Chapter 7

Absoluteness handout

Theorem 7.11 *The following formulas and class terms are all Σ_0^{ZF} (and hence Δ_1^{ZF}):*

1. $x = y$
2. $x \in y$
3. $x \subseteq y$
4. $F_n(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$ *(for each n)*
5. $F_n(x_1, \ldots, x_n) = \langle x_1, \ldots, x_n \rangle$ *(for each n)*
6. *(where $n \geq 1$ and $0 \leq i \leq n-1$) $F_i^n(x) = x_i$ if x is an n-tuple $\langle x_0, \ldots, x_{n-1} \rangle$, $\varnothing$ otherwise.*
7. $F(x, y) = x \cup y$.
8. $F(x, y) = x \cap y$.
9. $F(x) = \bigcup x$.
10. $F(x) = \bigcap x$ *if* $x \neq \varnothing$, $F(x) = \varnothing$ *otherwise.*
11. $F(x, y) = x \setminus y$.
12. *x is an n-tuple.*
13. *x is an n-ary relation on y.*
14. *x is a function.*
15. $F(x) = \mathrm{dom} x$ *if x is a function, $\varnothing$ otherwise.*
16. $F(x) = \mathrm{ran}\, x$ *if x is a function, $\varnothing$ otherwise.*
17. $F(x) = x(y)$ *if x is a function and* $y \in \mathrm{dom} x$, *$\varnothing$ otherwise.*
18. $F(x, y) = x[y]$ $(= \{x(t) : t \in y\})$ *if x is a function, $\varnothing$ otherwise.*
19. $F(x, y) = x \restriction y$ *if x is a function, $\varnothing$ otherwise.*
20. $F(x) = x^{-1}$ *if x is a function, $\varnothing$ otherwise.*
21. $F(x) = x \cup \{x\}$.
22. *x is transitive.*
23. *x is an ordinal.*
24. *x is a successor ordinal.*
25. *x is a limit ordinal.*
26. $x : y \to z$.

27. *$x : y \sim z$.*

28. *x is a natural number.*

29. *$x = \omega$.*

30. *x is a finite sequence of elements of y (i.e. $x \in {}^{<\omega}y$).*

Proof. (Selections) (3) $x \subseteq y \Leftrightarrow \forall z \in x(z \in y)$ which is Σ_0.

Note that all the class terms F above are in ZF provably class terms, so we only have to show that the statement $F(\mathbf{x}) = y$ can be put in Σ_0 form.

(4) $F_n(x_1, \ldots, x_n) = y \Leftrightarrow [x_1 \in y \wedge x_2 \in y \wedge \ldots \wedge x_n \in y \wedge \forall z \in y\,(z = x_1 \vee \ldots \vee z = x_n)]$.

(5) For $n = 2$, $F_2(x_1, x_2) = y \Leftrightarrow \exists z_1 \in y\, \exists z_2 \in y\,(z_1 = \{x_1\} \wedge z_2 = \{x_1, x_2\} \wedge \forall t \in y(t = z_1 \vee t = z_2))$, which is Σ_0 by (4).

(12) For $n = 2$, x is a 2-tuple iff $\exists z_1 \in x\, \exists x_1 \in z_1\, \exists x_2 \in z_1\,(x = \langle x_1, x_2 \rangle)$, which is Σ_0 by (5). (In this formula, $z_1 = \{x_1, x_2\}$.)

(13) For $n = 2$, x is a 2-ary relation on y iff $\forall z \in x\, \exists y_1 \in y\, \exists y_2 \in y\,(z = \langle y_1, y_2 \rangle)$, which is Σ_0 by (5).

(28) x is a natural number iff (x is an ordinal)$\wedge$(x is not a limit ordinal)$\wedge$($\forall y \in x$ y is not a limit ordinal), which is Σ_0 by (23), (25) and the fact that Σ_0^{ZF} formulas are closed under $\neg$. □

Lemma 7.12 *Suppose F and G are Δ_1^{ZF} class terms. Then "$F(\mathbf{x}) = G(\mathbf{y})$" is Δ_1^{ZF}.*

Proof. Let $\psi(\mathbf{x}, z)$ and $\chi(\mathbf{y}, t)$ be Σ_1 formulas defining (in ZF) $F(\mathbf{x}) = y$ and $G(\mathbf{y}) = t$ respectively. Then

$$F(\mathbf{x}) = G(\mathbf{y}) \underbrace{\Leftrightarrow}_{ZF} \exists z(\psi(\mathbf{x}, z) \wedge \chi(\mathbf{y}, z)),$$

which is Σ_1, and

$$F(\mathbf{x}) \neq G(\mathbf{y}) \underbrace{\Leftrightarrow}_{ZF} \exists z \exists t(\psi(\mathbf{x}, z) \wedge \chi(\mathbf{y}, t) \wedge \neg z = t),$$

which is Σ_1.

Hence "$F(\mathbf{x}) = G(\mathbf{y})$" is Δ_1^{ZF}. □

Theorem 7.13 *Suppose $F : V \times V \to V$ is a Δ_1^{ZF} class term. Then the class term G defined from F by recursion on On, ie:*

1. *$G(0, x) = x$*

2. *$G(\alpha + 1, x) = F(G(\alpha, x), x)$ for all $\alpha \in On$*

3. *$G(\delta, x) = \bigcup_{\alpha < \delta} G(\alpha, x)$ for all limit $\delta \in On$*

4. $G(y,x) = \varnothing$ for all $y \notin On$

is Δ_1^{ZF}.

Proof. As in the proof of 3.14 define $\phi(g, \alpha, x)$ by

$$\begin{array}{lcr} & On(\alpha) & \chi_1 \\ \wedge & g \text{ is a function} & \chi_2 \\ \wedge & \mathrm{dom} g = \alpha \cup \{\alpha\} & \chi_3 \\ \wedge & g(0) = x & \chi_4 \\ \wedge & \forall \beta \in \alpha \exists y_1 \exists y_2 (y_1 = \beta \cup \{\beta\} \wedge y_2 = g(\beta) \wedge g(y_1) = F(y_2)) & \chi_5 \\ \wedge & \forall \beta \in \alpha(\beta \text{ is a limit ordinal} \rightarrow g(\beta) = \bigcup\{g(\alpha) : \alpha \in \beta\}). & \chi_6 \end{array} \tag{7.1}$$

χ_1 is Σ_0^{ZF} by 7.11 (23); χ_2 is Σ_0^{ZF} by (14); χ_3 is by (15), (21) and 7.12; χ_4 can be rewritten as $\exists y((\forall z \in y(\neg z \in z) \wedge g(y) = x)$ so is Σ_1^{ZF} by (17); χ_5 is Σ_1^{ZF} by (21), (17) and the fact that F is Σ_1^{ZF}, and using 7.12; χ_6 is Σ_1^{ZF} by (25) and the fact that "$g(\beta) = \bigcup\{g(\alpha) : \alpha \in \beta\}$" is equivalent to $\exists y \exists z(y = g[\beta] \wedge z = \bigcup y \wedge g(\beta) = z)$, which is Σ_1^{ZF} by (18), (9) and (17).

Hence $\phi(g, \alpha, x)$ is Σ_1^{ZF}.

Now recall from the proof of 3.14 that G can be defined by:

$$G(\alpha, x) = y \Leftrightarrow \exists g(\phi(g, \alpha, x) \wedge g(\alpha) = y) \vee (\neg On(\alpha) \wedge y = \varnothing).$$

This shows G is Σ_1^{ZF}, and hence Δ_1^{ZF} by 7.8. $\square$