Overview & Plan for Lectures

Aim: To give an introduction to regularity theory in the context of Calculus of Variations.

Prerequisites:

- familiarity with measure theory, Lebesgue spaces and strong & weak convergence
- familiarity with Sobolev spaces

Further background material will be recalled as we go along if necessary.

Paradigm:

Weyl's Lemma (TH 1)

A weakly harmonic function is harmonic and C^∞-smooth (even real analytic).

$\Omega \subset \mathbb{R}^n$ open bounded domain

The Dirichlet integral for mappings $u : \Omega \to \mathbb{R}^N$ of Sobolev class $W^{1,2}$ is

$$\boxed{\mathrm{Dir}[u,\Omega] := \int_\Omega |Du(x)|^2\, dx} \quad (1)$$

Here $Du(x) = \left\{ \frac{\partial u^r}{\partial x^s}(x) \right\}_{1 \le s \le n}^{1 \le r \le N} \in \mathbb{R}^{N \times n}$

and the norm is the usual euclidean.

u is weakly harmonic iff it's a minimizer for $\mathrm{Dir}[\cdot,\Omega]$ subject to its own boundary values:

$$\mathrm{Dir}[u,\Omega] \leqq \mathrm{Dir}[v,\Omega] \qquad (2)$$

for all $v \in W^{1,2}_u(\Omega, \mathbb{R}^N)$.

Because $\mathrm{Dir}[\cdot,\Omega]$ is <u>convex</u> minimality (2) is equivalent to being a weak solution to the

$\boxed{\text{Euler–Lagrange equation}}$

$$\boxed{\operatorname{div} Du = \Delta u = 0 \text{ in } \Omega} \qquad (3)$$

(3) is of course Laplace's equation, in coordinates:

$$\begin{cases} u^1_{x_1x_1} + \cdots + u^1_{x_nx_n} = 0 \\ \vdots \qquad\qquad\qquad \vdots \\ u^N_{x_1x_1} + \cdots + u^N_{x_nx_n} = 0 \end{cases} \quad \text{in } \Omega$$

(3) is understood as

$$\int_\Omega Du \cdot D\varphi \, dx = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N).$$

In this course we intend to generalize Weyl's Lemma to minimizers of <u>regular variational integrals</u> (essentially Hilbert's 19th Problem):

$$\mathcal{F}[u, \Omega] := \int_\Omega F(Du(x)) \, dx \qquad (4)$$

is a regular variational iff $F : \mathbb{R}^{N \times n} \to \mathbb{R}$ is C^2 and there exists constants $0 < \ell \leq L < \infty$ such that

$$\boxed{\ell |\lambda|^2 \leq F''(\xi)[\lambda, \lambda] \leq L |\lambda|^2} \qquad (5)$$

for all $\xi, \lambda \in \mathbb{R}^{N \times n}$.

Again, because $\mathcal{F}[\cdot,\Omega]$ is convex (even strongly convex) being an $\mathcal{F}[\cdot,\Omega]$ minimizer is equivalent to being a weak solution to the

$\boxed{\text{Euler-Lagrange equation for } \mathcal{F}[\cdot,\Omega]}$

$$\boxed{\operatorname{div} F'(Du) = 0 \quad \text{in } \Omega} \tag{6}$$

(6) means: $\displaystyle\int_\Omega F'(Du) \cdot D\varphi \, dx = 0$

$$\forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N).$$

Note: N coupled nonlinear PDE of 2^{nd} order (in divergence form).

Weak solutions to the Euler-Lagrange equation are often called <u>extremals</u>.

We shall see that there is a big difference between the cases

$N = 1$ (1 unknown function, 'the scalar case')

$N > 1$ (more than 1 unknown function, 'the vectorial case')

We always assume that $n \geqslant 2$ (still $n = 2$ is special).

Tentative schedule

L1 & 2: - Harmonic functions & Weyl's Lemma
- Difference-quotient method
- Existence of minimizers by the Direct Method

- Discussion of conditions in scalar case

- Extending the scope: quasiminimizers

L3&4: Next week

- Quasiminimizers
- Hölder continuity; theorems of Campanato & Morrey, and embedding theorems, Oscillation Lemma
- Schauder estimates for linear elliptic systems in divergence form

Intend to cover during lectures:

- Campanato's regularity results for minimizers of (regular) variational problems
- DeGiorgi (-Nash-Moser) theorem
- Counter-examples to full regularity
- Integral estimates & Gehring's Lemma

Focus in course on interior regularity.

Hence it's less important what Ω is and we will occasionally assume it's an open ball or cube in $\mathbb{R}^n$.

Assume $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is harmonic, that is:

$$\int_\Omega |Du|^2 \leqq \int_\Omega |Dv|^2 \quad \forall v \in W^{1,2}_u \qquad (7)$$

Writing $v = u + \varphi$ for $\varphi \in W^{1,2}_0$ we see that (7) is equivalent to

$$\int_\Omega Du . D\varphi = 0 \quad \forall \varphi \in W^{1,2}_0 \qquad (8)$$

(Laplace's equation in weak or variational form.)

We shall prove Weyl's Lemma by way of the $\boxed{\text{Difference-quotient Method}}$

(Due to Shiffman and Nirenberg, 1940-50s.)

Background result

Let $f : \Omega \to \mathbb{R}^N$, $e_1, \dots, e_n$ canonical basis for $\mathbb{R}^n$ and $h \in \mathbb{R}$.

For $x \in \Omega$ such that $x + he_s \in \Omega$ define

$$\Delta_{s,h} f(x) := f(x + he_s) - f(x).$$

$\boxed{\text{TH 2}}$ (Difference-quotients characterization of $W^{1,p}_{loc}$): Let $1 < p < \infty$.

If $f \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ and $\Omega' \Subset \Omega'' \Subset \Omega$, then

$$\| \Delta_{s,h} f \|_{L^p(\Omega')} \leq \| D_s f \|_{L^p(\Omega'')} |h| \quad (9)$$

for $|h| < \mathrm{dist}(\Omega', \partial\Omega'')$ and $1 \leq s \leq n$

Conversely, if $f \in L^p_{loc}(\Omega, \mathbb{R}^N)$, $\Omega' \Subset \Omega$ and for some $0 < \delta < \mathrm{dist}(\Omega', \partial\Omega)$ and $M < \infty$ we have

$$\| \Delta_{s,h} f \|_{L^p(\Omega')} \leq M |h| \quad (10)$$

for $|h| < \delta$ and $1 \leq s \leq n$, then $f \in W^{1,p}(\Omega', \mathbb{R}^N)$ and $\| D_s f \|_{L^p(\Omega')} \leq M$, $1 \leq s \leq n$.

<u>Note</u> Conditions of the form

$$\| \Delta_{s,h} f \|_{L^p(\Omega')} \leq M |h|^\alpha$$

are called <u>Nikolskii conditions</u>. Can be used to define fractional order Sobolev spaces.

$\boxed{\text{Pf}}$ '$\Rightarrow$' Use FTC for the case of C^1 functions, then approximate general $W^{1,p}$ functions by mollification.

'$\Leftarrow$' Fix e_s. By assumption $(j \Delta_{s,\frac{1}{j}} f)$ is a bounded sequence in $L^p(\Omega')$ so by reflexivity ($1<p<\infty$!) it has a weakly convergent subsequence (still denoted by full sequence for convenience)

$$j \Delta_{s,\frac{1}{j}} f \longrightarrow f_s \quad \text{weakly in } L^p(\Omega').$$

Hence for $\varphi \in C_c^\infty(\Omega')$: $\int_{\Omega'} f_s(x)\varphi(x)\,dx =$

$$\lim_{j\to\infty} \int_{\Omega'} j\Delta_{s,\frac{1}{j}} f(x)\,\varphi(x)\,dx = \lim_{j\to\infty} \int_{\Omega'} f(x) . j\left(\varphi(x - \tfrac{1}{j}e_s) - \varphi(x)\right) dx = -\int_{\Omega'} f(x)\, D_s\varphi(x)\,dx.$$

By definition of weak derivative

$$D_s f = f_s \in L^p(\Omega'),$$

hence $f \in W^{1,p}(\Omega')$. (The inequality $\|D_s f\|_{L^p(\Omega')} \leq M$ follows too.)

$\boxed{\text{Pf of Weyl's Lemma}}$

Fix $\rho \in C_c^1(\Omega)$ with $\rho \geqslant 0$.

For $h \in \mathbb{R}$ with $|h| < \operatorname{dist}(\operatorname{spt}\rho, \partial\Omega)$,

$$\varphi := \Delta_{s,-h}(\rho^2 \Delta_{s,h} u) \in W_0^{1,2}(\Omega, \mathbb{R}^N)$$

hence from (8):

$$0 = \int_\Omega \Delta_{s,h} Du \cdot D(\rho^2 \Delta_{s,h} u)$$

$$= \int_\Omega \left(|\Delta_{s,h} Du|^2 \rho^2 + \Delta_{s,h} Du \cdot \Delta_{s,h} u \otimes D(\rho^2) \right).$$

$\boxed{\text{Notation}}$ If $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$, then

$$a \otimes b = \{a^r b^s\}_{1 \leq s \leq n}^{1 \leq r \leq N} \in \mathbb{R}^{N \times n}.$$

Estimate

Cauchy-Schwarz

$$|\Delta_{s,h} Du \cdot \Delta_{s,h} u \otimes D(\rho^2)| \leq$$

$$2\rho |\Delta_{s,h} Du| \cdot |\Delta_{s,h} u| \cdot |D\rho| \leq$$

$$\tfrac{1}{2} \rho^2 |\Delta_{s,h} Du|^2 + 2 |\Delta_{s,h} u|^2 |D\rho|^2 .$$

Hereby

$$\int_\Omega \rho^2 |\Delta_{s,h} Du|^2 \leq 4 \int_\Omega |\Delta_{s,h} u|^2 |D\rho|^2$$

Take $B_{3r} \subset \Omega$, $\mathbb{1}_{B_r} \leq \rho \leq \mathbb{1}_{B_{2r}}$ and $|D\rho| \leq \frac{2}{r}$ to get for $|h| < r$,

$$\int_{B_r} |\Delta_{s,h} Du|^2 \leq \left(\frac{4}{r}\right)^2 \int_{B_{2r}} |\Delta_{s,h} u|^2$$

Invoke TH2 (both ways!) to get

$$\int_{B_r} |D_s Du|^2 \leq \frac{c}{r^2} \int_{B_{3r}} |Du|^2 ,$$

hence $Du \in W^{1,2}_{loc}$.

Note: We can add up over $1 \leq s \leq n$ to get

$$\int_{B_r} |D^2 u|^2 \leq \frac{c}{r^2} \int_{B_{3r}} |Du|^2 \quad (11)$$

which is an example of a <u>Caccioppoli Inequality</u>.

In particular, $D_s u \in W^{1,2}_{loc}$ for each $1 \leq s \leq n$. Fix $\Omega' \Subset \Omega$ and $\psi \in C_c^\infty(\Omega', \mathbb{R}^N)$. Put $\varphi = D_s \psi$ in (8) and integrate by parts:

$$\int_{\Omega'} D(D_s u) \,.\, D\psi = 0.$$

Because $D(D_s u) \in L^2(\Omega', \mathbb{R}^{N \times n})$ and $C_c^\infty(\Omega', \mathbb{R}^N)$ is dense in $W_0^{1,2}(\Omega', \mathbb{R}^N)$ we deduce

$$\int_{\Omega'} D(D_s u) \,.\, D\varphi = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega', \mathbb{R}^N)$$

$\therefore$ $D_s u$ is weakly harmonic.

By induction $u \in W^{k,2}_{loc}$ for all $k \in \mathbb{N}$ and hence

$$u \in \bigcap_{k \in \mathbb{N}} W^{k,2}_{loc} = C^\infty . \qquad \square$$

Note: By Sobolev's embedding theorem

$W^{k,2}_{loc} \hookrightarrow C^j$ with $j = \lfloor k - \frac{n}{2} \rfloor$

provided that $n < 2k$.

Assume $\mathcal{F}[u,\Omega] = \int_\Omega F(Du)$

regular variational integral and let

$g \in W^{1,2}(\Omega, \mathbb{R}^N)$.

Consider the <u>regular variational problem</u>

$$(*) \begin{cases} \text{Minimize } \mathcal{F}[u,\Omega] \text{ over the} \\ \text{Dirichlet class } W^{1,2}_g(\Omega, \mathbb{R}^N). \end{cases}$$

- Existence of minimizers?
(uniqueness?)
- Regularity?

Direct method of the calculus of variations

Based on

- coercivity
- (sequential) lower semicontinuity (lsc)

Recall

$$W^{1,2} = W^{1,2}(\Omega, \mathbb{R}^N) = \{u \in L^2 : D_s u \in L^2, 1 \leq s \leq n\}$$

is a separable Hilbert space.

— The weak topology has good compactness properties: any closed ball ~~the~~ is sequentially compact in the weak topology.

— (u_j) sequence in $W^{1,2}$

$u_j \rightharpoonup u$ weakly in $W^{1,2}$ iff

$u_j \rightharpoonup u$ & $Du_j \rightharpoonup Du$ weakly in L^2.

Discussion of conditions for existence

Assume $G : \mathbb{R}^{N\times n} \to \mathbb{R}$ is Borel and

$$|G(\xi)| \leq c(|\xi|^2 + 1) \quad \forall \xi \qquad (12)$$

Note: $\int_\Omega G(Du)$ well-defined for all $u \in W^{1,2}$ when (12) holds.

$\boxed{\text{TH 3}}$ If G is convex, then $u \mapsto \int_\Omega G(Du)$ is weakly lsc on $W^{1,2}(\Omega, \mathbb{R}^N)$.

Moreover $\boxed{\text{when } N=1}$: Let $g \in W^{1,2}(\Omega)$.

$u \mapsto \int_\Omega G(Du)$ is weakly (seq.) lsc on $W^{1,2}_g(\Omega)$ iff G is convex.

$\boxed{\text{Pf}}$ Exercise (see for instance Dacorogna's book 'Direct Methods in CoV'). □

Note: When $N>1$ "convexity" should be replaced by "quasiconvexity". We shall not discuss quasiconvexity at this point.

Coercivity?

— that is, do there exist positive constants $c_1, c_2 > 0$ such that

$$\int_\Omega G(Du) + c_1 \int_\Omega (1+|g|^2 + |Dg|^2) \geqslant c_2 \int_\Omega |Du|^2 \quad (13)$$

for all $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$?

"Gårding Inequality"

(13) holds trivially when for some $\varepsilon > 0$

$$G(\xi) \geqslant \varepsilon |\xi|^2 - \frac{1}{\varepsilon} \quad \forall \xi \qquad (14).$$

Note that (14) implies that

$$\xi \mapsto G(\xi) - \varepsilon' |\xi|^2$$

is convex <u>at some $\xi_0 \in \mathbb{R}^{N \times n}$</u> when $0 < \varepsilon' < \varepsilon$

(Pf is an exercise. HINT: Consider the

convex envelope of $G - \varepsilon' |\cdot|^2$.)

$\boxed{\text{TH 4}}$ Assume $N = 1$.

(8) holds iff for some $\varepsilon > 0$

$$\xi \longmapsto G(\xi) - \varepsilon|\xi|^2 \qquad (15)$$

is convex at some $\xi_0 \in \mathbb{R}^N$.

$\boxed{\text{Sketch of Pf}}$ $\Leftarrow$ If $G - \varepsilon|\cdot|^2$ is convex at ξ_0, then

$$G(\xi) - \varepsilon|\xi|^2 \geqslant a(\xi) \qquad \forall \xi$$

for an affine function $a : \mathbb{R}^N \to \mathbb{R}$.

But $|a(\xi)| \leqslant \frac{\varepsilon}{2}|\xi|^2 + C_\varepsilon \ \ \forall \xi$ and so (13) follows.

$\Rightarrow$ If (13) holds then by relaxation formula the convex envelope of

$$\xi \mapsto G(\xi) - \frac{c_2}{2}|\xi|^2$$

cannot be identically $-\infty$. Hence it must be real-valued and then (15) must hold (details omitted). $\square$

Summarizing : when $N=1$

$\Rightarrow$ $\int_\Omega G(Du)$ weakly lsc iff G convex

$\Rightarrow$ $\int_\Omega G(Du)$ coercive iff for some $\varepsilon>0$

$G - \varepsilon|\cdot|^2$ convex at some ξ_0

$\boxed{\text{DEF 5}}$ $G : \mathbb{R}^{N\times n} \to \mathbb{R}$ is strongly convex iff $G - \varepsilon|\cdot|^2$ is convex for some $\varepsilon>0$.

Note : When $G \in C^2$ then $G - \varepsilon|\cdot|^2$ is strongly convex iff

$$G''(\xi)[\lambda,\lambda] \geqslant 2\varepsilon|\lambda|^2$$

for all $\xi, \lambda \in \mathbb{R}^{N\times n}$.

Hence the regular variational problem $(*)$ admits a solution.

It's unique and coincides with the unique solution to the Euler-Lagrange equation

$$(**)\quad \begin{cases} \operatorname{div} F'(Du) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

<u>Regularity?</u>

$\boxed{\text{TH 6}}$ Let $u \in W^{1,2}_g(\Omega, \mathbb{R}^N)$ be the $\mathcal{F}$-minimizer (= solution of $(**)$).

Then $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$ and for $B_{3r} \subset \Omega$,

$$\int_{B_r} |D^2 u|^2 \leq \frac{c}{r^2} \int_{B_{3r}} |Du|^2 . \qquad (16)$$

Furthermore, for each $1 \leq s \leq n$,

$$\int_{\Omega} F''(Du)\,[D D_s u, D\varphi] = 0 \qquad (17)$$

for all $\varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N)$ with $\operatorname{dist}(\operatorname{spt}\varphi, \partial\Omega) > 0$.

$\boxed{\text{DEF } 7}$ Let $Q \geqslant 1$. Then $w \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a Q-quasiminimizer (briefly: Q-minimizer) for $\mathcal{F}[\cdot, \Omega]$ iff

$$\mathcal{F}[w, \Omega'] \leqq Q\,\mathcal{F}[w+\varphi, \Omega'] \qquad (18)$$

for all $\varphi \in W_0^{1,2}(\Omega', \mathbb{R}^N)$ and all $\Omega' \Subset \Omega$, open subsets.

Note: From (5) and (17) it follows that $D_s u$ $(1 \leqq s \leqq n)$ is a Q-minimizer for $\mathrm{Dir}[\cdot, \Omega]$ when $Q = \left(\frac{L}{\ell}\right)^2$:

Fix an open subset $\Omega' \subset \Omega$ and let $v \in W^{1,2}_{D_s u}(\Omega', \mathbb{R}^N)$ be harmonic.

Then taking $\varphi = D_s u - v$ in (17) and rearranging:

$$\ell \int_{\Omega'} |DD_s u|^2 \leqq \int_{\Omega'} F''(Du)[DD_s u, DD_s u]$$

$$= \int_{\Omega'} F''(Du)[DD_s u, Dv] \overset{\text{Cauchy-Schwarz}}{\leqq}$$

$$\int_{\Omega'} F''(Du)[DD_s u, DD_s u]^{\frac{1}{2}}\, F''(Du)[Dv, Dv]^{\frac{1}{2}}$$

$$\leq L \int_{\Omega'} |DD_s u|\,|Dv|$$

and thus $\quad \mathrm{Dir}[D_s u, \Omega'] \leq \left(\frac{L}{\ell}\right)^2 \mathrm{Dir}[v, \Omega']$.

$\boxed{\text{Pf of TH6}}$ We start from $(**)$.

Fix $1 \leq s \leq n$, $B_{3r} \subset \Omega$ and Lipschitz cut-off $\rho \in W^{1,\infty}$ such that

$$\mathbb{1}_{B_r} \leq \rho \leq \mathbb{1}_{B_{2r}} \quad \text{and} \quad |D\rho| \leq \frac{1}{r} \text{ a.e.}$$

For $|h| < r$, $\quad \varphi = \Delta_{s,-h}(\rho^2 \Delta_{s,h} u) \in W_0^{1,2}$

and hence from $(**)$

$$0 = \int_\Omega \Delta_{s,h} F'(Du) \cdot \left(\rho^2 \Delta_{s,h} Du + \Delta_{s,h} u \otimes D\rho^2\right).$$

Since $\quad |F'(\xi) - F'(\eta)| \leq L|\xi - \eta|$ and

$$(F'(\xi) - F'(\eta)) \cdot (\xi - \eta) = \int_0^1 F''(\eta + t(\xi - \eta))[\xi - \eta, \xi - \eta]\, dt \overset{15)}{\geq} \ell |\xi - \eta|^2$$

we get

$$\ell \int_\Omega \rho^2 |\Delta_{s,h} Du|^2 \leq 2L \int_\Omega |\Delta_{s,h} Du|\,|\Delta_{s,h} u|\,|D\rho|$$

hence

$$\int_{B_r} |\Delta_{s,h} Du|^2 \leq \left(\frac{2L}{\ell}\right)^2 \frac{1}{r^2} \int_{B_{2r}} |\Delta_{s,h} u|^2$$

and so invoking TH 2 we conclude that $u \in W^{2,2}_{loc}$ and (16) holds.

(17) follows by taking $\varphi = D_s \psi$ and integration by parts. $\square$