Chapter 8

Gödel numbering and the construction of Def

(Throughout, if we say "$F : U_1 \times \cdots \times U_n \to V$ is a Δ_1^{ZF} term" we mean that the classes $U_1, \ldots, U_n$ are Δ_1^{ZF} (ie. defined by Δ_1^{ZF} formulas) and that "$F(x_1, \ldots, x_n) = y$" can be expressed by a Σ_1 formula. This clearly guarantees that the extension $F' : V^n \to V$ of F defined by $F'(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)$ if $x_1 \in U_1$, …,$x_n \in U_n$ and $= \varnothing$ otherwise, is Δ_1^{ZF} in the sense given.)

To give numbers to formulas we first define $F : \omega^3 \to \omega$ by $F(n, m, l) = 2^n 3^n 5^l$. Then F is injective and easily seen to be Δ_1^{ZF}. Write $[n, m, l]$ for $F(n, m, l)$. We now define $\ulcorner\phi\urcorner$ by induction on ϕ:

$$\begin{aligned}
\ulcorner v_i = v_j \urcorner &= [0, i, j]; \\
\ulcorner v_i \in v_j \urcorner &= [1, i, j]; \\
\ulcorner \phi \vee \psi \urcorner &= [2, \ulcorner\phi\urcorner, \ulcorner\psi\urcorner]; \\
\ulcorner \neg\phi \urcorner &= [3, \ulcorner\phi\urcorner, \ulcorner\phi\urcorner]; \\
\ulcorner \forall v_i \phi \urcorner &= [4, i, \ulcorner\phi\urcorner].
\end{aligned}$$

Of course this definition does not take place in ZF and is not actually used in the following definition of Def. However it should be borne in mind in order to see what's going on.

Now defined the class term $Sub : V^4 \to V$ by $Sub(a, f, i, c) = f(c/i)$ if $f \in {}^{<\omega}a$, $c \in a$ and $i \in \omega$ and $= \varnothing$ otherwise; where if $f \in {}^{<\omega}a$, $c \in a$ and $i \in \omega$, $f(c/i) \in {}^{<\omega}a$ is defined by $\mathrm{dom}(f(c/i)) = \mathrm{dom}f$, and for $j \in \mathrm{dom}f$, $f(c/i)(j) = f(j)$ if $j \neq i$, and c if $j = i$.

It's easy to check that Sub is Δ_1^{ZF}.

We now define a class term $Sat : \omega \times V \to V$. The idea is that if $m \in \omega$ and $m = \ulcorner\phi(v_0, \ldots, x_{n_1})\urcorner$, for some formula ϕ of LST, and $a \in V$, then (*) $Sat(m, a) = \{f \in {}^{<\omega}a : \mathrm{dom}f \geq n \wedge \langle a, \in\rangle \vDash \phi(f(0), \ldots, f(n-1))\}$. We simply mimic the definition of satisfaction from predicate logic. (This definition uses a version of the recursion theorem which is slightly different from the usual one, and which I give later.)

Definition 8.1 *Firstly if $a \in V$, $m \in \omega$ but m is not of the form $[i,j,k]$, for any $i,j,k \in \omega$ with $i < 5$, then $Sat(m,a) = \varnothing$. Otherwise, if $a \in V$ and $m = [i,j,k]$ with $i < 5$, then*

$$\begin{aligned}
Sat([0,j,k],a) &= \{f \in {}^{<\omega}a : j,k \in \mathrm{dom} f \wedge f(j) = f(k)\}.\\
Sat([1,j,k],a) &= \{f \in {}^{<\omega}a : j,k \in \mathrm{dom} f \wedge f(j) \in f(k)\}.\\
Sat([2,j,k],a) &= Sat(j,a) \cup Sat(k,a).\\
Sat([3,j,k],a) &= ({}^{<\omega}a \setminus Sat(j,a)) \cap \{g \in {}^{<\omega}a : \exists f \in Sat(j,a), \mathrm{dom} f \leq \mathrm{dom} g\}.\\
Sat([4,j,k],a) &= \{f \in {}^{<\omega}a : j \in \mathrm{dom} f \wedge \forall x \in a, Sub(a,f,j,x) \in Sat(k,a)\}.
\end{aligned}$$

The generalized version of the recursion theorem (on ω) required here is:

Lemma 8.2 *Suppose that $\pi_1, \pi_2, \pi_3 : \omega \to \omega$ are Δ_1^{ZF} class terms and $H : V^4 \times \omega \to V$ is a Δ_1^{ZF} class term. Suppose further that $\forall n \in \omega \setminus \{0\}$ $\pi_i(n) < n$ for $i = 1,2,3$. Then there is a Δ_1^{ZF} class term $F : \omega \times V \to V$ such that*

1. $F(0,a) = 0$
2. *and* $\forall n \in \omega \setminus \{0\}$

$$F(n,a) = H(F(\pi_1(n),(a)), F(\pi_2(n),(a)), F(\pi_3(n),(a)), a, n).$$

(Thus instead of defining $F(n,a)$ in terms of $F(n-1,a)$, we are defining $F(n,a)$ in terms of three specified previous values.)

Proof. Similar to the proof of the usual recursion theorem on ω. □

Thus the definition of Sat in 8.1 is an application of 8.2 with $\pi_1(n) = i$ if for some $j,k < n$, $[i,j,k] = n$, $= 0$ otherwise; and π_2 and π_3 are defined similarly, picking out j and k respectively from $[i,j,k]$, and with $H : V^4 \times \omega \to V$ defined so that

$$H(x,y,z,a,n) = \begin{cases}
\{f \in {}^{<\omega}a : \pi_2(n), \pi_3(n) \in \mathrm{dom} f \wedge f(\pi_2(n)) = f(\pi_3(n))\} & \text{if } \pi_1(n) = 0,\\
\{f \in {}^{<\omega}a : \pi_2(n), \pi_3(n) \in \mathrm{dom} f \wedge f(\pi_2(n)) \in f(\pi_3(n))\} & \text{if } \pi_1(n) = 1,\\
y \cup z & \text{if } \pi_1(n) = 2,\\
({}^{<\omega}a \setminus y) \cap \{g \in {}^{<\omega}a : \exists f \in y \mathrm{dom} f \leq \mathrm{dom} g\} & \text{if } \pi_2(n) = 3,\\
\{f \in {}^{<\omega}a : \pi_2(n) \in \mathrm{dom} f \wedge \forall x \in a Sub(a,f,\pi_2(n),x) \in z\} & \text{if } \pi_1(n) = 4,\\
0 & \text{otherwise.}
\end{cases}$$

(The F got from this H, π_1, π_2, π_3 (in 8.2) is Sat.)

It is completely routine to show that Sat so defined satisfies the required statement (*) (just before 8.1)—by induction on ϕ.

Before defining G we must introduce a term that picks out the largest $m \in \omega$ such that "v_m occurs free" in the "formula coded by n". We denote this n by

$\theta(n)$. We first define $Fr(m)$ ("the set of i such that v_i occurs free in the formula coded by m") as follows (again using 8.2):

$$\begin{aligned}
Fr([0,i,j]) &= \{i,j\};\\
Fr([1,i,j]) &= \{i,j\};\\
Fr([2,i,j]) &= Fr(i) \cup Fr(j);\\
Fr([3,i,j]) &= Fr(i);\\
Fr([4,i,j]) &= Fr(j) \setminus i;\\
Fr(x) &= \varnothing,\ \text{if } x \text{ not of the above form.}
\end{aligned} \tag{8.1}$$

Clearly one can prove in ZF that $Fr(x)$ is a finite set of natural numbers for any set x, and we defined

$$\theta(x) = \max(Fr(x)).$$

θ is Δ_1^{ZF}.

It is easy to show that if ϕ is any formula of LST and $m = \ulcorner\phi\urcorner$, then $\theta(m)$ is the largest n such that v_n occurs as a free variable in ϕ, and that if $f \in Sat(m,a)$, for any $a \in V$, then $\mathrm{dom} f \geq 1+\theta(m)$ (ie. $0, 1, \ldots, \theta(m) \in \mathrm{dom} f$). This is proved by induction on ϕ and it is for this reason that we defined $Sat([3,j,k],a)$ as we did (rather than just as ${}^{<\omega}a \setminus Sat(j,a)$).

We can now define G by

$$G(m,a,s) = \begin{cases} \{b \in a : (s \cup \{\langle\theta(m),b\rangle\}) \in Sat(m,a)\} & \text{if } s \in {}^{<\omega}a \text{ and } \mathrm{dom} s = \theta(m) (= \{0,\ldots,\theta(m)-1\}),\\ \varnothing & \text{otherwise.}\end{cases}$$

Then G is easily seen to be Δ_1^{ZF} (since θ, Sat are), and has the required properties mentioned at the beginning of chapter 6, because of (*) (just before 8.1).