Recall from L1

Šverák & Yan, 2002: There exists a C^∞ function $F : \mathbb{R}^{3\times 3} \to \mathbb{R}$ satisfying for some constants $0 < \ell < L < \infty$

$$\ell |\lambda|^2 \leq F''(\xi)[\lambda,\lambda] \leq L|\lambda|^2, \ \forall \xi, \lambda$$

s.t. the mapping

$$u : B(0,1) \ni x \longmapsto |x|^{-\varepsilon}\left(\frac{x_i x_j}{|x|} - \frac{|x|}{n}\delta_{ij} \right)$$

(considered as a mapping into the symmetric trace-free 3×3 matrices) is F-minimizing for some $\varepsilon \in (0, \frac{1}{2})$.

In particular: Du is unbounded near 0, and so $u \notin W^{1,\infty}_{loc}$.

➡ Are there conditions on F that ensure local Lipschitz regularity of F-minimizers?

$\boxed{\text{TH}}$ (Chipot & Evans 1986)

Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be C^2 and assume that

$$F''(\xi) \to \mathbb{A} \qquad \text{as} \qquad |\xi| \to \infty,$$

where $\mathbb{A}$ is a (constant coefficient) bilinear form on $\mathbb{R}^{N\times n}$ satisfying

$$\begin{cases} \mathbb{A}[\xi, \eta] \leq L|\xi||\eta| & \forall \xi, \eta \\ \mathbb{A}[a\otimes b, a\otimes b] \geq \ell |a|^2 |b|^2 & \forall a \in \mathbb{R}^N, b \in \mathbb{R}^n, \end{cases}$$

where $0 < \ell \leq L < \infty$ are constants.

Then any F-minimizer $u \in W^{1,2}$ is locally Lipschitz.

$\boxed{\text{Pf:}}$ We use the linearization strategy at infinity (following Chipot & Evans).

Put $\quad k := \sup_{\xi \in \mathbb{R}^{N\times n}} |F''(\xi) - \mathbb{A}|$

and note that $k < \infty$ since $\xi \mapsto |F''(\xi) - \mathbb{A}|$ is continuous and $|F''(\xi) - \mathbb{A}| \to 0$ as $|\xi| \to \infty$.

We need the following:

Lemma $\forall \delta > 0 \ \exists m < \infty$ s.t.

$$(*) \quad |F(\xi+\xi_0) - F(\xi_0) - F'(\xi_0)[\xi] - \tfrac{1}{2}\mathbb{A}(\xi,\xi)| \leq 2\delta(|\xi|^2 + \lambda|\xi|)$$

for all $\xi_0, \xi \in \mathbb{R}^{N\times n}$ and $\lambda \geq 0$ provided that

$$(**) \quad \lambda^2 + |\xi_0|^2 \geq M^2 := \left(1 + \frac{2k}{\delta}\right)^2 m^2 + \left(\frac{2k}{\delta}\right)^2 m^2.$$

Pf of Lemma: Let $\delta > 0$ and take $m > 0$ s.t. $|F''(\xi) - \mathbb{A}| \leq \delta$ for $|\xi| \geq m$.

For ξ_0, λ satisfying $(**)$ and any ξ we have

$$|F(\xi+\xi_0) - F(\xi_0) - F'(\xi_0)[\xi] - \tfrac{1}{2}\mathbb{A}[\xi,\xi]|$$

$$\leq \int_0^1 (1-t)|F''(\xi_0 + t\xi) - \mathbb{A}|\,dt\,|\xi|^2$$

$$\leq \int_0^1 |F''(\xi_0 + t\xi) - \mathbb{A}|\,dt\,|\xi|^2.$$

Put $I = \{t \in [0,1] : |\xi_0 + t\xi| < m\}$.

There is nothing to prove for $\xi = 0$, so assume WLOG that $\xi \neq 0$.

Note $\quad \mathcal{L}^1(I) \leq \dfrac{\mathcal{H}^1([\xi_0, \xi_0+\xi] \cap B(0,m))}{|\xi|}$

$$\leq \frac{2m}{|\xi|}.$$

We have: $\quad \Delta :=$

$$|F(\xi+\xi_0) - F(\xi_0) - F'(\xi_0)[\xi] - \tfrac{1}{2}\mathbb{A}[\xi,\xi]| \leq$$

$$\left[\int_I |F''(\xi_0+t\xi) - \mathbb{A}|\,dt + \int_{[0,1]\setminus I} |F''(\xi_0+t\xi) - \mathbb{A}|\,dt\right]|\xi|^2$$

$$\leq \left(\delta(1-\mathcal{L}^1(I)) + k\mathcal{L}^1(I)\right)|\xi|^2$$

$$\leq \left(\delta + k\mathcal{L}^1(I)\right)|\xi|^2.$$

If $|\xi| \geq \dfrac{2km}{\delta}$, then $\mathcal{L}^1(I) \leq \dfrac{2m}{|\xi|} \leq \dfrac{\delta}{k}$

and so $\quad \Delta \leq 2\delta|\xi|^2 \leq 2\delta(|\xi|^2 + \lambda|\xi|)$,

which is the desired bound.

If $|\xi| < \dfrac{2km}{\delta}$ we must invoke the assumption

$$\lambda^2 + |\xi_0|^2 \geq M^2 = \left(1+\frac{2k}{\delta}\right)^2 m^2 + \left(\frac{2k}{\delta}\right)^2 m^2$$

which gives that

$$|\xi_0| \geqslant \left(1+\frac{2k}{\delta}\right)m \quad \text{or} \quad \lambda \geqq \frac{2k}{\delta}m.$$

If $\lambda \geqq \frac{2k}{\delta}m$, then $2m \leqq \frac{\lambda\delta}{k}$ and

so $\mathcal{L}^1(I) \leqq \frac{2m}{|\xi|} \leqq \frac{\lambda\delta}{k|\xi|}$, hence

$$\Delta \leqq (\delta + k\mathcal{L}^1(I))|\xi|^2 \leqq \delta(|\xi|^2 + \lambda|\xi|),$$

which is the desired bound.

If $|\xi_0| \geqslant \left(1+\frac{2k}{\delta}\right)m$, then for all $t \in [0,1]$: (recall: $|\xi| < \frac{2km}{\delta}$)

$$|\xi_0 + t\xi| \geqslant |\xi_0| - |\xi| > m,$$

hence $I = \emptyset$ and therefore

$$\Delta \leqq \delta|\xi|^2 \leqq \delta(|\xi|^2 + \lambda|\xi|). \quad \square$$

We return to the proof of the TH.

Fix $B(x,r) \subset \Omega$. Put $\xi_0 := (Du)_{x,r}$

and $\lambda := E(x,r)^{\frac{1}{2}} = \left(\fint_{B(x,r)} |Du - (Du)_{x,r}|^2 \right)^{\frac{1}{2}}$.

Let

$$\tilde{F}(\xi) := F(\xi + \xi_0) - F(\xi_0) - F'(\xi_0)[\xi]$$

and $\tilde{u} = u - a$ for an affine map a with $Da = \xi_0$. Then $\tilde{u}$ is $\tilde{F}$-minimizing, and given $\delta > 0$ we find

$m = m(\delta) < \infty$ s.t.

$$\circledast \quad \left| \tilde{F}(\xi) - \tfrac{1}{2} A[\xi, \xi] \right| \leq 2\delta \left(|\xi|^2 + \lambda |\xi| \right)$$

provided $\fint_{B(x,r)} |Du|^2 = E(x,r) + |(Du)_{x,r}|^2 =$

$$\lambda^2 + |\xi_0|^2 \geq M^2 := \left(1 + \tfrac{2k}{\delta}\right)^2 m^2 + \left(\tfrac{2k}{\delta}\right)^2 m^2 .$$

We assume $\fint_{B(x,r)} |Du|^2 \geq M^2$, and take $\xi = D\tilde{u}(x)$ above, whereby:

$$(\otimes) \quad \left| \tilde{F}(D\tilde{u}) - \tfrac{1}{2} A[D\tilde{u}, D\tilde{u}] \right| \leq 2\delta \left(|D\tilde{u}|^2 + E(x,r)^{\frac{1}{2}} |D\tilde{u}| \right)$$

$$\leq 2\delta \left(\tfrac{3}{2} |D\tilde{u}|^2 + \tfrac{1}{2} E(x,r) \right) \leq 3\delta \left(|D\tilde{u}|^2 + E(x,r) \right)$$

a.e. in Ω. Let $h \in W^{1,2}_{\tilde{u}}(B(x,r), \mathbb{R}^N)$ be A-harmonic. Then as $\tilde{u} - h \in W^{1,2}_0$:

$$\frac{\ell}{2}\fint_{B(x,r)} |D\tilde{u} - Dh|^2 \le \fint_{B(x,r)} \frac{1}{2} A[D\tilde{u} - Dh, D\tilde{u} - Dh]$$

$$\overset{\text{why?}}{=} \frac{1}{2}\fint_{B(x,r)} \Big(A[D\tilde{u}, D\tilde{u}] - A[Dh, Dh]\Big) \overset{\otimes}{\le}$$

$$\fint_{B(x,r)} \Big(\tilde{F}(D\tilde{u}) + 3\delta\big(|D\tilde{u}|^2 + E(x,r)\big) - \frac{1}{2} A[Dh, Dh]\Big)$$

$$\overset{\tilde{u}\ \tilde{F}\text{-min}}{\le} \fint_{B(x,r)} \Big(\tilde{F}(Dh) + 3\delta\big(|D\tilde{u}|^2 + E(x,r)\big) - \frac{1}{2} A[Dh, Dh]\Big)$$

Taking $\xi = Dh$ in $\circledast$ we get $\otimes$ with Dh replacing $D\tilde{u}$:

$$\Big|\tilde{F}(Dh) - \frac{1}{2} A[Dh, Dh]\Big| \le 3\delta\big(|Dh|^2 + E(x,r)\big),$$

hence

$$\frac{\ell}{2}\fint_{B(x,r)} |D\tilde{u} - Dh|^2 \le$$

$$3\delta \fint_{B(x,r)} \Big(|D\tilde{u}|^2 + 2E(x,r) + |Dh|^2\Big),$$

Exercise

Show that $\displaystyle \fint_{B(x,r)} |Dh|^2 \leq c \fint_{B(x,r)} |D\tilde{u}|^2$,

where $c = c(\frac{L}{l})$ (in fact, we can take $c = 4(3 + \frac{L}{l} + (\frac{L}{l})^2)$).

Thus: $\displaystyle \frac{l}{2} \fint_{B(x,r)} |D\tilde{u} - Dh|^2 \leq$

$$3\delta \fint_{B(x,r)} \left(|D\tilde{u}|^2 + 2E(x,r) + c(\tfrac{L}{l}) |D\tilde{u}|^2 \right)$$

$\leq c_0 \delta E(x,r)$, where $c_0 = c_0(\frac{L}{l})$, and

we used that $\displaystyle \fint_{B(x,r)} |D\tilde{u}|^2 = E(x,r)$.

Consequently, $\displaystyle \fint_{B(x,r)} |D\tilde{u} - Dh|^2 \leq c_1 \delta E(x,r)$

provided $\displaystyle \fint_{B(x,r)} |Du|^2 \geq M^2$, where

$c_1 = \frac{2c_0}{l}$ and $M = M(\delta) < \infty$.

By the generalized Weyl Lemma

$\exists\ \tilde{c} = \tilde{c}(n, N, \frac{L}{\ell}) < \infty$ s.t. for $0 < \sigma \leqq 1$:

$$\fint_{B(x,\sigma r)} |Dh - (Dh)_{x,\sigma r}|^2 \leqq \tilde{c}\,\sigma^2 \fint_{B(x,r)} |Dh - (Dh)_{x,r}|^2$$

As in above exercise $\exists\ c(\frac{L}{\ell}) < \infty$ s.t.

$\fint_{B(x,r)} |Dh - (Dh)_{x,r}|^2 \leqq c(\frac{L}{\ell}) E(x,r)$, and

hence for $0 < \sigma \leqq 1$:

$$\fint_{B(x,\sigma r)} |Dh - (Dh)_{x,\sigma r}|^2 \leqq \tilde{c}\, c(\tfrac{L}{\ell})\sigma^2 E(x,r)$$
$$=: c_2 \sigma^2 E(x,r).$$

For $0 < \sigma \leqq 1$: $E(x, \sigma r) \leqq$

$$2\Big(\fint_{B(x,\sigma r)} |Dh - (Dh)_{x,\sigma r}|^2 + \fint_{B(x,\sigma r)} |D\tilde{u} - Dh|^2\Big)$$

$$\leqq 2c_2\sigma^2 E(x,r) + \frac{2}{\sigma^n} \fint_{B(x,r)} |D\tilde{u} - Dh|^2$$

$$\leqq 2c_2\sigma^2 E(x,r) + \frac{2}{\sigma^n} c_1 \delta E(x,r)$$

$$\leqq c(\sigma^2 + \delta\sigma^{-n}) E(x,r) \text{ where}$$

$c := \max(2c_2, 2c_1)$.

$\boxed{\text{We have shown:}}$

$\forall \delta > 0 \ \exists M = M(\delta) < \infty$ s.t. when

$B(x,r) \subset \Omega$ and $\displaystyle\fint_{B(x,r)} |Du|^2 \geq M$,

then $E(x,\sigma r) \leq c(\sigma^2 + \delta\sigma^{-n}) E(x,r)$

for all $\sigma \in (0,1]$.

We simplify the last inequality:

First take $\sigma \in (0,1]$ s.t. $c\sigma^2 = \frac{1}{4}$, that is, $\sigma = (4c)^{-\frac{1}{2}}$. Then take $\delta > 0$ s.t. $c\delta\sigma^{-n} = \frac{1}{4}$, that is, $\delta = \sigma^{n+2} = (4c)^{-\frac{n+2}{2}}$.

$\boxed{\text{Consequently:}}$ $\exists M = M(\delta) < \infty$ corresponding to $\delta = (4c)^{-\frac{n+2}{2}} = \sigma^{n+2}$ s.t. when

$B(x,r) \subset \Omega$ and $\displaystyle\fint_{B(x,r)} |Du|^2 \geq M$,

then $E(x,\sigma r) \leq \frac{1}{2} E(x,r)$.

Fix $\Omega' \Subset \Omega$. Let $x_0 \in \Omega'$ be an L^2-Lebesgue point for Du:

$$Du(x_0) = \lim_{r \to 0} (Du)_{x_0,r} \quad \& \quad \lim_{r \searrow 0} E(x_0,r) = 0.$$

Put $d := \operatorname{dist}(\Omega', \partial\Omega)$.

If $\displaystyle ⨍_{B(x_0,r)} |Du|^2 \geqq M^2 \quad \forall r \in (0,d]$,

then define $r_0 = r_0(x_0) := d$. Otherwise

define

$$r_0 := \inf\left\{ r > 0 : ⨍_{B(x_0,r)} |Du|^2 < M^2 \right\}.$$

If $r_0 = 0$, then $|Du(x_0)| \leqq M^{\cancel{2}}$.

If $0 < r_0 < d$, then by continuity

$$⨍_{B(x_0,r_0)} |Du|^2 = M^2.$$

Finally we estimate in the case $r_0 = d$:

$$⨍_{B(x_0,r_0)} |Du|^2 \leqq \frac{1}{|B| d^n} \int_\Omega |Du|^2.$$

Put $\displaystyle \Lambda := \max\left\{ M^2, \frac{1}{|B| d^n} \int_\Omega |Du|^2 \right\}$

and observe: when $0 < r_0 \leq d$

$$\fint_{B(x_0,r_0)} |Du|^2 \leq \Lambda$$

and $\fint_{B(x_0,r)} |Du|^2 \geq M^2 \quad \forall r \in (0, r_0)$,

while for $r_0 = 0$, $|Du(x_0)| \leq M \leq \Lambda^{\frac{1}{2}}$.

Consider the case $0 < r_0 \leq d$. Then we get from above:

$$E(x_0, \sigma^j r_0) \leq \tfrac{1}{2} E(x_0, \sigma^{j-1} r_0) \leq \cdots$$

$$\leq 2^{-j} E(x_0, r_0) \leq 2^{-j} \Lambda, \quad \forall j \in \mathbb{N}.$$

Hence

$$|(Du)_{x_0,\sigma^j r_0}| \leq |(Du)_{x_0,r_0}| + \sum_{i=1}^{j} |(Du)_{x_0,\sigma^i r_0} - (Du)_{x_0,\sigma^{i-1} r_0}|$$

$$\leq \Big\{\fint_{B(x_0,r_0)} |Du|^2\Big\}^{\frac{1}{2}} + \sigma^{-\frac{n}{2}} \sum_{i=1}^{j} E(x_0, \sigma^{i-1} r_0)^{\frac{1}{2}}$$

$$\leq \Lambda^{\frac{1}{2}} + \sigma^{-\frac{n}{2}} \sum_{i=1}^{j} (2^{1-i} \Lambda)^{\frac{1}{2}}$$

$$\leq \Lambda^{\frac{1}{2}}\Big(1 + \sigma^{-\frac{n}{2}} \frac{1}{1 - \frac{1}{\sqrt{2}}}\Big) = \Lambda^{\frac{1}{2}}\Big(1 + \sigma^{-\frac{n}{2}} \frac{\sqrt{2}}{\sqrt{2} - 1}\Big)$$

and thus $\quad |Du(x_0)| \leqq \Lambda^{\frac{1}{2}} \left(1 + \sigma^{-\frac{n}{2}} \frac{\sqrt{2}}{\sqrt{2}-1}\right).$

We already saw that $|Du(x_0)| \leqq \Lambda^{\frac{1}{2}}$ when $r_0 = 0$, so we have shown that

$$|Du| \leqq \Lambda^{\frac{1}{2}} \left(1 + \sigma^{-\frac{n}{2}} \frac{\sqrt{2}}{\sqrt{2}-1}\right) \quad \text{a.e. in } \Omega',$$

that is, since $\Omega' \Subset \Omega$ was arbitrary,

$u \in W^{1,\infty}_{loc}$. $\square$

Put $D(\xi) := |\xi|^2$. Then the Chipot–Evans result says in particular that when

⊛ F is C^2 and $\lim_{|\xi| \to \infty} |F''(\xi) - D''(\xi)| = 0$,

then F-minimizers are locally Lipschitz.

If ⊛ holds, then we say that F resembles the Dirichlet integrand in a C^2 sense at infinity.

It is then natural to say that when

$$\frac{F(\xi)}{|\xi|^2} \to 1 \quad \text{as} \quad |\xi| \to \infty,$$

then F resembles the Dirichlet integrand in a C^0 sense at infinity.

Likewise, when $\dfrac{|F'(\xi) - D'(\xi)|}{|\xi|} \to 0$ as $|\xi| \to \infty$

then F resembles the Dirichlet integrand in a C^1 sense at infinity.

TH (G. Dolzmann & JK, 2005)

Let $F : \mathbb{R}^{N \times n} \to \mathbb{R}$ be C^0 and assume that $\lim_{|\xi| \to \infty} \dfrac{F(\xi)}{|\xi|^2} = \ell > 0$.

Then F-minimizers $u \in W^{1,2}$ satisfy $u \in W^{1,p}_{loc}$ for all $p < \infty$.

Pf Omitted.

More general results subsequently obtained by Scheven & Schmidt 2009, 2010.

➡ Can we take $p = \infty$ in above result?

No — there exists $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ that resembles the Dirichlet integrand in a C^1 sense at infinity, and that admits non-Lipschitz minimizers.

(G. Dolzmann, K. Zhang & JK, 2010)