \mathbb{P}A$. (Cantor))