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Recall from L2

Let $F : \mathbb{R}^{N \times n} \to \mathbb{R}$ be continuous and of quadratic growth: $|F(\xi)| \leq c(|\xi|^2 + 1)$ $\forall \xi$

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and boundary datuum $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ consider

$$\mathcal{F}[v] := \int_\Omega F(Dv), \quad v \in W^{1,2}_g(\Omega, \mathbb{R}^N).$$

Then $\mathcal{F}$ is swlsc on $W^{1,2}_g$ iff F is <u>quasiconvex</u> (= QC) in the sense of Morrey:

$$\text{(QC)} \quad \fint_{B(0,1)} F(\xi + D\varphi(x))\,dx \geq F(\xi)$$

holds for all $\xi \in \mathbb{R}^{N \times n}$ and $\varphi \in W^{1,\infty}_0(\Omega, \mathbb{R}^N)$.

Convex functions are QC by virtue of <u>Jensen's inequality</u> for convex functions.

More generally <u>polyconvex</u> (= PC) functions are QC. Recall that F is PC iff

$F(\xi)$ = convex functions of all minors of ξ, so that for $n = N = 2$:

$F(\xi) = f(\xi, \det \xi)$ for $\xi \in \mathbb{R}^{2\times 2}$, where $f : \mathbb{R}^{2\times 2} \times \mathbb{R} \cong \mathbb{R}^5 \to \mathbb{R}$ is convex.

The relevance of minors:

F is quasiaffine iff $F(\xi)$ is an affine function of the minors of ξ.

We have: $C \underset{\not\Leftarrow}{\Longrightarrow} PC \Longrightarrow QC$,

so far.

Also recall that we discuss swlsc because of its relevance in connection with proving existence of minimizers by the direct method — so far we have:

If $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is QC, of quadratic growth and (pointwise) coercive

$\exists \, \varepsilon > 0$ s.t. $F(\xi) \geqslant \varepsilon |\xi|^2 - \frac{1}{\varepsilon} \quad \forall \xi$,

then $\mathcal{F}[\cdot]$ admits a minimizer $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$.

We return to discussion of QC, and aim for a <u>necessary condition for QC.</u>

Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be QC.

Let $(\Phi_\varepsilon)_{\varepsilon>0}$ be a standard C^∞ smooth mollifier: $\Phi \in C_c^\infty(B(0,1))$, $\Phi \geqq 0$, $\int \Phi = 1$

and $\Phi_\varepsilon(\xi) := \varepsilon^{-nN}\Phi\left(\frac{\xi}{\varepsilon}\right)$. Put $F_\varepsilon := \Phi_\varepsilon * F$ (convolution). Then F_ε is QC:

for $\xi \in \mathbb{R}^{N\times n}$, $\varphi \in W_0^{1,\infty}(B(0,1), \mathbb{R}^N)$

$$\fint_{B(0,1)} F_\varepsilon(\xi + D\varphi(x))\,dx = \fint_{B(0,1)} \int_{\mathbb{R}^{N\times n}} \Phi_\varepsilon(\eta) F(\xi - \eta + D\varphi(x))\,d\eta\,dx$$

$$= \int_{\mathbb{R}^{N\times n}} \fint_{B(0,1)} F(\xi-\eta+D\varphi(x))\,dx\ \Phi_\varepsilon(\eta)\,d\eta$$

$$\geqslant \int_{\mathbb{R}^{N\times n}} F(\xi-\eta)\Phi_\varepsilon(\eta)\,d\eta = F_\varepsilon(\xi).$$

Now for $\xi \in \mathbb{R}^{N\times n}$, $\varphi \in W_0^{1,\infty}(B(0,1), \mathbb{R}^N)$ and $0<\delta<1$:

$$0 \leqq \fint_{B(0,1)} \left(F_\varepsilon(\xi+\delta D\varphi) - F_\varepsilon(\xi)\right) \overset{\text{Taylor}}{=}$$

$$\fint_{B(0,1)} \left(F_\varepsilon'(\xi)[\delta D\varphi] + \int_0^1 (1-t) F_\varepsilon''(\xi+t\delta D\varphi)[\delta D\varphi, \delta D\varphi]\,dt\right)$$

$$= \delta^2 \fint_{B(0,1)} \int_0^1 (1-t) F_\xi''(\xi + \delta t D\varphi)[D\varphi, D\varphi]\,dt$$

ie

$$0 \leq \fint_{B(0,1)} \int_0^1 (1-t) F_\xi''(\xi + \delta t D\varphi)[D\varphi, D\varphi]\,dt$$

for all $0 < \delta < 1$. Take $\delta \searrow 0$ to get

$$0 \leq \fint_{B(0,1)} F_\xi''(\xi)[D\varphi, D\varphi].$$

By arbitrariness of φ, we deduce that the quadratic form $\eta \mapsto F_\xi''(\xi)[\eta, \eta]$ is QC at 0.

Exercise Prove that a quadratic form $Q : \mathbb{R}^{N\times n} \to \mathbb{R}$ is QC iff it is QC at 0.

Def. An integrand $G : \mathbb{R}^{N\times n} \to \mathbb{R}$ is rank-1 convex (= RC) iff for all $\xi, \eta \in \mathbb{R}^{N\times n}$ with $\text{rank}(\eta) = 1$ we have that

$$\mathbb{R} \ni t \longmapsto G(\xi + t\eta)$$

is convex.

Notation $\eta \in \mathbb{R}^{N\times n}$ has rank one iff $\exists\, a \in \mathbb{R}^N \setminus \{0\}$, $b \in \mathbb{R}^n \setminus \{0\}$ s.t. $\eta = a \otimes b$:

$$\eta x = a \langle b, x \rangle \quad \forall x \in \mathbb{R}^n.$$

$\operatorname{im} \eta = \operatorname{span}\{a\}$, $\ker \eta = \operatorname{span}\{b\}^\perp$

$|\eta| = |a| \cdot |b|$ and we can always assume that $|a| = 1$ or $|b| = 1$.

$\boxed{\text{Legendre – Hadamard's condition}}$

Assume $G : \mathbb{R}^{N \times n} \to \mathbb{R}$ is C^2.

Then G is RC iff $G''(\xi)[\eta, \eta] \geq 0$ for all $\xi, \eta \in \mathbb{R}^{N \times n}$ with $\operatorname{rank}(\eta) = 1$.

$\boxed{\text{Exercise}}$ Prove that a quadratic form $Q : \mathbb{R}^{N \times n} \to \mathbb{R}$ is RC iff $Q(\eta) \geq 0$ for all $\eta \in \mathbb{R}^{N \times n}$ with $\operatorname{rank}(\eta) = 1$.

$\boxed{\text{Th}}$ Let $Q : \mathbb{R}^{N \times n} \to \mathbb{R}$ be a quadratic form. Then Q is RC iff Q is QC.

Before proving the theorem we'll use it — recall $F_\varepsilon = \Phi_\varepsilon * F$ is QC and that $\eta \mapsto F_\varepsilon''(\xi)[\eta, \eta]$ is QC.

By above theorem $\eta \mapsto F_\varepsilon''(\xi)[\eta, \eta]$ is RC, hence $F_\varepsilon''(\xi)[\eta, \eta] \geqslant 0$ for all ξ, η

with $\operatorname{rank}(\eta)=1$. But this is the Legendre–Hadamard condition for F_ε'' so F_ε is RC, and since $F_\varepsilon(\xi)\to F(\xi)$ pointwise in ξ as $\varepsilon\searrow 0$, also F is RC. Hence we have proved

$\boxed{\text{Lemma}}$ QC $\Longrightarrow$ RC.

Note that when $n=1$ or $N=1$, then <u>all</u> $N\times n$ — matrices have rank ≤ 1 and so in this (scalar!) case we have

PC = QC = RC = convexity. However, when $n, N > 1$ we have:

$$C \underset{\not\Leftarrow}{\Longrightarrow} PC \Longrightarrow QC \Longrightarrow RC$$

We return to the proof of the theorem that QC = RC for quadratic forms. It uses the <u>Fourier transformation</u>.

When $f \in \mathcal{S}(\mathbb{R}^n)$ $\Big(=$ Schwartz test functions$\Big)$

$$\hat{f}(y) := \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i\, x\cdot y}\, dx$$

Then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{\hat{f}}(x) = f(-x)$.

Also $\widehat{D_k f}(y) = -2\pi i\, \hat{f}(y) y_k$ and so if $f : \mathbb{R}^n \to \mathbb{R}^N$ (each component in $\mathcal{S}(\mathbb{R}^n)$), then

$$\widehat{Df}(y) = -2\pi i\, \hat{f}(y) \otimes y$$

Fourier transform defined componentwise

Writing $\hat{f}(y) = A(y) + iB(y)$, where $A(y), B(y) \in \mathbb{R}^N$, we get

$$\boxed{\widehat{Df}(y) = -2\pi i\,(A + iB) \otimes y}$$

Plancherel's theorem

For $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^{N\times n})$ we have $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.

Hence the Fourier transformation extends by continuity to a surjective isometry (= unitary transformation) of $L^2(\mathbb{R}^n, \mathbb{R}^{N\times n})$ onto itself. Furthermore (denoting the unique extension by the same symbol):

$$\langle f, g \rangle_{L^2} = \langle \hat{f}, \hat{g} \rangle_{L^2} \quad \forall f, g \in L^2.$$

$\boxed{\text{Pf}}$ of Th of equivalence between RC & QC for quadratic forms:

Let $\tilde{Q}: \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ be a symmetric linear mapping so $Q(\xi) = \langle \tilde{Q}\xi, \xi \rangle \ \forall \xi$.

Note that it suffices to show:

$$Q(a\otimes b) \geq 0 \quad \forall a \in \mathbb{R}^N, b \in \mathbb{R}^n \iff Q \text{ QC at } 0.$$

Let $\varphi \in W_0^{1,\infty}(B(0,1), \mathbb{R}^N)$, and extend φ to $\mathbb{R}^n \setminus B(0,1)$ by $0 \in \mathbb{R}^N$. We still denote this extension by φ, and note that in particular $D\varphi \in L^2(\mathbb{R}^n, \mathbb{R}^{N\times n})$, and

so $$\widehat{D\varphi}(y) = -2\pi i (A + iB) \otimes y \quad \text{a.e.}.$$

By Plancherel:

$$\int_{B(0,1)} Q(D\varphi(x))\,dx = \int_{\mathbb{R}^n} \langle \tilde{Q} D\varphi(x), D\varphi(x) \rangle\,dx$$

$$= \sum_{k,l,r,s} \tilde{Q}^{kl}_{rs} \int_{\mathbb{R}^n} D_k\varphi_l\, D_r\varphi_s\,dx = \sum_{k,l,r,s} \tilde{Q}^{kl}_{rs} \int_{\mathbb{R}^n} \widehat{D_k\varphi_l}\, \overline{\widehat{D_r\varphi_s}}\,dy$$

$$= \sum_{k,l,r,s} \tilde{Q}^{kl}_{rs} (2\pi)^2 \int_{\mathbb{R}^n} (A+iB)_l y_k (A-iB)_s y_r \, dy$$

$$= (2\pi)^2 \int_{\mathbb{R}^n} \sum_{k,l,r,s} \tilde{Q}^{kl}_{rs} \{ A_l y_k A_s y_r + B_l y_k B_s y_r$$

$$+ i(A_s y_r B_l y_k - A_l y_k B_s y_r) \} \, dy$$

$$= (2\pi)^2 \int_{\mathbb{R}^n} \{ < \tilde{Q}(A \otimes y), A \otimes y > + < \tilde{Q}(B \otimes y), B \otimes y >$$

$$+ i (< \tilde{Q}(B \otimes y), A \otimes y > - < \tilde{Q}(A \otimes y), B \otimes y >) \} dy$$

$$\overset{\tilde{Q} \text{ symm}}{=} (2\pi)^2 \int_{\mathbb{R}^n} \big(Q(A \otimes y) + Q(B \otimes y) \big) \, dy,$$

that is,

$$(*) \quad \boxed{\int_{B(0,1)} Q(D\varphi) \, dx = (2\pi)^2 \int_{\mathbb{R}^n} \big(Q(A \otimes y) + Q(B \otimes y) \big) \, dy}$$

Consequently, Q RC $\Longrightarrow$ Q QC.

Assume Q is QC. Then by def,

$$0 \leq \int_{B(0,1)} Q(D\varphi) \, dx \quad \text{for all } \varphi \in W^{1,\infty}_0(B(0,1), \mathbb{R}^N).$$

By dilation (change variables $x \mapsto Rx$) it follows that

$$0 \leq \int_{B(0,R)} Q(D\varphi)\,dx \quad \forall \varphi \in W_0^{1,\infty}(B(0,R), \mathbb{R}^N)$$

for any $R > 0$, and hence that

$$0 \leq \int_{\mathbb{R}^n} Q(D\varphi)\,dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^N).$$

Given $\varphi \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^N)$ we can find $\varphi_j \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^N)$ s.t. $\int_{\mathbb{R}^n} Q(D\varphi_j)\,dx \to \int_{\mathbb{R}^n} Q(D\varphi)\,dx$ as $j \to \infty$ (eg take $\varphi_j = \varphi \cdot \rho_j$, where $\rho_j \in C_c^\infty(\mathbb{R}^n)$ and $\rho_j(x) = 1$ for $|x| \leq j$, $\rho_j(x) = 0$ for $|x| \geq j+1$ and $|D\rho_j| \leq 2$).

Consequently, $0 \leq \int_{\mathbb{R}^n} Q(D\varphi)\,dx \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^N)$.

Now fix $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$ with $|a| = 1$, $|b| = 1$.

Take $\varphi_j \in \mathcal{S}$ s.t. $\widehat{\varphi_j}(y) = a\, j^{\frac{n}{2}} \eta(j(y-b))$

where $\eta \in C_c^\infty(B(0,1))$ and $\int_{\mathbb{R}^n} \eta^2 = 1$.

This corresponds to

$$\widehat{D\varphi_j}(y) = -2\pi i \, \hat{\varphi}_j(y) \otimes y$$

$$= -2\pi i \cdot j^{\frac{n}{2}} \eta(j(y-b)) \cdot a \otimes y \,,$$

so $A_j = j^{\frac{n}{2}} \eta(j(y-b))$, $B_j = 0$ in the notation above. Now by QC and ⊛:

$$0 \leq \int_{\mathbb{R}^n} Q(D\varphi_j)\,dx = (2\pi)^2 \int_{\mathbb{R}^n} Q(a\otimes y)\, j^n \eta(j(y-b))^2 dy$$

$$= (2\pi)^2 \int_{|y-b|<\frac{1}{j}} Q(a\otimes y)\, j^n \eta(j(y-b))^2 dy$$

$$\underset{j\to\infty}{\longrightarrow} (2\pi)^2 Q(a\otimes b) \,, \qquad \square$$

$\boxed{\text{Remark}}$ (Terpstra, D. Serre)

One can go one step further:

Let $Q : \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quadratic form. Then Q is RC iff Q is PC <u>when</u> $\min\{n, N\} \leq 2$.

This is false for $n, N \geq 3$: there exists a quadratic form $Q : \mathbb{R}^{3\times 3} \to \mathbb{R}$ that is

RC, but <u>not</u> PC.

Note that for $n=N=2$ the above result states: a quadratic form $Q:\mathbb{R}^{2\times 2}\to\mathbb{R}$ is RC iff $\exists$ positive semidefinite quadratic form $q:\mathbb{R}^{2\times2}\to\mathbb{R}$ and a number $\delta\in\mathbb{R}$, s.t. $Q(\xi)=q(\xi)+\delta\det\xi$, $\forall\xi$.

So far we have seen that in general

$$C \Longrightarrow PC \Longrightarrow QC \Longrightarrow RC$$

$\not\Longleftarrow$ when $n,N\geqq 2$

When $n=1$ or $N=1$ (corresponding to the scalar case) all matrices in $\mathbb{R}^{N\times n}$ have rank at most 1, and so in this case

$$\boxed{\text{Convexity} = PC = QC = RC}$$ When $\min\{n,N\}=1$

We have also mentioned that $QC \not\Rightarrow PC$ for quadratic forms on $\mathbb{R}^{N\times n}$ when $N,n\geqq 3$.

$\boxed{\text{Ex}}$ (Alibert, Dacorogna & Marcellini)

Let $f_t(\xi) = |\xi|^4 + 2t|\xi|^2 \det\xi$, $\xi \in \mathbb{R}^{2\times 2}$,

where $t \in \mathbb{R}$ is a parameter.

Note that f_t is a homogeneous polynomial of degree 4, which is also <u>isotropic</u>: $f_t(R\xi S) = f_t(\xi)$ for all $\xi \in \mathbb{R}^{2\times 2}$ and $R, S \in SO(2)$.

For isotropic functions on $\mathbb{R}^{2\times 2}$ one can show that they are PC iff they are PC on diagonal matrices:

f_t is PC iff $\exists$ convex $g : \mathbb{R}^3 \to \mathbb{R}$ s.t. $f_t\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = g(x, y, xy)$ for all $x, y \in \mathbb{R}$.

(Dacorogna & Koshigoe). Using this observation it is easy to show that

$$\boxed{f_t \text{ is PC} \iff |t| \leq 1}$$

Note that also

$$f_t(\xi) \geq 0 \;\; \forall \xi \iff |t| \leq 1 .$$

This can be easily deduced by use of <u>Hadamard's inequality</u>:

$$|\det\xi| \leq n^{-\frac{n}{2}} |\xi|^n \quad \forall \xi \in \mathbb{R}^{n\times n},$$

and '=' holds iff $\xi = 0$ or ξ is conformal or ξ is anti-conformal.

<u>Recall</u>: ξ is conformal iff $\xi = rR$, where $r>0$ and $R \in SO(n)$. It is anti-conformal iff $\xi = rS$, where $r>0$ and $S \in O(n)\setminus SO(n)$.

We remark that it is impossible for f_t to be PC for any $|t|>1$: for such t we can find $\xi \in \mathbb{R}^{2\times 2}$ s.t. $f_t(\xi)<0$ and then for $s>0$,

$$f_t(s\xi) = f_t(\xi)\cdot s^4 < 0.$$

But a PC function on $\mathbb{R}^{2\times 2}$ must be bounded below by a quadratic polynomial:

Assume $G(\xi) = g(\xi, \det\xi)$, where $g : \mathbb{R}^5 \cong \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ is convex.

Since g is convex we can find $\xi_0 \in \mathbb{R}^{2\times 2}$, $t_0 \in \mathbb{R}$ s.t. $g(\xi,t) \geqslant g(0,0) + \langle \xi_0, \xi \rangle + t_0 t$

and hence $G(\xi) \geqslant \underbrace{g(0,0) + \langle \xi_0, \xi \rangle + t_0 \det \xi}_{\text{quadratic pol.}} \quad \forall \xi$

$\boxed{\text{Exercise}}$ Show that a subquadratic PC function $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ must be convex. [Subquadratic means here that

$$\lim_{|\xi| \to \infty} \frac{|F(\xi)|}{|\xi|^2} = 0 .]$$

Šverák (1991) gave examples of nonconvex RC functions on $\mathbb{R}^{2\times 2}$.

For the function f_t we have:

$$f_t \text{ RC} \iff |t| \leqq \frac{2}{\sqrt{3}} \qquad \left(\begin{array}{c}\text{Dacorogna \&} \\ \text{Marcellini}\end{array}\right)$$

$\exists\, \varepsilon \in \left(0, \frac{2}{\sqrt{3}} - 1\right]$ s.t. f_t is QC when $|t| \leqq 1 + \varepsilon$.
(Alibert & Dacorogna)

The precise value of ε is not known.

$\boxed{\text{In particular:} \quad \text{QC} \not\Rightarrow \text{PC} \quad \text{for } n, N \geqslant 2}$

$\boxed{\text{Morrey's conjecture (1952): RC} \not\Rightarrow \text{QC}}$

$\boxed{\text{Ex}}$ (Šverák 1992): RC $\not\Rightarrow$ QC when $N \geq 3$, $n \geq 2$

(whereas the question remains open for $N=2$, $n \geq 2$).

We focus on the case $N=3$, $n=2$. The general case can be deduce from this. Define

$$u(x,y) = \frac{1}{2\pi}\begin{bmatrix} \sin 2\pi x \\ \sin 2\pi y \\ \sin 2\pi(x+y) \end{bmatrix}$$

Then
$$Du(x,y) = \begin{bmatrix} \cos 2\pi x & 0 \\ 0 & \cos 2\pi y \\ \cos 2\pi(x+y) & \cos 2\pi(x+y) \end{bmatrix},$$

and
$$\int_{(-\frac{1}{2},\frac{1}{2})^2} Du(x,y)\,dx\,dy = 0 \quad \& \quad Du \in L,$$

where
$$L = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix} : r,s,t \in \mathbb{R} \right\}.$$

Observe that L is a subspace of $\mathbb{R}^{3\times 2}$ and that for $\xi, \eta \in L$ we have $\operatorname{rank}(\xi-\eta) = 1$ iff
$$\xi - \eta \in \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \cup \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle \cup \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

where $\langle \cdots \rangle$ denotes the linear span.

Define $f : L \to \mathbb{R}$ by $f\begin{pmatrix} r & 0 \\ 0 & s \\ t & t \end{pmatrix} := -rst$.

Then f is RC. We also calculate that

$$\int_{(-\frac{1}{2},\frac{1}{2})^2} f\big(Du(x,y)\big)\,dx\,dy = -\frac{1}{4} < 0 = f(0)$$

so f is not QC on L.

$\boxed{\text{Exercise}}$ Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous. Then F is QC iff

$$\int_{(-\frac{1}{2},\frac{1}{2})^n} F(\xi + D\varphi(x))\,dx \geq F(\xi)$$

for all $\varphi \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^N)$ that are $(-\frac{1}{2},\frac{1}{2})^n$ — periodic. ~~and~~

It is not known if f can be extended to a RC function on $\mathbb{R}^{3\times 2}$. Such an extension cannot be a polynomial of degree 3 :

$\boxed{\text{Exercise}}$ Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be a polynomial of degree 3. Show that F is RC iff F is QC.

Instead we let for $\varepsilon, k > 0$

$$F(\xi) = F_{\varepsilon,k}(\xi) := f(P\xi) + \varepsilon(|\xi|^2 + |\xi|^4) + k|P\xi - \xi|^2, \quad \xi \in \mathbb{R}^{3\times 2},$$

where $P : \mathbb{R}^{3\times 2} \to L$ is orthogonal projection.

$\boxed{\text{Lemma}}$ For each $\varepsilon > 0$ there exists $k = k_\varepsilon < \infty$ s.t. $F_{\varepsilon,k}$ is RC.

Pf is omitted.

When $\varepsilon > 0$ is small $F_{\varepsilon,k}$ is RC, but not QC.