Recall from L1

When the dimensions n, N are sufficiently high, then a regular variational problem can admit singular solutions:

There exists a C^∞ smooth integrand

$$F : \mathbb{R}^{5\times 2} \longrightarrow \mathbb{R}$$

satisfying for some constants $0<\ell<L<\infty$

$$(*) \qquad \ell|\lambda|^2 \leq F''(\xi)[\lambda,\lambda] \leq L|\lambda|^2 \quad \forall \xi, \lambda$$

and such that for some constant $\varepsilon>0$ the $W^{1,2}$ mapping $u : B(0,1) \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^5$ defined by

$$u : B(0,1) \ni x \longmapsto |x|^{-\varepsilon}\left(\frac{x_i x_j}{|x|} - \frac{|x|}{n}\delta_{ij}\right)$$

is an F-minimizer, i.e.

$$\int_{B(0,1)} F(Du) \leq \int_{B(0,1)} F(Du + D\varphi) \quad \forall \varphi \in W_0^{1,2}$$

(Šverák & Yan, 2002)

— Is all lost? Can we expect no regularity?

Note u is smooth away from $\Sigma_u := \{0\}$, a relatively closed subset of $\Omega = B(0,1)$. Besides this Σ_u is a 'small' set. This isn't a coincidence!

$\boxed{\text{TH}}$ $\left\{\begin{array}{l}\text{Almgren, DeGiorgi, } \ldots \quad 1968 \\ \text{Morrey, Giusti \& Miranda, Giaquinta\&Giusti}\end{array}\right\}$

Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be C^k $(k \geqq 2)$ and

$$(*) \quad \ell|\lambda|^2 \leqq F''(\xi)[\lambda,\lambda] \leqq L|\lambda|^2 \quad \forall \xi, \lambda,$$

where $0 < \ell \leqq L < \infty$ are constants.

If $u \in W^{1,2}$ is an F-minimizer, then

$$\Sigma_u := \left\{ x \in \Omega : \begin{array}{l} \overline{\lim_{r \searrow 0}} \, |(Du)_{x,r}| = \infty \text{ or } \\ \underline{\lim_{r\searrow 0}} \fint_{B(x,r)} |Du - (Du)_{x,r}|^2 > 0 \end{array} \right\}$$

is relatively closed in Ω, and u is $C^{k-1,\alpha}_{loc}$ on $\Omega \setminus \Sigma_u$ for each $\alpha < 1$.

$\boxed{\text{Remarks}}$ We say that u is 'partially regular' and refer to the above result as a 'partial regularity result'.

The set Σ_u of singularities is

called the singular set.

From Lebesgue's differentiation theorem follows that $\mathcal{L}^n(\Sigma_u) = 0$, but much more can be said.

We shall prove a partial regularity result for minimizers of variational integrals under 'natural conditions'.

In the remainder of this lecture and part of the next we shall discuss these natural conditions — their motivation comes from the question of existence of minimizers, and our discussion starts with the direct method:

<u>Existence of minimizers by the direct method</u>

$\boxed{\text{Set-up}}$ Assume $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ continuous

and $$|F(\xi)| \leq c(|\xi|^2 + 1) \quad \forall \xi$$

Consider $$\mathcal{F}[v] := \int_\Omega F(Dv), \quad v \in W^{1,2}_g(\Omega, \mathbb{R}^N)$$

Assume to start with that $\exists\, \varepsilon > 0$ s.t.

$$F(\xi) \geqslant \varepsilon |\xi|^2 - \frac{1}{\varepsilon} \quad \forall \xi .$$

Then $\quad \mathcal{F}[v] \geqslant \varepsilon \int_\Omega |Dv|^2 - \frac{\mathcal{L}^n(\Omega)}{\varepsilon} \quad \forall v \in W^{1,2}_g$,

so $\mathcal{F}$ is in particular bounded below on $W^{1,2}_g$ and if $(u_j) \subset W^{1,2}_g$ is a minimizing seq.

$$\mathcal{F}[u_j] \longrightarrow \inf_{v \in W^{1,2}_g} \mathcal{F}[v],$$

then (Du_j) is bounded in L^2. Since $u_j - g \in W^{1,2}_0$ we have $\int_\Omega |u_j - g|^2 \leqslant c \int_\Omega |Du_j - Dg|^2$ by Poincaré, so (u_j) is bounded in $W^{1,2}$.

<u>Recall</u> $W^{1,2}$ is a Hilbert space, so a (norm-) bounded seq (u_j) admits a weakly cvgt subseq: $\exists\, (u_{j_k}), u \in W^{1,2}$ s.t. $u_{j_k} \rightharpoonup u$ in $W^{1,2}$.

By Mazur's Lemma $u \in W^{1,2}_g$.

Since Ω is a <u>bounded Lipschitz domain</u> $u_{j_k} \rightharpoonup u$ means:

$u_{j_k} \to u$ in L^2 (in fact, in L^p, $p<2^*$, by Rellich)

&

$Du_{j_k} \rightharpoonup Du$ in L^2.

$\boxed{\text{Brief review of weak/weak* crg in } L^p}$

<u>Def.</u> $f_j \rightharpoonup f$ in L^p ($\overset{*}{\rightharpoonup}$ when $p=\infty$) iff

$\int_\Omega f_j \cdot g \to \int_\Omega f \cdot g$ pointwise in $g \in L^{p'}(\Omega, \mathbb{R}^k)$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

<u>Lemma:</u> Assume $1 < p \leq \infty$ and $\mathcal{L}^n(\Omega) < \infty$.

$f_j \rightharpoonup f$ in L^p iff ($\overset{*}{\rightharpoonup}$ if $p=\infty$)

(i) $\sup_j \|f_j\|_{L^p} < \infty$ & (ii) $(f_j)_B \to f_B$ for all balls $B \subset \Omega$.

(Pf: exercise)

<u>Lemma:</u> Assume $1 \leq p < \infty$ and $\mathcal{L}^n(\Omega) < \infty$.

$f_j \to f$ in L^p iff

(I) $(|f_j|^p)$ equi-integrable

&

(II) $f_j \to f$ in measure

(Pf: exercise) (Vitali's convergence th.)

$\boxed{\text{De la Vallée-Poussin}}$ (I) holds iff

$\exists$ increasing (& convex) function $\Phi : [0,\infty) \to [0,\infty)$

s.t. $\dfrac{\Phi(t)}{t} \to \infty$ as $t \to \infty$

and $\sup_j \int_\Omega \Phi(|f_j|^p) < \infty$.

(Pf: exercise)

$\boxed{\text{Obstructions to strong convergence}}$

when $\mathcal{L}^n(\Omega) < \infty$

oscillations: not convergent in measure

ex: $\Omega = (0,1)$, $f_j(x) = \sin(jx)$

concentrations: not equi-integrable

ex: $\Omega = (0,1)$, $f_j(x) = j^{\frac{1}{p}} \mathbb{1}_{(0,\frac{1}{j})}(x)$

Vitali's convergence theorem is in a sense a qualitative statement: we have strong L^p-convergence iff the sequence does not oscillate and does not concentrate.

A more quantitative statement is

$\boxed{\text{Riesz-Kolmogorov compactness criterion}}$

Assume $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain. Let $\mathcal{F} \subset L^p(\Omega, \mathbb{R}^k)$ be norm-bounded, where $1 \leq p < \infty$.

Then $\mathcal{F}$ is precompact in $L^p(\Omega, \mathbb{R}^k)$ iff

$$\lim_{h \to 0} \sup_{f \in \mathcal{F}} \int_{\Omega \cap (\Omega - h)} |f(x+h) - f(x)|^p \, dx = 0.$$

(Pf: exercise)

Compare with the difference-quotient characterization of Sobolev spaces given in L1: the above result states that compactness in L^p is closely related to regularity! Hence it is not surprising when the difficulties in regularity proofs often stems from compactness issues.

A device for generating oscillating sequences:

The Riemann-Lebesgue Lemma

Let $1 \leqq p \leqq \infty$ and $\mathcal{L}^n(\Omega) < \infty$.

Assume $f : \mathbb{R}^n \to \mathbb{R}^k$ is Q-periodic and L^p_{loc}, where $Q = \prod_{j=1}^{n} (a_j, b_j)$ is a rectangle (or more generally $Q = T(\prod_{j=1}^{n} (a_j, b_j))$ for some $T \in GL_n$).

Define $f_j(x) := f(jx)$, $x \in \Omega$, $j \in \mathbb{N}$.

Then $f_j \longrightarrow \fint_Q f$ in $L^p(\Omega, \mathbb{R}^k)$ ($\overset{*}{\rightharpoonup}$ if $p=\infty$)

understood as a constant map.

(Pf: exercise, or B. Dacorogna: Direct methods in the Calculus of Variations.)

$\boxed{\text{Ex}}$ Let $f : \mathbb{R} \to \mathbb{R}$ be 1-periodic and $f = -117\,\mathbb{1}_{(0,\frac{1}{2})} + 117\,\mathbb{1}_{(\frac{1}{2},1)}$. Put $f_j(x) = f(jx)$, $x \in \Omega$.

Then $f_j \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$ ($\Omega \subset \mathbb{R}$ any measurable set with $\mathcal{L}^1(\Omega) < \infty$)

Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be cont. and

$$|F(\xi)| \leq c(|\xi|^2+1) \quad \forall \xi,$$

$$\mathcal{F}[v] = \int_\Omega F(Dv), \quad v \in W^{1,2}.$$

Assume $\mathcal{F}[v]$ is swlsc (sequentially weakly lower semicontinuous) on $W^{1,2}_g$:

If $v_j, v \in W^{1,2}_g$ and $v_j \rightharpoonup v$ in $W^{1,2}$, then $\underline{\lim}\, \mathcal{F}[v_j] \geq \mathcal{F}[v]$.

What does it mean for F?

Let $\omega \subset \mathbb{R}^n$ be bounded, open and $\mathcal{L}^n(\partial\omega) = 0$.

Take $a<b$ so $\omega \subset (a,b)^n =: Q$.

Let $\varphi \in W^{1,2}_0(\omega, \mathbb{R}^N)$ and extend φ to $Q \setminus \omega$ by 0, and then to $\mathbb{R}^n$ by Q-periodicity. Hereby $\varphi|Q \in W^{1,2}_0(Q, \mathbb{R}^N)$ and $\varphi \in W^{1,2}_{loc}$.

Take $x_0 \in \Omega$, $r>0$ s.t. $x_0 + 2rQ \subset \Omega$, and put

$$\phi_j(x) := \begin{cases} \frac{r}{j}\varphi\left(j\,\frac{x-x_0}{r}\right), & x \in x_0 + rQ \\ 0, & \text{else}. \end{cases}$$

Then $\phi_j \in W^{1,2}_0(\Omega, \mathbb{R}^N)$, $\phi_j \to 0$ in L^2

and $\qquad D\phi_j(x) = \begin{cases} D\varphi(j\frac{x-x_0}{r}) \,, & x \in x_0 + rQ \\ 0 \,, & \text{else} \,, \end{cases}$

so $D\phi_j \rightharpoonup 0 \; (= \fint_Q D\varphi)$ in L^2 by Riemann-Lebesgue. Take $\rho \in W_0^{1,\infty}(\Omega)$ s.t. $\mathbb{1}_{x_0+rQ} \leq \rho \leq \mathbb{1}_{x_0+2rQ}$ and $|D\rho| \leq \frac{2}{r(b-a)}$ a.e.

For $\xi \in \mathbb{R}^{N\times n}$ put $a(x) := \xi x$, $x \in \mathbb{R}^n$, and

$$v_j := \rho(a + \phi_j) + (1-\rho)g \in W_g^{1,2}(\Omega, \mathbb{R}^N).$$

Note $v_j \rightharpoonup v := \rho a + (1-\rho)g$ in $W_g^{1,2}$,

so $\underline{\lim} \int_\Omega F(Dv_j) \geq \int_\Omega F(Dv)$.

Since $\qquad v_j = \begin{cases} a + \phi_j & \text{on } x_0 + rQ \\ v & \text{on } x_0 + rQ \end{cases}$

and $v = a$ on $x_0 + rQ$,

$$\underline{\lim} \int_{x_0+rQ} F(\xi + D\phi_j) \geq F(\xi)\, \mathcal{L}^n(x_0+rQ) = F(\xi)\, r^n \mathcal{L}^n(Q)$$

$$\int_{x_0+rQ} F(\xi + D\phi_j) = r^n \int_Q F(\xi + D\varphi(jx))\,dx.$$

$x \mapsto F(\xi + D\varphi(x))$ is Q-periodic and L^1_{loc},

so by Riemann-Lebesgue,

$$\int_Q F(\xi + D\varphi) \geq F(\xi)\mathcal{L}^n(Q),$$

and since $\varphi = 0$ on $Q \setminus \omega$, we get

$$F(\xi) \leq \fint_\omega F(\xi + D\varphi(x))\,dx \quad (+)$$

This is a necessary condition for swlsc on $W^{1,2}_g$ — it's also sufficient!

$\boxed{\text{TH}}$ (Meyers 1965, Fusco 1981)

Assume $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is continuous, and $|F(\xi)| \leq c(|\xi|^2+1)$ $\forall \xi$. Let $g \in W^{1,2}(\Omega, \mathbb{R}^N)$.

Then $\mathcal{F}[v] = \int_\Omega F(Dv)$ is swlsc on $W^{1,2}_g$

iff

(QC) $$\fint_{B(0,1)} F(\xi + D\varphi(x))\,dx \geq F(\xi)$$

for all $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W^{1,\infty}_0(B(0,1), \mathbb{R}^N)$.

We omit pf of $\Leftarrow$.

<u>Note</u> If (QC) holds, then (+) holds (ie, ω bounded & open with $\mathcal{L}^n(\partial\omega) = 0$ & $\varphi \in W^{1,2}_0$)

<u>Def.</u> (C.B. Morrey, Jr ; 1952)

F is <u>quasiconvex</u> at $\xi \in \mathbb{R}^{N\times n}$ iff

(QC) $$\fint_{B(0,1)} F(\xi + D\varphi(x))\,dx \geqq F(\xi)$$

for all $\varphi \in W_0^{1,\infty}(B(0,1), \mathbb{R}^N)$.

F is quasiconvex iff it's quasiconvex at all $\xi \in \mathbb{R}^{N\times n}$.

<u>Note</u> If $|F(\xi)| \leqq c(|\xi|^p + 1)$ $\forall \xi$, then F is QC at ξ iff (QC) holds for all $\varphi \in W_0^{1,p}(B(0,1), \mathbb{R}^N)$.

<u>Exx</u> If $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is convex, then it is QC. This follows from <u>Jensen's inequality</u>: If μ is a probability measure on $\mathbb{R}^{N\times n}$ with centre of mass $\bar{\mu} := \int_{\mathbb{R}^{N\times n}} \eta \, d\mu(\eta)$, then

$$\int_{\mathbb{R}^{N\times n}} F\,d\mu \geqq F(\bar{\mu}).$$

$\therefore$ QC is merely Jensen's inequality for a special class of probability measures :

$\mu = \mu_{\xi,\varphi}$ defined via Riesz' represen= tation theorem as

$$\int_{\mathbb{R}^{N\times n}} \Phi \, d\mu := \fint_{B(0,1)} \Phi(\xi + D\varphi(x))\,dx$$

for $\Phi \in C_0^0(\mathbb{R}^{N\times n})$ (= continuous functions vanishing at infinity)

$\boxed{\text{EX}}$ For $n = N = 2$ (or more generally $n = N > 1$)

$F(\xi) = \det \xi$ 'determinant of square matrix ξ'

is not convex, but is quasiconvex.

It's even <u>quasiaffine</u>:

$$\fint_{B(0,1)} \det(\xi + D\varphi(x))\,dx = \det \xi$$

for all $\xi \in \mathbb{R}^{n\times n}$ and all $\varphi \in W_0^{1,\infty}(B(0,1), \mathbb{R}^n)$.

For $n = N = 2$: $\quad \xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}$

$$\det(\xi + D\varphi) = \begin{vmatrix} a + \varphi^1_{,x} & b + \varphi^1_{,y} \\ c + \varphi^2_{,x} & d + \varphi^2_{,x} \end{vmatrix} =$$

$$\det \xi + \langle \operatorname{cof} \xi, D\varphi \rangle + (\varphi^1 \varphi^2_{,x})_{,y} - (\varphi^1 \varphi^2_{,y})_{,x}$$

hence $\displaystyle\fint_{B(0,1)} \det(\xi + D\varphi) = \det\xi$ by Gauss.

In fact, $\displaystyle\int_{\Omega} \det Du = \int_{\Omega} \det Dv$ for all $u, v \in W^{1,n}(\Omega, \mathbb{R}^n)$ for which $u - v \in W_0^{1,n}(\Omega, \mathbb{R}^n)$.

$$\boxed{\begin{array}{l} W^{1,n}(\Omega, \mathbb{R}^n) \ni u \longmapsto \displaystyle\int_{\Omega} \det Du \\ \text{only depends on } u|_{\partial\Omega} \end{array}}$$

Since $\det' = \operatorname{cof}$ we have that the $\boxed{\text{Euler-Lagrange equation}}$ for $\mathcal{F}[u] = \displaystyle\int_{\Omega} \det Du$

is $\qquad \operatorname{div} \operatorname{cof} Du = 0$ in Ω

(divergence taken row-wise and in sense of distributions on $n \times n$ matrix $\operatorname{cof} Du$).

<u>All</u> $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ satisfy the above Euler-Lagrange equation! We say that $F = \det$ is a $\boxed{\text{Null Lagrangian}}$

$\boxed{\text{TH}}$ (Morrey 1952, 1966 ... Ball 1977, Dacorogna ...)

Let $F : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ be C^1.
Then the following statements are equiva=
lent:

(1) F is quasiaffine: '=' holds in QC

(2) F is a Null Lagrangian:
$\operatorname{div} F'(Du) = 0$ in Ω for all $u \in W^{1,\infty}(\Omega, \mathbb{R}^N)$

(3) $W^{1,\infty}(\Omega, \mathbb{R}^N) \ni u \longmapsto \int_\Omega F(Du)$
only depends on $u|_{\partial\Omega}$

(4) F is an affine function of the minors of ξ (for example, when $n = N = 2$, then this means that $F(\xi) = c_0 + \langle \xi_0, \xi \rangle + \delta_0 \det \xi$ where $c_0, \delta_0 \in \mathbb{R}$, $\xi_0 \in \mathbb{R}^{2\times 2}$).

(Pf: B. Dacorogna: Direct methods in the CoV.)

$\boxed{\text{Def.}}$ (Morrey, Ball 1975)

$F : \mathbb{R}^{N\times n} \to \mathbb{R}$ (or $\mathbb{R}\cup\{\infty\}$) is <u>polyconvex</u> iff $F(\xi)$ is a convex function of the minors of ξ. For example, when $n = N = 2$, then this means that
$F(\xi) = f(\xi, \det\xi)$ $\forall \xi$, where
$f : \mathbb{R}^5 \to \mathbb{R}$ is convex.

<u>Lemma</u>: If F is polyconvex, then it is also quasiconvex.

$\boxed{\text{Pf}}$ Jensen's inequality and the fact that minors are quasiaffine. $\square$