Recall

- When $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is cont. and $|F(\xi)| \leq c(|\xi|^2+1)$ $\forall \xi$, then

$$\mathcal{F}[v] = \int_\Omega F(Dv), \quad v \in W^{1,2}_g(\Omega, \mathbb{R}^N)$$

is swlsc on $W^{1,2}_g$ iff F is QC.

- Furthermore we have checked that

$$C \Longrightarrow PC \Longrightarrow QC \Longrightarrow RC,$$

and that none of the reverse implications hold in general. However, when $n=1$ or $N=1$ then all the convexity notions reduce to normal convexity.

- The motivation for investigating swlsc came from the problem of existence of minimizers. Using the direct method we have so far established:

If $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is QC, $|F(\xi)| \leq c(|\xi|^2+1)$ $\forall \xi$ and $\exists\, \varepsilon > 0$ s.t. $F(\xi) \geq \varepsilon|\xi|^2 - \frac{1}{\varepsilon}$ $\forall \xi$, then $\mathcal{F}(v) = \int_\Omega F(Dv)$ admits a minimizer on $W^{1,2}_g(\Omega, \mathbb{R}^N)$.

We now return to the issue of <u>coercivity</u>. From $F(\xi) \geqslant \varepsilon|\xi|^2 - \frac{1}{\varepsilon}$ $\forall \xi$ follows

$$(*) \quad \int_\Omega F(Dv) \geqq \varepsilon \int_\Omega |Dv|^2 - c_\varepsilon \quad \forall v \in W^{1,2}_g$$

An inequality of this type is often called a <u>Gårding inequality</u>. When it holds we say that F is <u>mean-coercive</u> on $W^{1,2}_g$.

$\boxed{\text{TH}}$ (B. Yan 2004, C-Y Chen & K 2011)

Assume $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is continuous and $|F(\xi)| \leqq c(|\xi|^2+1)$ $\forall \xi$. Let $g \in W^{1,2}(\Omega, \mathbb{R}^N)$.

Then F is mean-coercive on $W^{1,2}_g$ iff

$\exists\, \varepsilon > 0$ s.t. $F - \varepsilon|\cdot|^2$ is <u>QC at some $\xi \in \mathbb{R}^{N\times n}$</u>.

<u>Remark:</u> When $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ is QC and $|F(\xi)| \leqq c(|\xi|^2+1)$ $\forall \xi$, $\exists\, \varepsilon > 0$ s.t. $F - \varepsilon|\cdot|^2$ is QC somewhere, then $\mathcal{F}(v) = \int_\Omega F(Dv)$ admits a minimizer on $W^{1,2}_g$.

If $F(\xi) = \det \xi + \varepsilon |\xi|^2$, $\xi \in \mathbb{R}^{2\times 2}$, then F is not coercive when $0 < \varepsilon < \frac{1}{2}$, but it is mean-coercive.

The above assumptions are not sufficient for partial regularity though:

when F is a null-Lagrangian (eg $F = \det$), then $v \mapsto \int_\Omega F(Dv)$ only depends on $v|_{\partial\Omega}$, and hence <u>any</u> $v \in W^{1,2}_g$ is F-minimizing. F is not mean-coercive but if $F(\xi) = \det \xi + \Phi(|\xi|)$, where $\Phi \in C^\infty(\mathbb{R})$ and $\Phi(t) = \begin{cases} 0 & \text{for } t < 117 \\ t^2 & \text{for } t > 120, \end{cases}$

then F is coercive. If $g \in W^{1,2}$ satisfies $|Dg| < 117$ a.e., then g is F-minimizing on $W^{1,2}_g$. Of course g need not be partially regular.

We must exclude '=' in QC and demand a strict version.

Def. Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be continuous. Then F is <u>strongly quasiconvex</u> iff $\exists\, \varepsilon > 0$ s.t. $F - \varepsilon|\cdot|^2$ is QC, that is,

$$\fint_{B(0,1)} \Big(F(\xi + D\varphi(x)) - F(\xi) \Big)\, dx \geq \varepsilon \fint_{B(0,1)} |D\varphi(x)|^2 dx$$

for all $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W_0^{1,\infty}(B(0,1), \mathbb{R}^N)$.

<u>Natural conditions for partial regularity:</u>

$$\begin{cases} \bullet \quad F : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R} \quad \text{is } C^2 \\ \bullet \quad |F(\xi)| \leq c(|\xi|^2 + 1) \quad \forall \xi \\ \bullet \quad F \text{ strongly QC} \end{cases}$$

We aim to prove:

TH (Evans 1986, Acerbi & Fusco 1986)

Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be C^2, of quadratic growth and strongly QC. Then any F-minimizer is partially regular. More precisely:

Assume $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ is F-minimizing, and let

$$\Sigma_u := \left\{ x \in \Omega : \begin{array}{c} \overline{\lim}_{r \searrow 0} |(Du)_{x,r}| = \infty \\ \text{or} \\ \lim_{r \searrow 0} E(x,r) > 0 \end{array} \right\},$$

where $E(x,r) := \fint_{B(x,r)} |Du - (Du)_{x,r}|^2$.

Then Σ_u is relatively closed in Ω, and (the precise representative of) u is $C^{1,\alpha}_{loc}$ on $\Omega \setminus \Sigma_u$.

Hölder continuity

- $f : \Omega \to \mathbb{R}^k$ is Hölder continuous of order $\gamma \in (0,1]$ (briefly: γ-Hölder, or, $f \in C^{0,\gamma}(\Omega, \mathbb{R}^k)$) iff $\exists\, c < \infty$ s.t.

$$|f(x) - f(y)| \leq c|x-y|^\gamma \quad \forall x, y \in \Omega.$$

- $f : \Omega \to \mathbb{R}^k$ is locally γ-Hölder (or, $f \in C^{0,\gamma}_{loc}(\Omega, \mathbb{R}^k)$) iff $\forall x \in \Omega\ \exists r > 0$ s.t. $f|_{\Omega \cap B(x,r)}$ is γ-Hölder.

Precise representative

Let $f \in L^1_{loc}(\Omega, \mathbb{R}^k)$.

$x \in \Omega$ is a <u>good point</u> for f iff

$$\lim_{r \searrow 0} f_{x,r} \quad \text{exists in } \mathbb{R}^k.$$

It follows in particular from Lebesgue's differentiation theorem that $\mathcal{L}^n$-a.e. $x \in \Omega$ is good for f. Note that the notion doesn't depend on the particular representative of f.

The precise representative is the map $\bar{f} : \Omega \to \mathbb{R}^k$ defined everywhere by

$$\bar{f}(x) := \begin{cases} \lim_{r \searrow 0} f_{x,r}, & x \text{ good for } f \\ 0, & \text{otherwise.} \end{cases}$$

We usually skip the bar and write simply f also for the precise representative of f.

$\boxed{\text{TH 9}}$ (Campanato 1963, Meyers 1964)

Let $f \in L^p(\Omega, \mathbb{R}^k)$, $1 \leq p < \infty$, $0 < \alpha \leq 1$.

Assume $\exists\, m < \infty$ s.t.

$$\fint_{B(x,r)} |f - f_{x,r}|^p \leq m^p r^{\alpha p}$$

for all $B(x,r) \subset \Omega$. Then (the precise representative) $f \in C^{0,\alpha}_{loc}(\Omega, \mathbb{R}^k)$ and $\exists\ c = c(n, \alpha, p) < \infty$ s.t.

$$|f(x) - f(y)| \leqq c\,m\,|x-y|^{\alpha}$$

for $x, y \in B(x_0, r)$ when $B(x_0, 4r) \subset \Omega$.

Pf — See Giusti's book 'Direct methods in the Calculus of Variations'. $\square$

Alibert & Dacorogna (1992):

There exists $\varepsilon > 0$ s.t.

$$f_t(\xi) = |\xi|^4 + 2t|\xi|^2 \det \xi, \quad \xi \in \mathbb{R}^{2\times 2}$$

is QC for all $|t| \leq 1+\varepsilon$.

$\boxed{\text{Recall}}$

$$\left.\begin{array}{lll} f_t & PC \iff & |t| \leq 1 \\ f_t & RC \iff & |t| \leq \frac{2}{\sqrt{3}} \end{array}\right\} \text{Dacorogna \& Marcellini 1988}$$

Construction of nontrivial QC functions

(Iwaniec & K, 2005)

Assume $R : \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^3 and pos. p-homogeneous: (i) $R(t\xi) = t^p R(\xi)$ $\forall \xi \in \mathbb{R}^{N\times n}$, $t \geq 0$

(so $p > 3$)

and $\exists\, \delta > 0$ s.t. $R''(\xi)[a\otimes b, a\otimes b] \geq \delta |\xi|^{p-2} |a|^2 |b|^2$

Assume P is pos. p-homogeneous and strongly p-QC: (ii) $\exists\, \varepsilon > 0$ s.t.

$$\int_{\mathbb{R}^n} \Big(P(\xi + D\varphi(x)) - P(\xi) \Big)\, dx \geq \varepsilon \int_{\mathbb{R}^n} |D\varphi|^p$$

for all $\xi \in \mathbb{R}^{N\times n}$, $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^N)$.

Then $\exists\, t_0 < \infty$ s.t. $R + tP$ is QC for all $t \geq t_0$. Note: $t_0 = \max\left(1, 2^{(p-1)(p-2)}\right) \dfrac{\delta^{3-p}}{2\varepsilon} \sup_{|\xi| \leq 1} \|R'''(\xi)\|^{p-2}$ will do.

Pf: Fix $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W_0^{1,\infty}(B(0,1),\mathbb{R}^N)$.

By Taylor expansion:

$$R(\xi + D\varphi) - R(\xi) = R'(\xi)[D\varphi] + \tfrac{1}{2}R''(\xi)[D\varphi, D\varphi] + I$$

where $I = \int_0^1 (1-s)\big(R''(\xi + sD\varphi) - R''(\xi)\big)[D\varphi, D\varphi]\,ds.$

Using that $R \in C^3$ is p-homogeneous we get:

$$\| R''(\xi+\eta) - R''(\xi)\| \leqq \delta|\xi|^{p-2} + 2t_0\varepsilon|\eta|^{p-2}$$

for all $\xi, \eta \in \mathbb{R}^{N\times n}$.

Thus: $\| R''(\xi + sD\varphi) - R''(\xi)\| \leqq \delta|\xi|^{p-2} + 2t_0\varepsilon|D\varphi|^{p-2}$

hence $I \geqslant -\tfrac{1}{2}\delta|\xi|^{p-2}|D\varphi|^2 - t_0\varepsilon|D\varphi|^p$ a.e.

Now

$$\int_{B(0,1)} \big(R(\xi + D\varphi(x)) - R(\xi)\big)\,dx = \int_{B(0,1)} \Big(\underbrace{\cancel{R'(\xi)[D\varphi(x)]}}_{0} + \tfrac{1}{2}R''(\xi)[D\varphi(x), D\varphi(x)] + I\Big)\,dx \geqslant \int_{B(0,1)} \Big(\tfrac{1}{2}R''(\xi)[D\varphi, D\varphi] - \tfrac{1}{2}\delta|\xi|^{p-2}|D\varphi|^2 - \varepsilon t_0|D\varphi|^p\Big)\,dx.$$

Recall $R''(\xi)[a\otimes b, a\otimes b] \geqslant \delta|\xi|^{p-2}|a|^2|b|^2$

and so by use of the Fourier transform

$$\int_{B(0,1)} \tfrac{1}{2}R''(\xi)[D\varphi, D\varphi]\,dx \geqslant \frac{\delta}{2}\int_{B(0,1)} |\xi|^{p-2}|D\varphi|^2\,dx,$$

ie,

$$\int_{B(0,1)} \big(R(\xi+D\varphi)-R(\xi)\big)\,dx \geqslant$$

$$\int_{B(0,1)} \left(\frac{\delta}{2}|\xi|^{p-2}|D\varphi|^2 - \frac{\delta}{2}|\xi|^{p-2}|D\varphi|^2 - \varepsilon t_0|D\varphi|^p\right)dx$$

$$= -\varepsilon t_0\int_{B(0,1)} |D\varphi|^p\,dx.$$

Note that $\displaystyle\int_{B(0,1)} \big((R+tP)(\xi+D\varphi)-(R+tP)(\xi)\big)dx$

$$\geqslant \varepsilon(t-t_0)\int_{B(0,1)} |D\varphi|^p\,dx. \quad \square$$

Conformal & anticonformal parts

For $\xi \in \mathbb{R}^{2\times 2}$ define

$$\xi^+ := \tfrac{1}{2}(\xi + \operatorname{cof}\xi), \qquad \xi^- := \tfrac{1}{2}(\xi - \operatorname{cof}\xi)$$

where $\operatorname{cof}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ is the cofactor matrix.

Check that $\langle \xi^+, \xi^- \rangle = 0$ and that $(\mathbb{R}^{2\times 2})^{+/-} = \{ \xi^{+/-} : \xi \in \mathbb{R}^{2\times 2} \}$ is a 2-dimensional subspace. We have

$$\mathbb{R}^{2\times 2} = (\mathbb{R}^{2\times 2})^+ \oplus (\mathbb{R}^{2\times 2})^-.$$

Check that

- $|\xi|^2 = |\xi^+|^2 + |\xi^-|^2 \qquad \xi \in \mathbb{R}^{2\times 2}$
- $2\det\xi = |\xi^+|^2 - |\xi^-|$

$$f_t(\xi) = |\xi|^2 (|\xi|^2 + 2t\det\xi)$$
$$= |\xi|^2 \left((1+t)|\xi^+|^2 + (1+t)|\xi^-|^2 \right)$$

f_t PC iff $|t| \leq 1$

f_t RC iff $|t| \leq \frac{2}{\sqrt{3}}$

For $0 < t < \frac{2}{\sqrt{3}}$ we can write

$$f_t(\xi) = \left(1 - t\frac{\sqrt{3}}{2}\right)|\xi|^4 + \underbrace{f_{\frac{2}{\sqrt{3}}}(\xi)}_{RC}\, t\frac{\sqrt{3}}{2}$$

so this f_t satisfies (i).

Fix $1 < t < \frac{2}{\sqrt{3}}$.

Note $f_1(\xi) = 2|\xi|^2|\xi - 1|^2$ and

$$f_t(\xi) + \gamma f_1(\xi) = (1+\gamma)|\xi|^2\left(|\xi|^2 - 2\frac{t+\gamma}{1+\gamma}\det\xi\right)$$

If f_1 is strongly p-QC, then $\exists\, \gamma_0 < \infty$ s.t. $f_t - \gamma f_1$ is QC for $\gamma \geq \gamma_0$.

Note $\frac{t+\gamma}{1+\gamma} > 1$ so we have the assertion provided f_1 is strongly p-QC.

$\boxed{\text{Lemma}}$ $Q(\xi) = f_1(\xi) - \frac{1}{4}|\xi - 1|^4$ is PC (hence QC).

Beurling-Ahlfors' inequality

For $1<p<\infty$ there exists $c_p<\infty$ s.t.

$$\|(D\varphi)^+\|_{L^p} \leq c_p \|(D\varphi)^-\|_{L^p}$$

$$\|(D\varphi)^-\|_{L^p} \leq c_p \|(D\varphi)^+\|_{L^p} \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^2,\mathbb{R}^2)$$

<u>Remark:</u> If we identify $\mathbb{R}^2 \cong \mathbb{C}$, then $(D\varphi)^+$ corresponds to $\partial\varphi/\partial z$ and $(D\varphi)^-$ to $\partial\varphi/\partial \bar{z}$.

$\boxed{\text{Lemma}}$ $P_{+/-}(\xi) := |\xi^{+/-}|^p$ is strongly p-QC.

$\boxed{\text{Pf}}$ We have for $p \geq 2$:

$$|X|^p - |Y|^p \geq p|Y|^{p-2}\langle Y, X-Y\rangle + 2^{2-p}|X-Y|^p$$

In particular:

$$\int_{\mathbb{R}^n} \left(P_{+/-}(\xi + D\varphi) - P_{+/-}(\xi) \right) \geq p|\xi^{+/-}|^{p-2}\langle \xi^{+/-}, \int_{\mathbb{R}^n} D\varphi \rangle$$

$$+ 2^{2-p}\int_{\mathbb{R}^n} |D\varphi^{+/-}|^p$$

$$= 2^{2-p}\|D\varphi^{+/-}\|_{L^p}^p$$

and so by the Beurling-Ahlfors ineq we conclude with $\varepsilon = 2^{2-p}c_p^p$. $\square$