Recall from last lecture:

$\boxed{\text{TH 29}}$ De Giorgi's Theorem (1957).

Assume $A : \Omega \to \mathbb{R}^{n\times n}$ is measurable, $A(x) = A(x)^T$ and $\ell|\lambda|^2 \leq A(x)\lambda\cdot\lambda \leq L|\lambda|^2$ for all $\lambda\in\mathbb{R}^n$ and for a.e. $x\in\Omega$.

If $u\in W^{1,2}_{loc}(\Omega)$ and $\operatorname{div} A(x)Du = 0$ in Ω, i.e. $\int_\Omega A(x)Du\cdot D\varphi\,dx = 0 \quad \forall\varphi\in C^1_c(\Omega)$,

then $u\in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha(n,\frac{\ell}{L})\in(0,1)$.

We proved that

- $u\in DG(\Omega)$ ie

$$\begin{cases} (i') \displaystyle\int_{A(k,r)} |Du|^2 \leq \frac{c}{(R-r)^2}\int_{A(k,R)} |u-k|^2 \\ (ii') \displaystyle\int_{B(k,r)} |Du|^2 \leq \frac{c}{(R-r)^2}\int_{B(k,R)} |u-k|^2 \end{cases}$$

where $A(k,r) = \{x\in B_r : u(x) > k\}$,
$B(k,r) = \{x\in B_r : u(x) < k\}$.

- If $u \in DG(\Omega)$, then $u \in L^\infty_{loc}(\Omega)$ and for $B_R \Subset \Omega$,

$$\sup_{B_{R/2}} |u| \leq c \left(\fint_{B_R} |u|^2 \right)^{\frac{1}{2}} .$$

- The Oscillation Lemma implies Hölder continuity, hence pf of DeGiorgi's Th is complete provided we prove:

$\boxed{\text{Oscillation Lemma}}$

Assume $u \in (W^{1,2} \cap L^\infty)(B_4)$ is a weak solution, $\operatorname{div} A(x) Du = 0$ in B_4.

If

(i) $|\{x \in B_1 : u(x) \leq 0\}| \geq \frac{1}{2} |B_1|$,

then

(ii) $\sup_{B_1} u^+ \leq C |\{x \in B_2 : u(x) > 0\}|^\alpha \sup_{B_4} u^+$,

where $c = c(n, \frac{L}{\lambda}) > 0$ and $\alpha = \alpha(n, \frac{L}{\lambda}) \in (0,1)$.

Our proof follows P. Tilli : Calc. Var. (2006) 25(3): 395–401.

$\boxed{\text{Lemma 34}}$ Modified Caccioppoli Inequality

Under the assumptions of De Giorgi's Theorem, if $B_R \subset \Omega$ then

$$\int_{B_r} |D(u-k)^+|^2 dx \leq \int_{\partial B_r} |Du| (u-k)^+ \, d\mathcal{H}^{n-1}$$

for all $k \in \mathbb{R}$ and a.e. $r \in (0,R)$.

<u>Note</u> Right-hand side is interpreted by use of precise representatives.

$\boxed{\text{Pf:}}$ Let $\rho(x) := \min\{1, \frac{r-|x|}{\varepsilon}\}$ $(\varepsilon > 0)$

Then $\rho \in W_0^{1,\infty}(B_r)$ and so $\varphi = \rho(u-k)^+ \in W_0^{1,2}(B_r)$, ie

$$0 = \int_{B_r} ADu \cdot D\varphi = \int_{B_r} \left(\rho \, ADu \cdot D(u-k)^+ + ADu \cdot D\rho \, (u-k)^+ \right)$$

$$\geq \ell \int_{B_r} \rho |D(u-k)^+|^2 - \int_{B_r} (u-k)^+ (ADu \cdot Du)^{\frac{1}{2}} (AD\rho \cdot D\rho)^{\frac{1}{2}}$$

$$\geq \ell \int_{B_r} \rho |D(u-k)^+|^2 - L \int_{B_r} (u-k)^+ |Du| |D\rho| ,$$

hence $\displaystyle\int_{B_r} \rho |D(u-k)^+|^2 \leq \frac{L}{\ell} \int_{B_r} (u-k)^+ |Du| |D\rho|$.

Note $D\rho = 0$ on $B_{r-\varepsilon}$ and $|D\rho| \leq \frac{1}{\varepsilon}$ a.e.,

so

$$\int_{B_r} \rho |D(u-k)^+|^2 \leq \frac{L}{\ell}\frac{1}{\varepsilon}\int_{B_r \setminus B_{r-\varepsilon}} (u-k)^+ |Du|$$

$$= \frac{L}{\ell}\frac{1}{\varepsilon}\int_{r-\varepsilon}^{r}\int_{\partial B_t} (u-k)^+ |Du| \, d\mathcal{H}^{n-1} \, dt$$

and using Lebesgue's differentiation theorem (one-sided version – pf the same) we conclude. □

Pf of Oscillation Lemma:

Assume $u \in (W^{1,2} \cap L^\infty)(B_4)$,

$$\circledast \quad \int_{B_4} ADu \cdot D\varphi = 0 \qquad \forall \varphi \in W^{1,2}_0(B_4)$$

and (i) $|\{x \in B_1 : u \leq 0\}| \geq \frac{1}{2}|B_1|$.

Aim to prove:

(ii) $\sup_{B_1} u^+ \leq C |\{x \in B_2 : u > 0\}|^\alpha \sup_{B_4} u^+$.

Let $k > 0$ and define

$$g(r) := \int_{B_{2-r}} |Du| (u - kr)^+ , \qquad r \in [0,2].$$

Note $g(r) \geq 0$, g is non-increasing,

$g(0) = \int_{B_2} |Du| u^+$ is independent of k

and g is AC.

For the latter notice that when $0 \leq \alpha < \beta \leq 2$, then

$$0 \leq g(\alpha) - g(\beta) = \int_{B_{2-\alpha} \setminus B_{2-\beta}} |Du| (u - k\alpha)^+$$

$$+ \int_{B_{2-\beta}} |Du| \underbrace{\left((u-k\alpha)^+ - (u-k\beta)^+ \right)}_{\leq k(\beta - \alpha)}$$

$$\leq \int_{2-\beta}^{2-\alpha} \left(\int_{\partial B_t} |Du| u^+ \, d\mathcal{H}^{n-1} \right) dt$$

$$+ \int_{B_2} |Du| \cdot k(\beta - \alpha),$$

and recall

$\boxed{\text{BACKGROUND RESULTS}}$ (see for instance section on Fundamental Theorem of Calculus in W. Rudin 'Real and complex analysis', pp. 144–149 in 3rd edition) :

$\boxed{\text{DEF:}}$ $f : [a,b] \to \mathbb{R}^d$ is absolutely continuous (AC) iff $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$\sum_{j \in J} |f(\beta_j) - f(\alpha_j)| < \varepsilon$$

whenever $\{(\alpha_j, \beta_j) : j \in J\}$ is a finite disjoint collection of open intervals contained in $[a,b]$ with $\sum_{j \in J} (\beta_j - \alpha_j) < \delta$.

$\boxed{\text{TH:}}$ Let $f : [a,b] \to \mathbb{R}^d$. Then f is AC iff f is diff. a.e. in $[a,b]$, $f' \in L^1[a,b]$ and

$$f(x) - f(a) = \int_a^x f'(t)\,dt$$

for all $x \in [a,b]$.

Note: $f \in W^{1,1}(a,b)$ iff the precise representative of f is AC on $[a,b]$.

To calculate $g'(r)$ at a.e. $r \in (0,2)$ we fix $0 < r < 2$ and $\delta > 0$ with $r + \delta < 2$.

Then $\quad -\dfrac{g(r+\delta) - g(r)}{\delta} =$

$$\frac{g(r) - g(r+\delta)}{\delta} = \frac{1}{\delta}\int_r^{r+\delta} \int_{\partial B_t} |Du| \left(u - k(r+\delta)\right)^+$$

$$+ \frac{1}{\delta}\int_{B_{2-r}} |Du| \left((u-kr)^+ - (u - k(r+\delta))^+ \right)$$

$$=: \mathrm{I} + \mathrm{II}.$$

Here $\mathrm{I} \to \int_{\partial B_r} |Du| (u - kr)^+$ as $\delta \searrow 0$

for a.e. r, and since

$$(u-kr)^+ - (u - k(r+\delta)^+) = \begin{cases} 0 & u \leq kr \\ u - kr & kr < u \leq k(r+\delta) \\ k\delta & u > k(r+\delta) \end{cases}$$

we get

$$\mathrm{II} = \int_{B_{2-r} \cap \{kr < u \leq k(r+\delta)\}} |Du| \frac{u-kr}{\delta}$$

$$+ \int_{B_{2-r} \cap \{u > k(r+\delta)\}} |Du| \cdot k$$

$$=: \mathrm{II}_1 + \mathrm{II}_2 .$$

$$|\mathrm{II}_1| \leq k \int_{B_{2-r} \cap \{kr < u \leq k(r+\delta)\}} |Du|$$

$$\to k \int_{B_{2-r} \cap \emptyset} |Du| = 0 \quad \text{as} \quad \delta \searrow 0 .$$

$$\mathrm{II}_2 \to k \int_{B_{2-r} \cap \{u > kr\}} |Du| = \int_{B_{2-r}} |D(u-k)^+|$$

as $\delta \searrow 0$.

Consequently : $\quad -g'(r) = A(r) + k B(r) \quad$ a.e.

where $\quad A(r) := \int_{\partial B_{2-r}} |Du| (u-kr)^+ ,$

$$B(r) := \int_{B_{2-r}} |D(u-k)^+| .$$

Put $m := \sup_{B_4} u^+$.

Then $g(r)^2 \overset{\text{Cauchy-Schwarz}}{\leq} \int_{B_{2-r}} |D(u-kr)^+|^2 \int_{B_{2-r}} ((u-kr)^+)^2$

$\leq m^{\frac{n-2}{n-1}} \int_{B_{2-r}} |D(u-kr)^+|^2 \int_{B_{2-r}} ((u-kr)^+)^{\frac{n}{n-1}} \overset{\text{Modified Caccioppoli ineq.}}{\leq}$

$c\, m^{\frac{n-2}{n-1}} A(r) \int_{B_{2-r}} ((u-kr)^+)^{\frac{n}{n-1}}$.

In view of hypothesis (i): For $0<r\leq 1$,

$|\{x \in B_{2-r} : (u-kr)^+ = 0\}| \geq$

$|\{x \in B_1 : u \leq 0\}| \geq \frac{1}{2}|B_1|$,

so for a constant $C = C(n, \frac{|B_2|}{|B_1|}) \equiv c(n)$ we have by a Poincaré-Sobolev inequality (see Th. 3.16 in Giusti) that

$$\int_{B_{2-r}} ((u-kr)^+)^{\frac{n}{n-1}} \leq c\left(\int_{B_{2-r}} |D(u-kr)^+|\right)^{\frac{n}{n-1}}$$

for $0<r\leq 1$. Hence

$$g(r)^2 \leq c\, m^{\frac{n-2}{n-1}} A(r) \left(\int_{B_{2-r}} |D(u-kr)^+|\right)^{\frac{n}{n-1}}$$
$$= c\, m^{\frac{n-2}{n-1}} A(r)\, B(r)^{\frac{n}{n-1}}.$$

Using Young's Inequality in the form

$$\alpha\beta \leq \frac{1}{p}\left(\frac{\alpha}{\varepsilon}\right)^{p} + \frac{1}{p'}\left(\varepsilon\beta\right)^{p'} \qquad \left(\begin{array}{c} \varepsilon,\alpha,\beta>0 \\ \frac{1}{p}+\frac{1}{p'}=1 \end{array}\right)$$

$$\leq \varepsilon^{-p}\alpha^{p} + \varepsilon^{p'}\beta^{p'}$$

with $\alpha = A(r)$, $\beta = B(r)^{\frac{n}{n-1}}$ and $1<p<\infty$ so $\frac{n}{n-1}p' = p$, ie $p = \frac{2n-1}{n-1}$,

we get

$$A(r)B(r)^{\frac{n}{n-1}} \leq \varepsilon^{-\frac{2n-1}{n-1}} A(r)^{\frac{2n-1}{n-1}} + \varepsilon^{\frac{2n-1}{n}} B(r)^{\frac{2n-1}{n-1}}$$

$$\leq \left(\varepsilon^{-1}A(r) + \varepsilon^{\frac{n-1}{n}} B(r)\right)^{\frac{2n-1}{n-1}}$$

$$= \left(\varepsilon^{-1}A(r) + \frac{\varepsilon^{\frac{n-1}{n}}}{k}\, kB(r)\right)^{\frac{2n-1}{n-1}}.$$

Choose $\varepsilon>0$ s.t. $\varepsilon^{-1} = \frac{\varepsilon^{\frac{n-1}{n}}}{k}$, that is,

$\varepsilon = k^{\frac{n}{2n-1}}$, to get

$$A(r)B(r)^{\frac{n}{n-1}} \leq k^{-\frac{n}{n-1}}\left(A(r) + kB(r)\right)^{\frac{2n-1}{n-1}}.$$

Consequently : $g(r)^2 \leq C\, m^{\frac{n-2}{n-1}}\, k^{-\frac{n}{n-1}}\left(-g'(r)\right)^{\frac{2n-1}{n-1}}$

for a.e. $r \in (0,1]$.

For a.e. $r \in (0,1]$ with $g(r) > 0$ we can rewrite this as

$$\left(\frac{k^{\frac{n}{n-1}}}{c\, m^{\frac{n-2}{n-1}}} \right)^{\frac{n-1}{2n-1}} \leq - \frac{g'(r)}{g(r)^{\frac{2(n-1)}{2n-1}}} .$$

If $g(r) > 0$ for $r \in [0, r_0]$, then we have by FTC:

$$\left(\frac{k^{\frac{n}{2n-1}}}{c^{\frac{n-1}{2n-1}}\, m^{\frac{n-2}{2n-1}}} \right) r_0 \leq - \int_0^{r_0} \frac{g'(t)}{g(t)^{\frac{2n-2}{2n-1}}}\, dt$$

$$= (2n-1) \left(g(0)^{\frac{1}{2n-1}} - g(r_0)^{\frac{1}{2n-1}} \right),$$

or

$$\tilde{c}\, m^{-\frac{n-2}{2n-1}}\, k^{\frac{n}{2n-1}}\, r_0 \leq g(0)^{\frac{1}{2n-1}} - g(r_0)^{\frac{1}{2n-1}},$$

and therefore:

$$0 < g(r_0)^{\frac{1}{2n-1}} \leq g(0)^{\frac{1}{2n-1}} - \tilde{c}\, m^{-\frac{n-2}{2n-1}}\, r_0 \cdot k^{\frac{n}{2n-1}}.$$

But then

$$k < \left(g(0)^{\frac{1}{2n-1}} \frac{m^{\frac{n-2}{2n-1}}}{\tilde{c}\, r_0} \right)^{\frac{2n-1}{n}}$$

ie

$$k < c\left(\int_{B_2} |Du|\, u^+\right)^{\frac{1}{n}} m^{\frac{n-2}{n}}\, r_0^{-\frac{2n-1}{n}} .$$

If we therefore take

$$k := c\left(\int_{B_2} |Du|\, u^+\right)^{\frac{1}{n}} m^{\frac{n-2}{n}}$$

we see that necessarily $r_0 < 1$, ie we must have $g(1) = 0$. This amounts to $\int_{B_1} |Du|\,(u-k)^+ = 0$

and as $|Du|\,(u-k)^+ = \frac{1}{2}\left|D\left[((u-k)^+)^2\right]\right|$ we get $D\left[((u-k)^+)^2\right] = 0$ a.e. in B_1, ie $(u-k)^+ =$ constant $\equiv c_0$ a.e. in B_1.

If $c_0 > 0$ then $u = c_0 + k$ a.e. in B_1 and so $|\{x \in B_1 : u \leq 0\}| = 0$ contradicting hypothesis (i). Thus $c_0 = 0$ and therefore $u \leq k$ a.e. in B_1,

ie
$$\sup_{B_1} u^+ \leq k = c\, m^{\frac{n-2}{n}} \left(\int_{B_2} |Du|\, u^+\right)^{\frac{1}{n}}$$

$u^+ \leq m$

$$\leq \quad C\, m^{\frac{n-1}{n}} \left(\int_{B_2} |Du^+| \right)^{\frac{1}{n}}$$

Hölder

$$\leq \quad C\, m^{\frac{n-1}{n}} \left|\{x \in B_2 : u > 0\}\right|^{\frac{1}{2n}} \left(\int_{B_2} |Du^+|^2 \right)^{\frac{1}{2n}}$$

Caccioppoli inequality
on super-level set
(Lemma 31(i) with $r=2, R=4$)

$$\leq \quad \tilde{C}\, m^{\frac{n-1}{n}} \left|\{x \in B_2 : u > 0\}\right|^{\frac{1}{2n}} \left(\int_{B_4} |u^+|^2 \right)^{\frac{1}{2n}}$$

$$\leq \quad C \left|\{x \in B_2 : u > 0\}\right|^{\frac{1}{2n}} m. \quad \square$$

$\boxed{\text{Example 35}}$ (De Giorgi 1968)

De Giorgi's theorem (TH 29) is not true in the multi-dimensional vectorial case $n = N > 2$.

Fix $n = N > 2$. Define

$$u(x) = |x|^{-\mu} x, \quad x \in B(0,1) \quad \left(\begin{array}{c}\text{open unit} \\ \text{ball in } \mathbb{R}^n\end{array}\right)$$

where

$$\mu = \frac{n}{2}\left(1 - \frac{1}{\sqrt{(2n-2)^2+1}}\right).$$

Note $u(x) = x$ on $\partial B(0,1)$.

Then u is positively $(1-\mu)$-homogeneous and as $\mu < \frac{n}{2}$, $u \in W^{1,2}_{id}(B(0,1), \mathbb{R}^n)$.

Since $\mu > 1$ precisely when $n > 2$ we also have $u \notin L^\infty_{loc}(B(0,1), \mathbb{R}^n)$.

Let $\mathcal{F}[v, B(0,1)] = \int_{B(0,1)} \mathbb{A}(Dv, Dv)\, dx$,

where $\mathbb{A} = \mathbb{A}(x) : B(0,1) \to \mathcal{L}^2(\mathbb{R}^{n\times n})$ is given by (for $x \neq 0$)

$$\mathbb{A}(x)[\lambda, \lambda] = |\lambda|^2 + \left(\left((n-2)\mathbb{I} + n\frac{x\otimes x}{|x|^2}\right)\cdot\lambda\right)^2.$$

Note $\mathbb{A}$ is measurable, bounded and smooth away from $0 \in \mathbb{R}^n$. Also

$$|\lambda|^2 \leqq \mathbb{A}(x)[\lambda, \lambda] \leqq n^2(n-1)|\lambda|^2$$

for all $\lambda \in \mathbb{R}^{n\times n}$ and all $x \in B(0,1)\setminus\{0\}$.

Consequently there exists a unique minimizer for $\mathcal{F}[\cdot, B(0,1)]$ on $W^{1,2}_{id}(B(0,1), \mathbb{R}^n)$ and it coincides with the unique solution $v \in W^{1,2}_{id}$ to the Euler-Lagrange equation

$$\text{(EL)} \quad \int_{B(0,1)} A[Dv, D\varphi] = 0 \quad \forall \varphi \in W^{1,2}_0 .$$

Let

$$A^{ij}_{k\ell} = A^{ij}_{k\ell}(x) = \delta_{k\ell}\delta_{ij} + \left((n-2)\delta_{ki} + n\frac{x_k x_i}{|x|^2}\right) \times \left((n-2)\delta_{\ell j} + n\frac{x_\ell x_j}{|x|^2}\right).$$

Then (using summation convention) we can write (EL) as

$$\int_{B(0,1)} A^{ij}_{k\ell} D_i v_k D_j \varphi_\ell = 0 \quad \forall \varphi \in W^{1,2}_0 .$$

Because $u \in C^\infty(B(0,1)\setminus\{0\}, \mathbb{R}^n)$ a computation yields for each $1 \le \ell \le n$,

$$D_j(A^{ij}_{k\ell} D_i u_k) = 0 \quad \text{in } B(0,1)\setminus\{0\},$$

hence (EL) holds for u when $\varphi \in W^{1,2}_0$ is supported on $B(0,1)\setminus\{0\}$.

Now let $\varphi \in C_c^1(B(0,1), \mathbb{R}^n)$ and take $\rho \in C^1(B(0,1))$ satisfying

$$0 \leq \rho \leq 1, \quad \rho = 0 \text{ in } B_\varepsilon,$$

$$\rho = 1 \text{ in } B \setminus B_{2\varepsilon} \text{ and } |D\rho| \leq \frac{2}{\varepsilon},$$

where $0 < \varepsilon < 1$. Since $\varphi\rho$ is supported in $B \setminus \{0\}$ we have

$$0 = \int_{B(0,1)} \mathbb{A}[Du, D(\varphi\rho)]\,dx$$

$$= \int_{B(0,1)} \rho\,\mathbb{A}[Du, D\varphi]\,dx$$

$$+ \int_{B(0,1)} \mathbb{A}[Du, \varphi \otimes D\rho]\,dx =: I + II.$$

Clearly $I \longrightarrow \int_{B(0,1)} \mathbb{A}[Du, D\varphi]$ as $\varepsilon \searrow 0$.

For II we estimate using Cauchy-Schwarz:

$$|II| \leq \int_{B(0,1)} \mathbb{A}[Du, Du]^{\frac{1}{2}}\, \mathbb{A}[\varphi \otimes D\rho, \varphi \otimes D\rho]^{\frac{1}{2}}$$

$$\leq n^2(n-1) \int_{B(0,1)} |Du| \cdot |\varphi| \cdot |D\varphi|$$

$$\leq n^2(n-1) \|\varphi\|_{L^\infty} \int_{B_{2\varepsilon} \setminus B_\varepsilon} |Du| \frac{2}{\varepsilon}$$

$$\leq 2n^2(n-1) \|\varphi\|_{L^\infty} \frac{1}{\varepsilon} |B_{2\varepsilon} \setminus B_\varepsilon|^{\frac{1}{2}} \left(\int_B |Du|^2 \right)^{\frac{1}{2}}$$

$$\longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.$$

$\therefore$ $u \in W^{1,2}_{id}(B(0,1), \mathbb{R}^n)$ is the unique solution to EL (and the unique $W^{1,2}_{id}$ minimizer of $\mathcal{F}[\cdot, B(0,1)]$).

<u>Note</u> An example of a singular solution to a higher order linear elliptic equation was given simultaneously by V. Maz'ya.

Remarks 36 Further related examples

- Soucek (1984), John, Malý & Stará (1989)

When $n, N > 2$ there exists $\mathbb{A} = \mathbb{A}(x) \in \mathcal{B}(\ell, L)$ s.t. $\operatorname{div} \mathbb{A} Du = 0$ admits a nowhere continuous solution.

- Giusti & Miranda (1968)

When $n = N > 2$ is sufficiently large and

$$\mathbb{A}(v)[\lambda, \lambda] = |\lambda|^2 + \left(\left(\mathbb{I} + \frac{4}{n-2} \frac{v \otimes v}{1 + |v|^2} \right) \cdot \lambda \right)^2,$$

then $u(x) = \frac{x}{|x|}$, $x \in B(0,1)$, is the unique $W^{1,2}_{id}(B(0,1), \mathbb{R}^n)$ minimizer of

$$\mathcal{F}[v, B(0,1)] = \int_{B(0,1)} \mathbb{A}(v(x))[Dv, Dv]\, dx .$$

Note: u bounded, but discontinuous (at 0), and coefficients $\mathbb{A}$ are smooth <u>but</u>

depend on $v(x)$ (that as a function of x isn't necessarily continuous).

$\boxed{\text{Examples 37}}$

$$\mathcal{F}(v,\Omega) = \int_\Omega F(Dv) \quad , \quad v : \Omega \to \mathbb{R}^N$$

a regular variational integral.

Minimizers can be:

- Nečas (1975): Lipschitz but not C^1 when $n \geqslant 25$, $N \geqslant 625$
- Hao, Leonardi & Nečas (1996): Lipschitz but not C^1 when $n \geqslant 5$, $N \geqslant 25$
- Šverák & Yan (2000, 2002):
  * non-Lipschitz when $n \geqslant 3, N \geqslant 5$
  * unbounded when $n \geqslant 5, N \geqslant 14$