1. The language of set theory

In the lectures, the language of set theory is defined to be a language of first-order predicate calculus with equality, with a single binary predicate symbol $\in$.

So, why do we need a special language of set theory, why do we choose this one, and how much does it matter?

1.1. The need for a language of set theory

We use the language of set theory, for instance, in stating the Subset Axiom scheme:

Subset axiom scheme *Suppose A is a set and $\phi(x)$ is a statement in the language of set theory. Then*

$$\{x \in A : \phi(x)\}$$

is a set.

Why, in the above definition, do we not use English (or Russian, or Chinese) instead?

One reason is that English (and Russian, and Chinese) are too flexible. They can express vague concepts ("x is an interesting number"). Worse, they can easily express logical paradoxes, especially paradoxes of self-reference. Consider, for instance,

The smallest natural number not definable in fewer than thirteen English words.

We've just defined it in twelve, giving a contradiction, by exploiting the ability of English to talk explicitly about itself. The upshot is that, if $\phi(x)$ is vague, or paradoxical, we might be unsure what the elements of the set $\{x \in A : \phi(x)\}$ are, or even whether the set can exist.

A second problem with natural languages, like English, is that they are extraordinarily complex. There do not exist complete descriptions of the grammar of any natural language. By contrast, the grammars of languages of predicate calculus are extremely simple. This simplicity is one reason why these languages are so hard to use. But it does mean that we can prove theorems about these languages, and this makes the advanced study of set theory tractable.

1.2. Why this language in particular?

The first-order language just described has one big argument in its favour: its simplicity. It has only one non-logical symbol, $\in$. First-order logic is just about the simplest logic in which one can say anything interesting. So the traditional language of set theory is the simplest usable candidate.

However, is it the "right" language of set theory? Should we instead use a more complicated language? For example, should we have extra predicate letters? Or should we use second-order logic, in which we can quantify over predicates or functions, not just over variables?*

I think there is no convincing, generally agreed answer to this question.

Some professional set theorists would say yes, the traditional choice of the language of set theory is the correct one. Others would say that it is not powerful enough. And

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* So in second order logic, you can replace the Subset Axiom Scheme by a (considerably stronger) single axiom: $\forall x\, \forall U(\cdot)\, \exists y\, \forall z\, (z \in y \leftrightarrow (z \in x \wedge U(z)))$.

indeed, some researchers have recently proposed extra axioms (to be added to ZFC) which look to me a little like attempts to get around the problem of the weakness of first-order logic.

1.3. How much does it matter?

How much does it matter which language of set theory we choose?

For the purposes of B1, hardly at all. That is why, in the lectures, we will be expressing the axioms in mathematical English, rather than in the language of set theory. All the theorems in the course would still be true if the language of set theory were, for instance, second order rather than first order.

This is convenient for us. Here, for example, is the Axiom of Infinity in the language of set theory:

$$\exists x\Bigg(\exists y\big(y \in x \wedge (\forall z\, \neg z \in y)\big) \wedge \forall z\Bigg(z \in x \to \exists w\Big(w \in x \wedge \forall v\big(v \in w \leftrightarrow (v = z \vee v \in z)\big)\Big)\Bigg)\Bigg).$$

The version in mathematical English is certainly easier to understand.

However, when we advance beyond B1 to, for instance, Gödel's proof of the relative consistency of the Axiom of Choice†, it matters very much which language we are using, and, indeed, such things as what quantifiers appear in the prenex normal form of a statement become important.

† That is, the statement that if the ZF axioms are consistent, then so is ZF together with the Axiom of Choice. This is proved in the section C Axiomatic Set Theory course.