In this lecture we'll use the linearization strategy implemented by way of the A-harmonic approximation lemma. This variant of the proof goes back to Duzaar & Steffen 2002; our goal is:

$\boxed{\text{TH}}$ (L.C. Evans 1986, E. Acerbi & N. Fusco 1986)

Assume $F: \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies

(H1) F is C^2 (H2) $|F(\xi)| \leq L(|\xi|^2+1)$

(H3) $F - \ell|\cdot|^2$ is QC,

where $0 < \ell \leq L < \infty$ are constants.

Let $g \in W^{1,2}(\Omega, \mathbb{R}^N)$. If $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ is F-minimizing, then

$$\Sigma_u := \left\{ x \in \Omega : \overline{\lim_{r\searrow 0}} |(Du)_{x,r}| = \infty \text{ or } \underline{\lim_{r\searrow 0}} E(x,r) > 0 \right\}$$

is closed in Ω, and u is $C^{1,\alpha}_{loc}$ on $\Omega\setminus\Sigma_u$ for all $\alpha < 1$.

The above is a consequence of:

$\boxed{\text{TH}}$ Under the assumptions of the above TH:

For each $m > 0$ there exists $\varepsilon = \varepsilon(m) > 0$ s.t.

$$\Omega\setminus\Sigma_u = \bigcup_{m=1}^{\infty} \left\{ x \in \Omega : \exists r \in (0, \tfrac{1}{2}\operatorname{dist}(x,\partial\Omega)) \text{ s.t. } |(Du)_{x,r}| < m \text{ and } E(x,r) < \varepsilon(m) \right\}$$

ie,

$$\Sigma_u = \bigcap_{m=1}^{\infty} \left\{ x \in \Omega : \begin{array}{l} \forall r \in (0, \tfrac{1}{2}\mathrm{dist}(x, \partial\Omega)) \text{ s.t.} \\ |(Du)_{x,r}| \geqq m \text{ or } E(x,r) \geqq \varepsilon(m) \end{array} \right\}.$$

Furthermore, if for some $r \in (0, \frac{1}{2}\mathrm{dist}(x, \partial\Omega))$ we have $|(Du)_{x,r}| < m$ and $E(x,r) < \varepsilon(m)$, then u is $C^{1,\alpha}$ near x for each $\alpha < 1$.

This TH is a consequence of the following:

$\boxed{\text{Proposition}}$ Under the assumptions of the above TH: There exists a constant $1 \leqq c = c(F'', \frac{L}{\ell}, n, N) < \infty$ with the following property. For each $\sigma \in (0,1)$, $m < \infty$ there exists $\delta = \delta(F'', \frac{L}{\ell}, n, N, \sigma, m) > 0$ s.t.

$$E(x,r) \leqq c\left(\sigma \left(\frac{R}{r}\right)^{n+2} + \left(\frac{r}{R}\right)^2 \right) E(x,R)$$

for $0 < r < \frac{R}{4}$ provided that $B(x,R) \subset \Omega$ and

$$|(Du)_{x,R}| < m \quad \text{and} \quad E(x,R) < \delta.$$

Pf that Prop. $\Rightarrow$ TH

Fix $m>0$ and let $\varepsilon, \sigma \in (0,1)$ satisfy

(1) $$\varepsilon \leq \delta\left(F'', \tfrac{L}{\ell}, n, N, \sigma, m+1\right)$$

NOTE (and not m)

Let $x_0 \in \Omega$ and suppose $0 < R < \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$

(2) $|(Du)_{x_0,R}| < m$ and $E(x_0, R) < \varepsilon$,

Then also $\exists \varrho > 0$ s.t. for all $x \in B(x_0, \varrho)$

(2') $|(Du)_{x,R}| < m$ and $E(x,R) < \varepsilon$.

Combining (1), (2') we get for $x \in B(x_0, \varrho)$ and $0 < \tau < \frac{1}{4}$ from Prop.:

$$E(x, \tau R) \leq c(\sigma \tau^{-n-2} + \tau^2) E(x,R)$$
$$\leq \tau^{2\alpha} E(x,R)$$

for $0 < \alpha < 1$ provided we choose

(3) $$\tau = (2c)^{-\frac{1}{2-2\alpha}}, \quad \sigma \leq \frac{\tau^{n+2+2\alpha}}{2c}.$$

We now seek to iterate this. Note that for $j \in \mathbb{N}$ and $x \in B(x_0, \varrho)$ we have

$$E(x, \tau^j R) \leq \tau^{2\alpha} E(x, \tau^{j-1} R)$$

provided

(4) $|(Du)_{x,\tau^{j-1}R}| < m+1$ & $E(x, \tau^{j-1}R) < \delta$.

We know that (4) holds for $j=1$ (by (1), (2')). Let $k \in \mathbb{N}$ and suppose that (4) holds for $j \in \{1,2,\dots,k\}$. Then

$$E(x,\tau^k R) \leq \tau^{2\alpha k} E(x,R) \overset{(1),(2'),(3) \text{ and } \tau \leq \frac{1}{4}}{<} \delta,$$

$$|(Du)_{x,\tau^k R}| \leq |(Du)_{x,R}| + \sum_{j=1}^{k} |(Du)_{x,\tau^j R} - (Du)_{x,\tau^{j-1}R}|$$

$$\overset{(2')}{<} m + \sum_{j=1}^{k} \fint_{B(x,\tau^j R)} |Du - (Du)_{x,\tau^{j-1}R}|$$

$$\leq m + \sum_{j=1}^{k} \left(\fint_{B(x,\tau^j R)} |Du - (Du)_{x,\tau^{j-1}R}|^2\right)^{\frac{1}{2}}$$

$$\leq m + \sum_{j=1}^{k} \tau^{-\frac{n}{2}} E(x,\tau^{j-1}R)^{\frac{1}{2}}$$

$$\overset{(4) \text{ for } 1\leq j\leq k}{\leq} m + \sum_{j=1}^{k} \tau^{-\frac{n}{2}} \tau^{\alpha(j-1)} E(x,R)^{\frac{1}{2}}$$

$$\leq m + \tau^{-\frac{n}{2}} (1-\tau^{\alpha})^{-1} E(x,R)^{\frac{1}{2}}$$

$$\overset{(2')}{<} m + \tau^{-\frac{n}{2}} (1-\tau^{\alpha})^{-1} \varepsilon^{\frac{1}{2}} \leq m+1$$

provided $\tau^{-\frac{n}{2}}(1-\tau^{\alpha})^{-1}\varepsilon^{\frac{1}{2}} \leq 1$, ie,

(5) $$\varepsilon \leq \tau^{n}(1-\tau^{\alpha})^{2}.$$

Consequently if we take $\tau = (2c)^{-\frac{1}{2-2\alpha}}$, then $\sigma = \dfrac{\tau^{n+2+2\alpha}}{2c}$ and then ε satisfying

(1), (5), then we can iterate, and (2) implies

$$E(x, \tau^j R) \leq \tau^{2\alpha j} E(x,R) \quad \forall j \in \mathbb{N},\ \forall x \in B(x_0, \varrho).$$

For $0 < r \leq \frac{R}{4}$ take $j \in \mathbb{N}$ s.t. $\tau^j R < r \leq \tau^{j-1} R$.
Then for $x \in B(x_0, \varrho)$: $E(x,r) \leq \tau^{-n} \tau^{2\alpha(j-1)} E(x,R)$

$$\leq M \left(\frac{r}{R}\right)^{2\alpha}, \text{ where } M := \tau^{-n-2\alpha} 2^n E(x_0, 2R).$$

The α-Hölder continuity now follows from the theorem of Campanato, Meyers. □

We turn to the proof of the Proposition, and recall a key ingredient from (LS):

The A-harmonic approximation lemma

Let $0 < \ell \leq L < \infty$, $n, N \in \mathbb{N}$, $\varepsilon > 0$.
$\exists\, \delta = \delta(\frac{L}{\ell}, n, N, \varepsilon) > 0$ with the following property: If $A : \mathbb{R}^{N\times n} \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is symm, bilinear and $u \in W^{1,2}(B(x,R), \mathbb{R}^N)$ satisfy

(i) $A[\xi, \eta] \leq L|\xi||\eta|$, $\forall \xi, \eta$

(ii) $\ell \int_{B(0,1)} |D\varphi|^2 \leq \int_{B(0,1)} A[D\varphi, D\varphi]$, $\forall \varphi \in W_0^{1,\infty}$

(iii) $\fint_{B(x,R)} A[Du, D\varphi] \leq \delta \|D\varphi\|_{L^\infty} \sqrt{E(x,R)}$
$\forall \varphi \in W_0^{1,\infty}(B(x,R), \mathbb{R}^N)$,

then

$$R^{-2} \fint_{B(x,R)} |u-h|^2 \leq \varepsilon E(x,R),$$

where $h \in W^{1,2}_u(B(x,R), \mathbb{R}^N)$ is $\mathbb{A}$-harmonic.

Pf. of Proposition

Step 1. Implementation of linearization strategy by way of $\mathbb{A}$-harmonic approximation.

Fix $m>0$. Take $B_R = B(x_0, R)$ s.t. $|(Du)_{x_0,R}| < m$ and put $\xi_0 = (Du)_{x_0,R}$.

Observe that $F''(\xi_0)$ satisfies (i), (ii) with

$$L \sim k := 1 + \sup_{|\xi| \leq m+1} |F''(\xi)| , \quad \ell ,$$

in the $\mathbb{A}$-harmonic approximation lemma.

We now seek an estimate of

$$\int_{B_R} F''(\xi_0)[Du, D\varphi]$$

for $\varphi \in W^{1,\infty}_0(B_R, \mathbb{R}^N)$. We need to discuss the possible bounds on F'', but first note

For $\varphi \in W_0^{1,\infty}(B_R, \mathbb{R}^N)$:

$$\int_{B_R} F''(\xi_0)[Du, D\varphi] =$$

u is an F- extremal so integrates to 0

$$\int_{B_R} \Big\{ F''(\xi_0)[Du-\xi_0, D\varphi] + \overbrace{(F'(\xi_0)-F'(Du))[D\varphi]} \Big\} =$$

$$\int_{B_R}\int_0^1 \Big(F''(\xi_0) - F''(\xi_0 + t(Du-\xi_0))\Big)[Du-\xi_0, D\varphi]\,dt.$$

Recall from L5: H2, H3 $\Longrightarrow$ $\exists\, c = c(n,N) > 0$

s.t. $|F'(\xi)| \leqq cL(|\xi|+1)$, $\forall \xi$,

and $|F(\xi)-F(\eta)| \leqq cL(|\xi|+|\eta|+1)|\xi-\eta|$, $\forall \xi,\eta$.

However, we do not have global control on F''.

Instead we have:

$\boxed{\text{Auxiliary Lemma}}$ Assume $\begin{cases} F:\mathbb{R}^{N\times n}\to\mathbb{R} \text{ is } C^2 \\ |F(\xi)|\leqq L(|\xi|^2+1),\ \forall\xi, \\ F \text{ is QC}. \end{cases}$

Let $m>0$. There exist a constant ~~c = c(m,L)~~ $c = c(m, L, F'') > 0$ and an increasing, concave function $\omega = \omega_{m,F''} : [0,\infty) \to [0,1]$ with $\omega(0+) = 0$ and $\omega(t) = 1$ for $\forall t \geqq 1$ with the following property: For each $|\xi_0| \leq m$ and all $\xi, \eta \in \mathbb{R}^{N\times n}$

$$\left| \int_0^1 \left(F''(\xi_0) - F''(\xi_0 + t(\xi-\xi_0))\right)[\xi-\xi_0, \eta]\, dt \right| =$$

$$\left| F''(\xi_0)[\xi-\xi_0, \eta] + \left(F'(\xi_0) - F'(\xi)\right)[\eta] \right| \leq$$

$$c\,\omega(|\xi-\xi_0|)|\xi-\xi_0|\,|\eta|.$$

$\boxed{\text{Pf:}}$ Put $k := 1 + \sup_{|\xi| \leq m+1} |F''(\xi)|$ and

$$\tilde{\omega}(t) := \frac{1}{2k} \sup\left\{ |F''(\xi) - F''(\eta)| : \begin{array}{l} |\xi-\eta| \leq t \\ |\xi|, |\eta| \leq m+1 \end{array} \right\}.$$

We record: $\tilde{\omega} : [0,\infty) \to [0,1]$ is increasing $\tilde{\omega}(0+) = 0$ and $|F''(\xi) - F''(\eta)| \leq 2k\tilde{\omega}(|\xi-\eta|)$ for all $|\xi|, |\eta| \leq m+1$.

Put

$$\theta(t) := \max\{\tilde{\omega}(t), \min\{t,1\}\}, \quad t \geq 0.$$

Then θ has all of the above properties besides $\theta(t) = 1 \;\forall t \geq 1$. Finally put $\omega(t) =$ concave envelope of $\theta(t)$ on $[0,\infty)$, ie,

$$\omega(t) := \inf\left\{ c(t) : \begin{array}{l} c:[0,\infty)\to[0,\infty) \text{ concave} \\ c \geqq \theta \text{ on } [0,\infty) \end{array} \right\}.$$

<u>Exercise:</u> Show that $\omega : [0,\infty) \to [0,1]$ is increasing, concave, $\omega(0+) = 0$, $\omega(t) = 1 \ \forall t \geqq 1$, and

$$|F''(\xi) - F''(\eta)| \leqq 2k\,\omega(|\xi-\eta|)$$

for all ξ, η with $|\xi|, |\eta| \leqq m+1$.

Now for $|\xi| \leqq m+1$ we estimate:

$$\left| \int_0^1 \left(F''(\xi_0) - F''(\xi_0 + t(\xi-\xi_0)) \right) [\xi - \xi_0, \eta]\, dt \right| \leqq$$

$$\int_0^1 |F''(\xi_0) - F''(\xi_0 + t(\xi - \xi_0))|\, |\xi-\xi_0|\,|\eta|\, dt \leqq$$

$$\int_0^1 2k\,\omega(t|\xi-\xi_0|)\,|\xi-\xi_0|\,|\eta|\, dt \leqq 2k\omega(|\xi-\xi_0|)|\xi-\xi_0||\eta|.$$

For $|\xi| > m+1$ we have $|\xi - \xi_0| > 1$, and we estimate:

$$|F''(\xi_0)[\xi-\xi_0, \eta] + (F'(\xi_0) - F'(\xi))[\eta]| \leqq$$

$$|F''(\xi_0)| \cdot |\xi-\xi_0| \cdot |\eta| + \left(|F'(\xi_0)| + |F'(\xi)|\right)|\eta| \leqq$$

$$k|\xi-\xi_0||\eta| + cL(|\xi_0|+1 + |\xi|+1)|\eta| \leqq$$

$$k|\xi-\xi_0||\eta| + cL(2m+2+|\xi-\xi_0|)|\eta| \le$$

$$(k+2cL(m+2))|\xi-\xi_0||\eta| =$$

$$(k+2cL(m+2))\underbrace{\omega(|\xi-\xi_0|)}_{=1}|\xi-\xi_0||\eta| ,$$

and so the desired estimate follows with

$$c = \max\{2k,\ k+2cL(m+2)\}. \quad \square$$

Return to estimation: $\forall \varphi \in W_0^{1,\infty}(B_R,\mathbb{R}^N)$,

$$\fint_{B_R} F''(\xi_0)[Du, D\varphi] = \fint_{B_R}\int_0^1 \big(F''(\xi_0)-F''(\xi_0+t(Du-\xi_0))\big)[Du-\xi_0, D\varphi]\,dt$$

Auxiliary Lemma with $\xi = Du$, $\eta = D\varphi$

$$\le \fint_{B_R} c\,\omega(|Du-\xi_0|)|Du-\xi_0||D\varphi|$$

Hölder ~~$\omega\le 1$, ω concave, Hölder~~

$$\le c\Big(\fint_{B_R}\omega(|Du-\xi_0|)^2\Big)^{\frac12}\Big(\fint_{B_R}|Du-\xi_0|^2\Big)^{\frac12}\|D\varphi\|_{L^\infty}$$

$\omega\le 1$, ω concave

$$\le c\,\omega\Big(\fint_{B_R}|Du-\xi_0|\Big)^{\frac12} E(x_0,R)^{\frac12}\|D\varphi\|_{L^\infty}$$

Hölder, ω increasing

$$\le c\,\omega\big(E(x_0,R)^{\frac12}\big)^{\frac12} E(x_0,R)^{\frac12}\|D\varphi\|_{L^\infty}.$$

Consequently: Let $m > 0$.

If $B(x_0, R) \subset \Omega$ and $\xi_0 = (Du)_{x_0,R}$ satisfies $|\xi_0| < m$, then $\exists\ c = c_m$, $\omega = \omega_m$ s.t.

$$\left| \int_{B_R} F''(\xi_0)[Du, D\varphi] \right| \leq c\,\omega\left(E(x_0,R)^{\frac{1}{2}}\right)^{\frac{1}{2}} E(x_0,R)^{\frac{1}{2}} \|D\varphi\|_{L^\infty}$$

for all $\varphi \in W_0^{1,\infty}(B_R, \mathbb{R}^N)$.

By the A-harmonic approximation Lemma $\exists\ \tilde{\delta} = \tilde{\delta}(F'', n, N, \sigma) > 0$ s.t. if

(1) $$c\,\omega\left(E(x_0,R)^{\frac{1}{2}}\right)^{\frac{1}{2}} < \tilde{\delta},$$

then
(2) $$R^{-2} \fint_{B_R} |u - h|^2 \leq \sigma E(x_0, R),$$

where $h \in W_u^{1,2}(B_R, \mathbb{R}^N)$ is $F''(\xi_0)$-harmonic.

Solving (1) for $E(x_0,R)$ we find a $\delta = \delta(F'', n, N, \sigma, m) > 0$ s.t. (1), and hence (2), holds when $E(x_0, R) < \delta$.

Note: δ depends on m via $\omega = \omega_m$.

$\boxed{\text{Remark}}$ Step 1 (and the conclusion (2)) used only that

- u is an F-extremal
- F'' satisfies the strong Legendre-Hadamard condition

and (H1), (H2). We did not use F-minimality or (H3)!

We linearize around $\xi_0 = (Du)_{x_0,R}$ and require smallness of $E(x_0,R)$ to get the L^2-approximation

$$R^{-2} \fint_{B_R} |u-h|^2 \leq \sigma E(x_0,R).$$

This is why we don't attempt to prove lower order regularity (eg $C^{0,\alpha}_{loc}$) by this method. We must bound the left-hand side below by a 1^{st} order term in u — for that we use a Caccioppoli inequality of the second kind. It is the proof of this inequality that requires all the assumptions.

Step 2: Caccioppoli inequality of 2^{nd} kind

$\boxed{\text{TH}}$ (Evans 1986, Acerbi & Fusco 1986)

Under the assumptions of the TH:
For each $m > 0$ there exists a constant $c_m = c(F'', \frac{L}{\ell}, n, N, m) < \infty$ s.t. for any affine map $a : \mathbb{R}^n \to \mathbb{R}^N$ and any $B_R = B(x_0, R) \subset \Omega$,

$$\int_{B_{R/2}} |Du - Da|^2 \leqq \frac{c_m}{R^2} \int_{B_R} |u - a|^2$$

provided that $|Da| \leqq m + 1$.

$\boxed{\text{Pf:}}$ Put $\tilde{u} := u - a$,

$$\tilde{F}(\xi) := F(\xi + Da) - F(Da) - F'(Da)[\xi]$$
$$= \int_0^1 (1-t) F''(Da + t\xi)[\xi, \xi]\, dt.$$

As in Auxiliary Lemma from Step 1 we prove that $\exists\, k_m = k(F'', L, m) < \infty$ s.t.

$$\begin{cases} |\tilde{F}(\xi)| \leqq k|\xi|^2, & \forall \xi \\ |\tilde{F}'(\xi)| \leqq k|\xi| \end{cases}$$

Furthermore, we note that $\tilde{F} - \ell|\cdot|^2$ is QC, and that $\tilde{u}$ is $\tilde{F}$-minimizing.

Fix radii $\frac{R}{2} < r < s < R$ and let $\rho \in W^{1,\infty}$ be a cut-off function so $\mathbb{1}_{B_r} \leq \rho \leq \mathbb{1}_{B_s}$ and $|D\rho| \leq \frac{1}{s-r}$ a.e.

Put $\varphi := \rho\tilde{u}$, $\psi := (1-\rho)\tilde{u}$.

Then $\varphi + \psi = \tilde{u}$, $\varphi \in W_0^{1,2}(B_s, \mathbb{R}^N)$ and $\psi = \tilde{u}$ on $B_R \setminus B_s$.

Now use that $\tilde{F} - \ell|\cdot|^2$ is QC at 0:

$$\ell \int_{B_s} |D\varphi|^2 \leq \int_{B_s} \left(\tilde{F}(D\varphi) - \tilde{F}(0) \right) = \int_{B_s} \tilde{F}(D\tilde{u} - D\psi)$$

$$= \int_{B_s} \tilde{F}(D\tilde{u}) + \int_{B_s} \left(\tilde{F}(D\tilde{u} - D\psi) - \tilde{F}(D\tilde{u}) \right)$$

$$\overset{\tilde{u}\ \tilde{F}\text{-min}}{\leq} \int_{B_s} \tilde{F}(D\psi) + \int_{B_s} \int_0^1 \tilde{F}'(D\tilde{u} - tD\psi)[-D\psi]\,dt$$

$$\leq \int_{B_s} \left(k|D\psi|^2 + k(|D\tilde{u}| + |D\psi|)|D\psi| \right)$$

$D\psi = (1-\rho)D\tilde{u} - \tilde{u} \otimes D\rho$, ie, $D\psi = 0$ on B_r

so $|D\psi| \leq |D\tilde{u}| + \frac{|\tilde{u}|}{s-r}$ and $|D\psi| = 0$ on B_r.

As $D\tilde{u} = D\varphi$ on B_r we estimate:

$$\ell \int_{B_r} |D\tilde{u}|^2 \leq \ell \int_{B_s} |D\varphi|^2 \leq \int_{B_s \setminus B_r} \Big(k \left(|D\tilde{u}| + \frac{|\tilde{u}|}{s-r} \right)^2$$

$$+ k \left(2|D\tilde{u}| + \frac{|\tilde{u}|}{s-r} \right) \left(|D\tilde{u}| + \frac{|\tilde{u}|}{s-r} \right) \Big)$$

$$= k \int_{B_s \setminus B_r} \left(5 |D\tilde{u}|^2 + 5 \frac{|\tilde{u}|^2}{(s-r)^2} \right).$$

We use the <u>Widman hole filling trick</u> to conclude:

add $\frac{5k}{\ell} \int_{B_r} |D\tilde{u}|^2$ to both sides to get

$$\left(1 + \frac{5k}{\ell}\right) \int_{B_r} |D\tilde{u}|^2 \leq \frac{5k}{\ell} \int_{B_s} |D\tilde{u}|^2 + \frac{5k}{\ell} \int_{B_s} \frac{|\tilde{u}|^2}{(s-r)^2},$$

ie, with $\theta := \frac{\frac{5k}{\ell}}{1 + \frac{5k}{\ell}} \in (0,1)$ we have:

$$\int_{B_r} |D\tilde{u}|^2 \leq \theta \int_{B_s} |D\tilde{u}|^2 + \theta \int_{B_s} \frac{|\tilde{u}|^2}{(s-r)^2}$$

$$\leq \theta \int_{B_s} |D\tilde{u}|^2 + \frac{A}{(s-r)^2}$$

for all $\frac{R}{2} < r < s < R$, where $A := \theta \int_{B_R} |\tilde{u}|^2$.

We can now invoke

$\boxed{\text{Iteration Lemma}}$ Let $R > 0$ and $f : [\frac{R}{2}, R] \to [0, \infty)$ be a bounded function. Assume that for some constants $\theta \in (0,1)$, $A \geqq 0$ we have

$$f(r) \leqq \theta f(s) + \frac{A}{(s-r)^2}$$

for all $\frac{R}{2} < r < s < R$. Then there exists a constant $c = c(\theta)$ (depending on θ only!) s.t. $f(\frac{R}{2}) \leqq c \frac{A}{R^2}$.

$\boxed{\text{Pf:}}$ is an exercise (or see Giusti's book) $\square$

Step 3: Conclusion.

From Step 1,

$$R^{-2} \fint_{B_R} |u-h|^2 \leqq \sigma E(x_0, R) \tag{3.1}$$

provided $B_R = B(x_0, R) \subset \Omega$, $|(Du)_{x_0,R}| < m$ and $E(x_0, R) < \delta$, where $h \in W^{1,2}_u(B_R, \mathbb{R}^N)$ is $F''(\xi_0)$-harmonic ($\xi_0 := (Du)_{x_0,R}$).

Recall $k = 1 + \sup_{|\xi| \le m+1} |F''(\xi)|$. From the Generalized Weyl Lemma: for $0 < r < R$

$$\fint_{B_r} |Dh - (Dh)_{x_0,r}|^2 \le \tilde{c} \left(\frac{r}{R}\right)^2 \fint_{B_R} |Dh - (Dh)_{x_0,R}|^2$$

exercise

$$\le c \left(\frac{r}{R}\right)^2 \fint_{B_R} |Du - (Du)_{x_0,R}|^2$$

$$(3.2) \qquad = c \left(\frac{r}{R}\right)^2 E(x_0, R).$$

Let $a : \mathbb{R}^n \to \mathbb{R}^N$ be the affine map s.t. $a_{x_0,2r} = h_{x_0,2r}$ and $Da = (Dh)_{x_0,R}$ $\left(= (Du)_{x_0,R}\right)$.

Note: $|Da| < m$

Now for $r < \frac{R}{2}$: Step 2

$$\fint_{B_r} |Du - (Du)_{x_0,r}|^2 \le \fint_{B_r} |Du - Da|^2 \le$$

$$\frac{\tilde{C}_m}{r^2} \fint_{B_{2r}} |u - a|^2 \le \frac{2\tilde{C}_m}{r^2} \fint_{B_{2r}} \left(|u-h|^2 + |h-a|^2\right)$$

Poincaré
$$\le \frac{2\tilde{C}_m}{r^2} \frac{|B_R|}{|B_{2r}|} R^2 \frac{1}{R^2} \fint_{B_R} |u-h|^2 + \frac{2\tilde{C}_m}{r^2} c r^2 \fint_{B_{2r}} |Dh - Da|^2$$

(3.1), (3.2)
$$\le C_m \left(\left(\frac{R}{r}\right)^{n+2} \sigma + \left(\frac{r}{R}\right)^2 \right) E(x_0, R). \quad \square$$