Oxford 1/2/2011

$\boxed{\text{Aim}}$ Interior regularity of minimizers of variational integrals under 'natural conditions'.

http://www.maths.ox.ac.uk/~kristens

$\boxed{\text{Paradigm}}$ 'Weyl's Lemma'

(Harmonic functions are smooth ...)

The Dirichlet integral (& notation)

$$\mathrm{Dir}[v,\Omega] := \int_\Omega |Dv(x)|^2 \, dx$$

$$v = \begin{pmatrix} v^1 \\ \vdots \\ v^N \end{pmatrix} : \Omega \to \mathbb{R}^N$$

$\Omega \subset \mathbb{R}^n$ bounded, open Lipschitz domain

Sobolev spaces $\begin{pmatrix} k \in \mathbb{N}_0 \\ 1 \leq p \leq \infty \end{pmatrix}$

$$W^{k,p}(\Omega,\mathbb{R}^N) := \left\{ v \in L^p(\Omega,\mathbb{R}^N) : \begin{array}{l} D^\alpha v \in L^p(\Omega,\mathbb{R}^N) \\ \text{for } |\alpha| \leq k \end{array} \right\}$$

When $g \in W^{1,p}(\Omega,\mathbb{R}^N)$,

$$W^{1,p}_g(\Omega,\mathbb{R}^N) := \left\{ v \in W^{1,p}(\Omega,\mathbb{R}^N) : v = g \text{ on } \partial\Omega \right\}$$

$$Dv(x) = \left\{ \frac{\partial v^i}{\partial x^j} \right\}_{1 \leq j \leq n}^{1 \leq i \leq N} \in \mathbb{R}^{N \times n}$$

Jacobi matrix of v

$$|Dv(x)| = \left(\operatorname{trace}\left(Dv(x)^T Dv(x) \right) \right)^{\frac{1}{2}}$$

usual euclidean norm on $\mathbb{R}^{N \times n} \cong \mathbb{R}^{Nn}$

$\boxed{\text{Def}}$ Let $g \in W^{1,2}(\Omega, \mathbb{R}^N)$. Then $u \in W_g^{1,2}(\Omega, \mathbb{R}^N)$ is harmonic iff

$$\operatorname{Dir}[u, \Omega] \leq \operatorname{Dir}[v, \Omega] \quad \forall v \in W_g^{1,2}(\Omega, \mathbb{R}^N)$$

<u>Existence & Uniqueness:</u> $\forall g \in W^{1,2}\ \exists!\, u \in W_g^{1,2}$ that is harmonic

— proof is easy and follows from completeness (and strict convexity of norm) of $W^{1,2}$.

<u>Regularity:</u> $\boxed{\text{Weyl's Lemma}}$ If $u \in W_g^{1,2}$ is harmonic, then $u \in C^\infty(\Omega, \mathbb{R}^N)$ and $\Delta u = u_{x_1 x_1} + \ldots + u_{x_n x_n} = 0$ on Ω.

Want to generalize this to more general variational integrals modelled on the Dirichlet integral in some way.

What's possible?

$\boxed{\text{Set-up}}$ (that will be relaxed next week)

Given an <u>integrand</u> $F : \mathbb{R}^{N\times n} \to \mathbb{R}$
which is C^∞ and with

$$|F''(\xi)| \leq L \quad \text{for all } \xi$$

<u>Notation</u> for derivatives of integrands:

$$F'(\xi)[\eta] := \frac{d}{dt}\Big|_{t=0} F(\xi + t\eta)$$

We think of $F'(\xi)$ as $N\times n$ matrix or as linear functional on $\mathbb{R}^{N\times n}$

$$F''(\xi)[\eta,\eta] := \frac{d^2}{dt^2}\Big|_{t=0} F(\xi + t\eta)$$

$F''(\xi)$ symmetric bilinear form on $\mathbb{R}^{N\times n}$

$$|F'(\xi)| := \operatorname{trace}\left(F'(\xi)^T F'(\xi)\right)^{\frac{1}{2}}$$ always!

$$|F''(\xi)| := \sup_{\substack{\eta,\lambda \in \mathbb{R}^{N\times n} \\ |\eta|,|\lambda| \leq 1}} F''(\xi)[\eta,\lambda]$$

From $|F''(\xi)| \leq L$ follows $\exists c$ s.t.

$|F(\xi)| \leq c(|\xi|^2+1)$ for all ξ

'F has quadratic growth', hence

$\mathcal{F}[v] := \int_\Omega F(Dv)$ 'variational integral'

is well-defined on $v \in W^{1,2}(\Omega, \mathbb{R}^N)$.

$\boxed{\text{Def. 1.1}}$ Let $g \in W^{1,2}(\Omega, \mathbb{R}^N)$. Then $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ is an <u>F-minimizer</u> iff $\mathcal{F}[u] \leq \mathcal{F}[v]$ $\forall v \in W^{1,2}_g(\Omega, \mathbb{R}^N)$.

$\boxed{\text{Def. 1.2}}$ $\mathcal{F}$ <u>regular variational integral</u> iff the integrand F is C^2 <u>and</u>

$$(*) \qquad \ell |\lambda|^2 \leq F''(\xi)[\lambda, \lambda] \leq L|\lambda|^2$$

holds for all $\xi, \lambda \in \mathbb{R}^{N \times n}$, where $0 < \ell \leq L < \infty$ are constants.

<u>Notes</u>
- $\mathcal{F} = \frac{L}{2}\,\text{Dir}$ if $\ell = L$.
- $F''(\xi)[\lambda, \lambda] \geq \ell|\lambda|^2$ $\forall \xi, \lambda$ iff $F - \frac{\ell}{2}|\cdot|^2$ is convex.

A <u>regular variational problem</u>: to minimize a regular variational integral over $W^{1,2}_g$.

Existence & Uniqueness: Assume $\mathcal{F}$ is a regular variational integral. Then $\forall g \in W^{1,2}(\Omega, \mathbb{R}^N)$ $\exists!\, u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ that is F-minimizing.

– Proof is easy consequence of (*) and completeness of $W^{1,2}$.

Observe that since F (and hence $\mathcal{F}$) is convex we have:
u F-minimizing iff u F-extremal
(ie, $\int_\Omega F'(Du)[D\varphi] = 0 \quad \forall \varphi \in W^{1,2}_0$)

Regularity? $\boxed{\text{Hilbert's } 19^{th} \text{ Problem}}$

→ Are solutions to regular variational problems as regular as the data?

Answer depends on values of N, n.

Big difference between

N=1 (1 unknown function 'scalar case')

$\boxed{\text{YES}}$ Ladyzhenskaya & Ural'tseva (1960s)
based on De Giorgi, Nash, Moser

N>1 (more than 1 unknown function 'vectorial case')

NO Counter examples by

De Giorgi 1968, Maz'ya 1968

Nečas 1975, Šverák & Yan 2000/2002

... among others.

We always assume that $n \geqslant 2$, and we note that also $n=2$ is a special case as regards regularity.

TH 1.1 (Difference-quotient method)

Assume F is C^2 and

$$\circledast \quad \ell|\lambda|^2 \leqq F''(\xi)[\lambda,\lambda] \leqq L|\lambda|^2, \ \forall \xi, \lambda.$$

If $u \in W^{1,2}_g$ is F-minimizing, then $u \in W^{2,2}_{loc}$. Furthermore, for $B(x,3R) \subset \Omega$,

$$(1) \quad \int_{B(x,R)} |D^2 u|^2 \leqq \left(\frac{2L}{\ell}\right)^2 \int_{B(x,3R)} \frac{|Du - (Du)_{x,3R}|^2}{R^2}$$

(a 'Caccioppoli inequality') and for each $1 \leqq s \leqq n$,

$$(2) \quad \int_\Omega F''(Du)[D(D_s u), D\varphi] = 0 \quad \forall \varphi \in C^\infty_c(\Omega, \mathbb{R}^N).$$

<u>Notation</u> When $f: \Omega \to \mathbb{R}^k$, $1 \leqq s \leqq n$, $h \in \mathbb{R}$, put $\Delta_{s,h} f(x) := f(x + he_s) - f(x)$ for $x \in \Omega$ with $x + he_s \in \Omega$, where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$.

Lemma: Let $1 < p < \infty$ and $\Omega' \Subset \Omega$.

If $f \in W^{1,p}(\Omega, \mathbb{R}^N)$, then

$$\| \Delta_{s,h} f \|_{L^p(\Omega')} \leq \| D_s f \|_{L^p(\Omega)} |h| \qquad (3)$$

for $1 \leq s \leq n$, $|h| < \text{dist}(\Omega', \partial\Omega)$.

If $f \in L^p(\Omega, \mathbb{R}^N)$ and for some $\delta \in (0, \text{dist}(\Omega', \partial\Omega))$ and $M < \infty$,

$$\| \Delta_{s,h} f \|_{L^p(\Omega')} \leq M |h| \qquad (4)$$

for $1 \leq s \leq n$, $|h| < \delta$, then $f \in W^{1,p}(\Omega', \mathbb{R}^N)$ and $\| D_s f \|_{L^p(\Omega')} \leq M$, $1 \leq s \leq n$.

Proof can be found in Giusti's book 'Direct methods...' or in my notes on web.

$\boxed{\text{Proof of TH 1}}$ (Difference-quotient method probably due to Shiffman, Nirenberg 1940-50s)

Let $u \in W^{1,2}_g$ be F-minimizing: $\mathcal{F}[u] \leq \mathcal{F}[v]$ $\forall v \in W^{1,2}_g$. Fix $\varphi \in W^{1,2}_0$ and note $u + t\varphi \in W^{1,2}_g$ so that $\mathcal{F}[u + t\varphi]$ is min at $t = 0$.

Hence as $|F''| \leq L$,

$$0 = \frac{d}{dt}\Big|_{t=0} \mathcal{F}[u + t\varphi] = \int_\Omega F'(Du)[D\varphi].$$

The Euler-Lagrange equation for $\mathcal{F}$.

Fix $B_{3R} = B(x,3R) \subset \Omega$, $\rho \in W^{1,\infty}(\Omega)$ s.t.

$$\mathbb{1}_{B_R} \leq \rho \leq \mathbb{1}_{B_{2R}} \quad \text{and} \quad |D\rho| \leq \frac{1}{R} \text{ a.e.}$$

For $1 \leq s \leq n$, $|h| < R$, put $\varphi := \Delta_{s,-h}(\rho^2 \Delta_{s,h}(u-a))$ where $a : \mathbb{R}^n \to \mathbb{R}^N$ is affine. Then $\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^N)$ so

$$\int_\Omega F'(Du)[D\varphi] = 0,$$

hence

$$0 = \int_\Omega \Delta_{s,h} F'(Du)\left[\rho^2 \Delta_{s,h} Du + \Delta_{s,h}(u-a) \otimes D(\rho^2)\right]$$

Notation. For vectors $b \in \mathbb{R}^n$, $d \in \mathbb{R}^N$

$$d \otimes b = \{d^i b^j\}_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n}} \in \mathbb{R}^{N \times n},$$

$$|d \otimes b| = |d| \cdot |b|.$$

Note

$$\Delta_{s,h} F'(Du)[\Delta_{s,h} Du] = \int_0^1 F''(Du + t\Delta_{s,h} Du)[\Delta_{s,h} Du, \Delta_{s,h} Du]\,dt$$

$$\overset{(*)}{\geq} \ell |\Delta_{s,h} Du|^2,$$

$$\left|\Delta_{s,h} F'(Du)[\Delta_{s,h}(u-a) \otimes D(\rho^2)]\right| \overset{\text{Cauchy-Schwarz } \& \ (*)}{\leq}$$

$$2L|\Delta_{s,h} Du|\rho \cdot |\Delta_{s,h}(u-a)|\,|D\rho|$$

and hence

$$\ell \int_\Omega |\Delta_{s,h} Du|^2 \rho^2 \leq 2L \int_\Omega |\Delta_{s,h} Du| \rho \, |\Delta_{s,h}(u-a)| \, |D\rho|$$

$$\leq \frac{\ell}{2} \int_\Omega |\Delta_{s,h} Du|^2 \rho^2 + \frac{1}{2} \frac{(2L)^2}{\ell} \int_\Omega |\Delta_{s,h}(u-a)|^2 |D\rho|^2,$$

so

$$\int_{B_R} |\Delta_{s,h} Du|^2 \leq \left(\frac{2L}{\ell}\right)^2 R^{-2} \int_{B_{2R}} |\Delta_{s,h}(u-a)|^2$$

$$\leq \left(\frac{2L}{\ell}\right)^2 \int_{B_{3R}} \frac{|D_s(u-a)|^2}{R^2} \cdot h^2 .$$

$\therefore D_s Du \in L^2(B_R)$ and

$$\int_{B_R} |D_s Du|^2 \leq \left(\frac{2L}{\ell}\right)^2 \int_{B_{3R}} \frac{|D_s(u-a)|^2}{R^2},$$

hence $u \in W^{2,2}_{loc}$ and

$$\int_{B_R} |D^2 u|^2 \leq \left(\frac{2L}{\ell}\right)^2 \int_{B_{3R}} \frac{|Du - (Du)_{x,3R}|^2}{R^2} .$$

Return to E-L: Take $\psi \in C_c^\infty(\Omega, \mathbb{R}^N)$ and use $\varphi = D_s \psi$ to get (2).

$\boxed{\text{TH 2}}$ (De Giorgi, 1957)

Assume $A: \Omega \to \mathbb{R}^{n\times n}$ measurable,
$A(x) = A(x)^T$ and $\ell|\lambda|^2 \leqq A(x)\lambda\cdot\lambda \leqq L|\lambda|^2$
for all $\lambda \in \mathbb{R}^n$ and a.e. $x \in \Omega$.

If $u \in W^{1,2}_{loc}(\Omega)$ and

$$\int_\Omega A(x) Du \cdot D\varphi \, dx = 0 \quad \forall \varphi \in C^\infty_c(\Omega),$$

then $u \in C^{0,\alpha_0}_{loc}(\Omega)$ for some $\alpha_0 = \alpha_0(n, \frac{L}{\ell}) \in (0,1)$.

Apply to $D_s u$ $1 \leqq s \leqq n$: from TH1 $D_s u \in W^{1,2}_{loc}$

& $\int_\Omega F''(Du)[D(D_s u), D\varphi] = 0 \quad \forall \varphi \in C^\infty_c(\Omega)$

$\therefore$ $D_s u \in C^{0,\alpha_0}_{loc}$ and so $u \in C^{1,\alpha_0}_{loc}$.

We can now apply <u>Schauder estimates</u>
whereby $u \in C^{k-1,\alpha}_{loc}$ $\forall \alpha < 1$ when $F \in C^k$

Note: Once we know Du is Hölder cont.,
then the theory applies equally well for $N>1$.
It's only De Giorgi's theorem where $N=1$ is needed.

$\boxed{N>1}$

Example (De Giorgi 1968)

De Giorgi's theorem doesn't hold when $n = N > 2$.

$$u(x) = |x|^{-\mu} x \quad , \quad |x| < 1$$

$$\mu = \frac{n}{2}\left(1 - \frac{1}{\sqrt{(2n-2)^2+1}}\right)$$

Note: $u(x) = x$ for $|x| = 1$,

u positively $(1-\mu)$-homogeneous

and as $\mu < \frac{n}{2}$, $u \in W_x^{1,2}(B(0,1), \mathbb{R}^n)$.

Since $\mu > 1$ (iff $n > 2$) also

$u \notin L^\infty_{loc}(B(0,1), \mathbb{R}^n)$.

~~If $A(x)$:~~

$$A(x)[\lambda, \lambda] := |\lambda|^2 + \left\langle (n-2)I + n\frac{x \otimes x}{|x|^2}, \lambda \right\rangle^2$$

Note A measurable (smooth away from 0)

and

$$|\lambda|^2 \leq A(x)[\lambda,\lambda] \leq n^2(n-1)|\lambda|^2,$$

$$\int_{B(0,1)} A(x)[Du, D\varphi]\,dx = 0 \qquad \forall \varphi \in W_0^{1,2}(B(0,1), \mathbb{R}^n).$$

(or equivalently: u minimizes

$$\int_{B(0,1)} A(x)[Dv, Dv]\,dx \quad \text{over } v \in W^{1,2}_x(B(0,1), \mathbb{R}^n))$$

<u>Note</u>: An example of a singular solution to a higher order linear elliptic equation was given simultaneously by Maz'ya. Though <u>not</u> variational.

<u>Examples</u> $\mathcal{F}(v) = \int_\Omega F(Dv)$, $v \in W^{1,2}(\Omega, \mathbb{R}^N)$

Minimizers of a regular variational integral can be:

- Lipschitz but not C^1 — Nečas 1975

  $n \geqslant 25$, $N \geqslant 625$

- —"— — Hao, Leonardi & Nečas 1996

  $n \geqslant 5$, $N \geqslant 25$

- $$u_{ij}(x) = |x|^{-\varepsilon(n)}\left(\frac{x_i x_j}{|x|} - \frac{|x|}{n}\delta_{ij}\right), \quad \varepsilon = \varepsilon(n) > 0$$

  $u(x) = (u_{ij}(x))$, $|x| < 1$

  considered as map into symm. trace-free $n \times n$ matrices (so $u : B(0,1) \subset \mathbb{R}^n \to \mathbb{R}^N$, $N = \frac{n(n+1)}{2} - 1$)

  Šverák & Yan (2000, 2002)

$n \geqslant 3,\ N \geqslant 5$: non-Lipschitz

$n \geqslant 5,\ N \geqslant 14$: unbounded

$\vdots$

regularity deteriorates as $n \to \infty$

However, not all is lost :

Morrey : F-minimizers are regular when
1940s $n = 2,\ N \geqq 1$.

Campanato : F-minimizers $u \in W^{2,2+\delta}_{loc}$
1980s for some $\delta = \delta(n, N, \frac{L}{\ell}) > 0$.

... and therefore $u \in C^{0,\alpha}_{loc}$ for some $\alpha \in (0,1)$ when $n \leqq 4,\ N \geqq 1$.

K & Melcher (2008): F-minimizers $u \in W^{2,2+\frac{1}{50}\frac{\ell}{L}}_{loc}$

∴ Dimension-free integrability improvement

K & Mingione (2010): F-minimizers $u \in W^{1,2}_g$
satisfy

$\cancel{r \mapsto r^{-(n-1)}}$

$$r \mapsto r^{-(n-1)\frac{\ell}{L}} \int_{B(x,r)} |Du|^2 \quad \text{increasing}$$

Consequently,

$$u \in C^{1,\alpha}_{loc} \quad \text{if} \quad \frac{l}{L} \geqq \frac{n-1}{n-2+2\alpha}$$

and

$$u \in C^{0,\alpha}_{loc} \quad \text{if} \quad \begin{cases} n-4+2\alpha > 0 \ \& \ \dfrac{l}{L} \geqq \dfrac{n-1}{n-4+2\alpha} \\ \text{or} \\ n-4+2\alpha \leqq 0. \end{cases}$$