Recall from L4 that our aim is to prove

$\boxed{\text{TH}}$ (L.C. Evans 1986, E. Acerbi & N. Fusco 1986)

Let the integrand $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfy

(H1) F is C^2 (H2) $|F(\xi)| \leq L(|\xi|^2+1)$ $\forall \xi$

(H3) $F - \ell|\cdot|^2$ is QC,

where $0 < \ell, L < \infty$ are constants. Assume $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ and that $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ is F-minimizing. Then the set

$$\Sigma_u := \left\{ x \in \Omega : \overline{\lim_{r \searrow 0}} |(Du)_{x,r}| < \infty \text{ or } \underline{\lim_{r \searrow 0}} E(x,r) > 0 \right\}$$

is relatively closed in Ω, and (the precise representative of) u is $C^{1,\alpha}_{loc}$ on $\Omega \setminus \Sigma_u$ for all $\alpha < 1$. $\left(E(x,r) := \fint_{B(x,r)} |Du - (Du)_{x,r}|^2 \right)$

Remark The set Σ_u is called the singular set for u, and it follows from Lebesgue's differentiation theorem that it is $\mathcal{L}^n$-negligible

---

The method of proof can be traced back (at least) to Almgren's and De Giorgi's works on minimal surfaces. It is sometimes called a linearization strategy — we obtain the desired regularity by linearizing the problem. More precisely, let $B(x,r) \subset \Omega$ and let P denote the 2^{nd} Taylor polynomial for the integrand F about $\xi_0 = (Du)_{x,r}$. Then as $F \in C^2$, $F(\xi) = P(\xi) + o(|\xi - \xi_0|^2)$. Let $h \in W^{1,2}_u(B(x,r), \mathbb{R}^N)$ be P-minimizing. Then h is C^∞ smooth and we have good estimates that express the regularity of h (by the generalized Weyl Lemma proved below). Note that the Euler–Lagrange equation for P is $\operatorname{div} P'(Dv) = 0$ in $B(x,r)$ — a linear elliptic system with constant coefficients. We then try to transfer the estimates from h to u: if $E(x,r)$ is small then we expect that h is a good approximation of u in $W^{1,2}$. The main difficulty is to ensure suitable compactness in $W^{1,2}$.

The conditions H1–3 were natural: they were strict form (=(H3)) of what we required for proving existence of minimizers by the direct method. In particular, when H1–3 hold an F-minimizer $u \in W^{1,2}_g$ exists for any choice of $g \in W^{1,2}$:

$$\mathcal{F}(u) \leq \mathcal{F}(v) \quad \forall v \in W^{1,2}_g .$$

It follows that u is also an <u>F-extremal</u> (= a weak solution to the <u>Euler–Lagrange equation</u> for $\mathcal{F}$): $\operatorname{div} F'(Du)$ in Ω, the divergence is understood row-wise and in the distributional sense on Ω, ie,

$$\text{(EL)} \qquad \int_\Omega F'(Du)[D\varphi]\,dx = 0 \qquad \forall \varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N).$$

We justify this by use of

Lemma Assume that $G : \mathbb{R}^{N\times n} \to \mathbb{R}$ is C^1 and RC. Then $|G'(\xi)| \leq \sqrt{\min(n,N)}\,\dfrac{\operatorname{osc}(G, B(0,2R))}{R}$ for all $|\xi| < R$. In particular, if $|G(\xi)| \leq L(|\xi|^2+1)$, $\forall \xi$, then $|G'(\xi)| \leq cL(|\xi|+1)$, $\forall \xi$, where $c = c(\min(n,N))$ is a constant.

$\boxed{\text{Pf.}}$ Fix $|\xi| < R$ and $\eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\eta) = 1 = |\eta|$. Then by RC we get for $t > 0$: when $t < \min(t_-, t_+)$, then

$$-\frac{\operatorname{osc}(G, B_{2R})}{R} \leq \frac{G(\xi - t_-\eta) - G(\xi)}{-t_-} \leq \frac{G(\xi - t\eta) - G(\xi)}{-t}$$

$$\leq G'(\xi)[\eta] \leq \frac{G(\xi + t\eta) - G(\xi)}{t} \leq \frac{G(\xi + t_+\eta) - G(\xi)}{t^+}$$

$$\leq \frac{\operatorname{osc}(G, B_{2R})}{R},$$

where $t_-, t_+ > 0$ are chosen so $\xi - t_-\eta$, $\xi + t_+\eta \in \partial B_{2R}$. This yields

$|G'(\xi)[\eta]| \leq \dfrac{\operatorname{osc}(G, B_{2R})}{R}$ and hence

$|G'(\xi)| \leq \sqrt{\min(n,N)}\, \dfrac{\operatorname{osc}(G, B_{2R})}{R}$.

Finally, $|G'(\xi)| \leq \sqrt{\min(n,N)}\, \dfrac{\operatorname{osc}(G, B_{2|\xi|})}{|\xi|}$

$\leq \sqrt{\min(n,N)}\, \dfrac{2L(4|\xi|^2+1)}{|\xi|}$, and hence easily the desired conclusion. $\square$

Consequently we can justify the calculation

$$0 = \frac{d}{dt}\Big|_{t=0} \mathcal{F}(u + t\varphi) = \int_\Omega F'(Du)[D\varphi]\, dx$$

for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Note that H1–3 do not give any global control on F''.

While H1–3 give existence (and partial regularity) of minimizers they <u>do not</u> give any uniqueness (see E. Spadaro 2008).

The functional $\mathcal{F}$ is <u>not</u> convex on $W^{1,2}_g$ in general, even far from! There can be a huge difference between F-extremals and F-minimizers: F-minimizer $\Rightarrow$ F-extremal, but $\Leftarrow$ is false. In fact, there is <u>no partial regularity theory</u> for F-extremals when F is merely strongly QC: There exists a strongly QC integrand $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ of class C^∞ and with $\sup_\xi \|F''(\xi)\| < \infty$, s.t. the corresponding Euler–Lagrange equation

$$\operatorname{div} F'(Du) = 0 \quad \text{in } \Omega$$

admits weak solutions that are Lipschitz but nondifferentiable in a dense set of points. (See S. Müller & V. Šverák 2003 and also L. Szekelyhidi 2004 for PC case.)

The 'problem' with QC in this regard is that the derivative F' of a QC function F need not have any useful monotonicity properties.

A notion of <u>quasimonotonicity</u> has been introduced and considered by K. Zhang, R. Landes and M. Fuchs. However, as mentioned above, F QC $\not\Rightarrow$ F' quasimonotone.

The situation is reminiscent of what happens in the case of harmonic mappings into manifolds.

Let $F: \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfy conditions H1-3.

If $\xi_0 \in \mathbb{R}^{N\times n}$, $\varepsilon > 0$ and $\varphi \in W_0^{1,\infty}(B(0,1), \mathbb{R}^N)$, then by (H3):

$$\varepsilon^2 \ell \int_{B(0,1)} |D\varphi|^2 \leq \int_{B(0,1)} \left(F(\xi_0 + \varepsilon D\varphi) - F(\xi_0) \right)$$

$$= \int_{B(0,1)} \left[\cancel{F'(\xi_0)[\varepsilon D\varphi]} + \varepsilon^2 \int_0^1 (1-t) F''(\xi_0 + \varepsilon t D\varphi)[D\varphi, D\varphi] \right]$$

$$= \varepsilon^2 \int_{B(0,1)} \int_0^1 (1-t)\, F''(\xi_0 + \varepsilon t D\varphi)[D\varphi, D\varphi]\, dt ,$$

hence

$$\ell \int_{B(0,1)} |D\varphi|^2 \leq \int_{B(0,1)} \int_0^1 (1-t) F''(\xi_0 + \varepsilon t D\varphi)[D\varphi, D\varphi]\, dt$$

for all $\varepsilon > 0$. Taking $\varepsilon \searrow 0$ we get

$$\ell \int_{B(0,1)} |D\varphi|^2 \leq \frac{1}{2} \int_{B(0,1)} F''(\xi_0)[D\varphi, D\varphi] \,,$$

hence the quadratic form $\eta \mapsto F''(\xi_0)[\eta,\eta] - 2\ell|\eta|^2$ is QC (at 0 and therefore everywhere).

With this in mind we shall consider a <u>symmetric bilinear form</u> $\mathbb{A} : \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying for some constants $0 < \ell \leq L < \infty$ the conditions:

(1) $\mathbb{A}[\xi,\eta] \leq L|\xi|\,|\eta| \qquad \forall \xi, \eta$

(2) $\xi \mapsto \mathbb{A}[\xi,\xi] - \ell|\xi|^2$ is QC

$\boxed{\text{Def.}}$ A mapping $v \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ is called <u>$\mathbb{A}$-harmonic</u> iff

$$\int_\Omega \mathbb{A}[Dv, D\varphi]\, dx = 0$$

for all $\varphi \in W^{1,2}(\Omega, \mathbb{R}^N)$ <u>with compact support</u>.

Remark The condition (2) is equivalent to:
$$\mathbb{A}[a\otimes b, a\otimes b] \geqslant \ell |a|^2 |b|^2, \ \forall a \in \mathbb{R}^N, b \in \mathbb{R}^n.$$

As we have seen this does <u>not</u> mean that $\xi \mapsto \mathbb{A}[\xi,\xi]$ is convex, but only RC. However, the integrand is <u>functionally convex</u> in the sense that for each $g \in W^{1,2}(\Omega,\mathbb{R}^N)$ the functional
$$v \longmapsto \int_\Omega \mathbb{A}[Dv,Dv]$$
is convex on $W^{1,2}_g(\Omega,\mathbb{R}^N)$.

Pf: We show more, namely that
$$\mathcal{A}(v) := \int_\Omega \Big(\mathbb{A}[Dv,Dv] - \ell |Dv|^2\Big)\,dx$$
is a convex functional on $W^{1,2}_g$.

Fix $v \in W^{1,2}_g$ and $\varphi \in W^{1,2}_0$. We must show that $\mathbb{R} \ni t \mapsto \mathcal{A}(v+t\varphi)$ is convex.

This is so since
$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}(v+t\varphi) = 2\int_\Omega \Big(\mathbb{A}[D\varphi,D\varphi] - \ell|D\varphi|^2\Big) \overset{(2)}{\geqslant} 0. \ \square$$

Consequently, given $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ there exists a <u>unique</u> $\mathbb{A}$-harmonic map $v \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ (Exercise: check it.)

$\boxed{\text{TH}}$ (Generalized Weyl Lemma)

Let $\mathbb{A}$ be as above, and assume that $v \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ is $\mathbb{A}$-harmonic.
Then v is C^∞, and for $B_{3R} = B(x_0, 3R) \Subset \Omega$ we have

$$\sup_{B_R} |D^2 v|^2 \leq \frac{c}{R^2} \fint_{B_{3R}} |Dv - (Dv)_{x_0, 3R}|^2,$$

$$\fint_{B_r} |Dv - (Dv)_{x_0, r}|^2 \leq c \left(\frac{r}{R}\right)^2 \fint_{B_{3R}} |Dv - (Dv)_{x_0, 3R}|^2$$

for all $0 < r \leq 3R$, where $c = c(n, \frac{l}{L})$ is a constant.

$\boxed{\text{Pf:}}$ We use the difference-quotient method and the Sobolev embedding theorem.

Let $0 < \delta \leq 1$ and take $\rho \in W_0^{1,\infty}(B_{(1+\delta)R})$ satisfying $0 \leq \rho \leq 1$. For $1 \leq s \leq n$, $|h| \leq \delta R$ and an affine map $a : \mathbb{R}^n \to \mathbb{R}^N$ put $\varphi := \Delta_{s,-h}(\rho^2 \Delta_{s,h}(v-a))$. Observe that $\varphi \in W_0^{1,\infty}(B_{3R}, \mathbb{R}^N)$, and therefore

$$0 = \int_\Omega \mathbb{A}[Dv, D\varphi] = \int_\Omega \mathbb{A}[\Delta_{s,h} Dv, \rho^2 \Delta_{s,h} Dv + \Delta_{s,h}(v-a) \otimes D(\rho^2)]$$

$$=: \mathrm{I} + \mathrm{II}.$$

$$\mathrm{I} = \int \mathbb{A}[\rho \Delta_{s,h} Dv, \rho \Delta_{s,h} Dv] =$$

$$\int \Big\{ \mathbb{A}[D(\rho \Delta_{s,h} \cancel{D}(v-a)), D(\rho \Delta_{s,h}(v-a))]$$

$$-2\mathbb{A}[\rho \Delta_{s,h} Dv, \Delta_{s,h}(v-a) \otimes D\rho]$$

$$- \mathbb{A}[\Delta_{s,h}(v-a) \otimes D\rho, \Delta_{s,h}(v-a) \otimes D\rho] \Big\}$$

$$\overset{(1),(2)}{\geqslant} \int \Big\{ \ell |D(\rho \Delta_{s,h}(v-a))|^2$$

$$- 2L\rho |\Delta_{s,h} Dv| |\Delta_{s,h}(v-a)| |D\rho| - L |\Delta_{s,h}(v-a)|^2 |D\rho|^2 \Big\}$$

$$\geqslant \int \Big\{ \ell \rho^2 |\Delta_{s,h} Dv|^2 - 2(\ell + L)\rho |\Delta_{s,h} Dv| |\Delta_{s,h}(v-a)| |D\rho|$$

$$-(\ell + L) |\Delta_{s,h}(v-a)|^2 |D\rho|^2 \Big\} \overset{\text{Cauchy–Schwarz}}{\geqslant}$$

$$\ell \int \rho^2 |\Delta_{s,h} Dv|^2 - 2(\ell+L) \| \rho \Delta_{s,h} Dv \|_2 \cdot \| |\Delta_{s,h}(v-a)| |D\rho| \|_2$$
$$-(\ell+L) \int |\Delta_{s,h}(v-a)|^2 |D\rho|^2$$

$$\geqslant \frac{2\ell}{3}\int \rho^2 |\Delta_{s,h} Dv|^2 - \left(L+\ell+\frac{2}{\ell}(L+\ell)^2\right)\int |\Delta_{s,h}(v-a)|^2 |D\rho|^2 .$$

$$\mathrm{II} = \int A\left[\Delta_{s,h} Dv, \Delta_{s,h}(v-a)\otimes D(\rho^2)\right] =$$

$$\int 2A\left[\rho\Delta_{s,h} Dv, \Delta_{s,h}(v-a)\otimes D\rho\right] \overset{(1)}{\geqslant}$$

$$-2L\int \rho|\Delta_{s,h} Dv|\cdot|\Delta_{s,h}(v-a)|\cdot|D\rho| \geqq$$

$$-\frac{\ell}{3}\int \rho^2|\Delta_{s,h} Dv|^2 - \frac{3}{\ell}L^2\int |\Delta_{s,h}(v-a)|^2|D\rho|^2 .$$

Consequently,

$$0 \geqslant \frac{\ell}{3}\int \rho^2|\Delta_{s,h} Dv|^2 - \frac{3}{\ell}L^2\int |\Delta_{s,h}(v-a)|^2|D\rho|^2$$

$$-\left(L+\ell+\frac{2}{\ell}(L+\ell)^2\right)\int |\Delta_{s,h}(v-a)|^2|D\rho|^2 ,$$

or

$$\int \rho^2|\Delta_{s,h} Dv|^2 \leqq \left(9\left(\frac{L}{\ell}\right)^2 + 3\frac{L}{\ell} + 1 + 9\left(\frac{L}{\ell}+1\right)^2\right)\int |\Delta_{s,h}(v-a)|^2|D\rho|^2$$

and hence

$$\boxed{*} \quad \int_\Omega \rho^2|\Delta_{s,h} Dv|^2 \leqq c\int_\Omega |\Delta_{s,h}(v-a)|^2|D\rho|^2$$

where $c = c(\frac{L}{\ell}) \geqslant 1$.

We now take ρ s.t. $\mathbb{1}_{B_R} \leqq \rho \leqq \mathbb{1}_{B_{(1+\delta)R}}$ and $|D\rho| \leqq \frac{1}{\delta R}$ a.e., whereby

$$\int_{B_R} |\Delta_{s,h} Dv|^2 \leq \frac{c}{\delta^2 R^2} \int_{B_{(1+\delta)R}} |\Delta_{s,h}(v-a)|^2 \leq$$

$$\frac{c}{\delta^2 R^2} \int_{B_{(1+2\delta)R}} |D_s(v-a)|^2 \, h^2 .$$

Take a s.t. $D_s a = (D_s v)_{x_0,(1+2\delta)R}$. Hereby

$$\int_{B_R} |\Delta_{s,h} Dv|^2 \leq \frac{c}{\delta^2 R^2} \int_{B_{(1+2\delta)R}} |D_s v - (D_s v)_{x_0,(1+2\delta)R}|^2$$

for all $|h| \leq \delta R$ and $1 \leq s \leq n$, hence

$$\boxed{*} \int_{B_R} |D_s Dv|^2 \leq \frac{c}{\delta^2 R^2} \int_{B_{(1+2\delta)R}} |D_s v - (D_s v)_{x_0,(1+2\delta)R}|^2$$

for all $1 \leq s \leq n$, and so summing over s we get

$$\boxed{**} \int_{B_R} |D^2 v|^2 \leq \frac{c}{\delta^2 R^2} \int_{B_{(1+2\delta)R}} |Dv - (Dv)_{x_0,(1+2\delta)R}|^2 .$$

As $B_{3R} \Subset \Omega$ was arbitrary we conclude that $v \in W^{2,2}_{loc}$. Taking $\phi = D_j \varphi$, $\varphi \in C_c^\infty$, in

$$0 = \int_\Omega A[Dv, D\phi]$$

and integrating by parts we get

$$0 = \int_\Omega A[D(D_j v), D\varphi] .$$

$\therefore$ $D_j v \in W^{1,2}_{loc}$ is A-harmonic in Ω and so $\boxed{*}$, $\boxed{**}$ apply to $D_j v$, hence

$D_j v \in W^{2,2}_{loc}$. Since this holds for each $1 \leq j \leq n$, $v \in W^{3,2}_{loc}$. By induction on $k \in \mathbb{N}$ we get $v \in W^{k,2}_{loc}$ and taking appropriate $0 < \delta < 1$ above also

$$\text{⊕} \quad \int_{B_R} |D^k v|^2 \leq \frac{c_k}{R^{2k-2}} \int_{B_{3R}} |Dv - (Dv)_{x_0,3R}|^2$$

where the constant $c_k = c_k(\delta)$ and $\delta = \delta(k)$.

Now if $2(k-2) > n$, ie, $k > 2 + \frac{n}{2}$, then by Sobolev embedding $W^{k,2}(B_R) \hookrightarrow W^{2,\infty}(B_R)$ and so by an interpolation inequality (or by using ⊕ for different values of k) to simplify the $W^{k,2}$-norm we get

$$\begin{aligned} \sup_{B_R} |D^2 v|^2 &\leq \tilde{c} R^{2(k-2)} \fint_{B_R} |D^k v|^2 + \tilde{c} \fint_{B_R} |D^2 v|^2 \\ &\leq \frac{\tilde{c}_k}{R^2} \fint_{B_{3R}} |Dv - (Dv)_{x_0,3R}|^2 . \end{aligned}$$

Finally the second inequality follows from the first: when $0 < r \leq R$ this is clear. When $R < r \leq 3R$, then

$$\int_{B_r} |Dv - (Dv)_{x_0,r}|^2 \leq \int_{B_r} |Dv - (Dv)_{x_0,3R}|^2 \leq$$

$$\left(\frac{r}{R}\right)^{n+2} \int_{B_{3R}} |Dv - (Dv)_{x_0,3R}|^2 . \quad \square$$

$\boxed{\text{TH}}$ (A-harmonic approximation lemma)

(De Giorgi, Simon ; Duzaar & Steffen 2002)

Let $0 < \ell \leq L < \infty$, $n, N \in \mathbb{N}$ and $\varepsilon > 0$.

$\exists\, \delta = \delta\left(\frac{L}{\ell}, n, N, \varepsilon\right) > 0$ with the following property: If $A : \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is symmetric and bilinear, and $u \in W^{1,2}(B(x,R), \mathbb{R}^N)$ satisfy:

(i) $\quad A[\xi, \eta] \leq L|\xi||\eta| \qquad \forall \xi, \eta$

(ii) $\quad \ell \int_{B(0,1)} |D\varphi|^2 \leq \int_{B(0,1)} A[D\varphi, D\varphi] \qquad \forall \varphi \in W_0^{1,\infty}$

(iii) $\quad \fint_{B(x,R)} A[Du, D\varphi] \leq \delta \|D\varphi\|_{L^\infty} \sqrt{E(x,R)}$

for all $\varphi \in W_0^{1,\infty}(B(x,R), \mathbb{R}^N)$, where

$E(x,R) := \fint_{B(x,R)} |Du - (Du)_{x,R}|^2$, then

$$+ R^{-2} \fint_{B(x,R)} |u-h|^2 \leq \varepsilon E(x,R),$$

where $h \in W_u^{1,2}(B(x,R), \mathbb{R}^N)$ is A-harmonic.

Pf: The proof is an indirect compactness argument (as in Duzaar & Steffen 2002).
WLOG we can assume $x=0$, $R=1$ and $E(x,R)=1$.
Write $B=B(0,1)$. Suppose the claim is false.
Then $\exists\, \varepsilon>0$ s.t. for each $\delta=\frac{1}{j}$ we can find $\mathbb{A}_j$, u_j satisfying (i)–(iii), but where

$$\circledast \quad \fint_B |u_j-h_j|^2 > \varepsilon$$

for $\mathbb{A}_j$–harmonic $h_j \in W^{1,2}_{u_j}(B,\mathbb{R}^N)$.
Subtracting affine maps we can assume that $(u_j)_B = 0$, $(Du_j)_B = 0$. Hence (u_j) is bounded in $W^{1,2}$ by Poincaré, and so for a subsequence (not relabelled) $u_j \rightharpoonup u$ in $W^{1,2}$ where $u \in W^{1,2}$. By swlsc of norms $\int_B |Du|^2 \leq \underline{\lim} \int_B |Du_j|^2 = 1$. For a further subsequence (again not relabelled) $\mathbb{A}_j \to \mathbb{A}$ in operator norms (as bilinear maps). It is easy to check that also $\mathbb{A}$ satisfies (i), (ii). For $\varphi \in C^1_c(B,\mathbb{R}^N)$:

$$\left|\int_B \mathbb{A}[Du, D\varphi]\right| \leq \left|\int_B \mathbb{A}[Du-Du_j, D\varphi]\right| + \left|\int_B (\mathbb{A}-\mathbb{A}_j)[Du_j, D\varphi]\right| + \left|\int_B \mathbb{A}_j[Du_j, D\varphi]\right| \xrightarrow[j\to\infty]{} 0$$

so u is $\mathbb{A}$-harmonic. Now

$$\ell \int_B |D(u_j - h_j)|^2 \overset{(ii)}{\leq} \int_B \mathbb{A}[D(u_j - h_j), D(u_j - h_j)]$$

$$= \int_B \mathbb{A}[Du_j, D(u_j - h_j)] \overset{(i)}{\leq} L \int_B |Du_j||D(u_j - h_j)|,$$

so by Cauchy-Schwarz:

$$\int_B |Du_j - Dh_j|^2 \leq \left(\frac{L}{\ell}\right)^2 \int_B |Du_j|^2 = \left(\frac{L}{\ell}\right)^2 |B|.$$

$\therefore$ (Dh_j) is bounded in L^2 and so as $h_j = u_j$ on ∂B, (h_j) is bounded in $W^{1,2}$ too.

For a further subsequence (again not relabelled) $h_j \rightharpoonup h$ in $W^{1,2}$, where $h \in W^{1,2}$. By Mazur's Lemma, $h \in W^{1,2}_u$, and as it is easy to check that h is also $\mathbb{A}$-harmonic it follows that $u = h$ by uniqueness of $\mathbb{A}$-harmonic maps. Finally we get the desired contradiction from Rellich's compactness theorem:

$$u_j - h_j = (u_j - u) + (h - h_j) \to 0 \text{ strongly in } L^2$$

which is impossible by $(*)$. $\square$

so u is A-harmonic. Now

$$\int_\Omega |D(u_j - h_j)|^2 \overset{(ii)}{\leq} \int_\Omega A\big(D(u_j - h_j), D(u_j - h_j)\big)$$

$$= \int_\Omega A[Du_j, D(u_j - h_j)] \overset{(iii)}{\leq} \frac{1}{j}\int_\Omega |Du_j|\,|D(u_j - h_j)|$$

by Cauchy-Schwarz:

$$\int_\Omega |Du_j - Dh_j|^2 \leq \left(\frac{1}{\lambda}\right)^2 \int_\Omega |Du_j|^2 = \left(\frac{1}{\lambda}\right)^2 |\Omega| .$$

So (Dh_j) is bounded in L^2 and so as $h_j = u_j$ on $\partial\Omega$, (h_j) is bounded in $W^{1,2}$ too.

For a further subsequence (which we relabel) $h_j \rightharpoonup h$ in $W^{1,2}$, where $h \in W^{1,2}$. By Mazur's Lemma, $h \in W^{1,2}_u$ and as it is easy to check that h is also A-harmonic it follows that $u = h$ by uniqueness of A-harmonic maps. Finally we get the desired contradiction from Rellich's compactness theorem:

$$u_j - h_j = (u_j - u) + (u - h_j) \to 0 \text{ strongly in } L^2$$

which is impossible by ⊛. □