Let $\mathcal{F}[v,\Omega] = \int_\Omega F(Dv)$ be a regular variational integral:

- $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ C^2
- $\ell|\lambda|^2 \leq F(\xi)[\lambda,\lambda] \leq L|\lambda|^2 \quad \forall \xi, \lambda$

When $N=1$ (scalar case) minimizers of $\mathcal{F}[\cdot,\Omega]$ are regular (Ladyzhenskaya & Ural'tseva based on DeGiorgi-Nash-Moser)

When $N>1$ (vectorial case) minimizers need not be regular (everywhere) (Nečas, ...)

$\boxed{\text{TH 38}}$ (Campanato)

Minimizers of regular variational problems are locally Hölder continuous when $n \leq 4$ (any $N \in \mathbb{N}$).

Note: Šverák & Yan's examples show that this isn't true when $n>4$.

For the pf of TH 38 we recall that if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a minimizer for $\mathcal{F}[\cdot, \Omega]$, then $u \in W^{2,2}_{loc}(\Omega, \mathbb{R}^N)$ and

$$\int_\Omega F''(Du)[D(D_s u), D\varphi] = 0 \quad \forall \varphi \in C^1_c$$

for $1 \leq s \leq n$ (see TH 6) and hence using the bounds on F'' (see Note following DEF 7) it follows that $D_s u$ is a Q-minimizer for $\mathrm{Dir}[\cdot, \Omega]$:

$$\int_{\operatorname{spt}\varphi} |D(D_s u)|^2 \leq Q \int_{\operatorname{spt}\varphi} |D(D_s u) + D\varphi|^2$$

for all $\varphi \in W^{1,2}(\Omega, \mathbb{R}^N)$ with $\operatorname{spt}\varphi$ compact and contained in Ω, where $Q = \left(\frac{L}{\ell}\right)^2$.

We can therefore conclude the pf by use of Morrey's embedding theorem and

$\boxed{\text{TH 39}}$ (Ziemer)

Let $Q \geq 1$ and assume $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ is a Q-minimizer for $\mathrm{Dir}[\cdot, \Omega]$. Then $r \mapsto r^{-\frac{n-1}{Q}} \int_{B(x_0, r)} |Du|^2 \, dx$ is nondecreasing on $(0, \mathrm{dist}(x_0, \partial\Omega))$.

In particular, $Du \in L^{2,\frac{n-1}{Q}}_{loc}(\Omega, \mathbb{R}^{N\times n})$, and when $n=2$, $u \in C^{0,\frac{1}{2Q}}_{loc}(\Omega, \mathbb{R}^{N\times 2})$.

$\boxed{\text{Pf of TH 38:}}$ (Based on TH 39.)

We have $|D^2u| \in L^{2,\frac{n-1}{Q}}_{loc}(\Omega)$ and hence by Poincaré's inequality $Du \in L^{2,2+\frac{n-1}{Q}}_{loc}(\Omega, \mathbb{R}^{N\times n})$

Now if $\alpha = \dfrac{2+(2+\frac{n-1}{Q})-n}{2} > 0$, ie

$n < 4 + \frac{n-1}{Q}$ (always true when $n \leq 4$)

that is, $n < \dfrac{4-\frac{1}{Q}}{1-\frac{1}{Q}} \overset{Q=(\frac{L}{l})^2}{=} \dfrac{4-(\frac{l}{L})^2}{1-(\frac{l}{L})^2}$,

we have by Morrey's embedding theorem that $u \in C^{0,\alpha}_{loc}(\Omega, \mathbb{R}^N)$. $\square$

<u>Note</u> Minimizers are Hölder continuous provided $\frac{L}{l} < \left(\frac{n-1}{n-4}\right)^{\frac{1}{2}}$ when $n \geq 5$.

We turn to the proof of TH 39 and remark that it only requires that

$u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ be a spherical Q-minimizer:

for all balls $B = B(x_0, R) \Subset \Omega$

$$\int_B |Du|^2 \leqq Q \int_B |Du + D\varphi|^2 \quad \forall \varphi \in W^{1,2}_0(B, \mathbb{R}^N).$$

Note This class is strictly smaller than the class of Q-minimizers (see discussion in Giusti's book 'Direct methods in the CoV' pp. 188-189).

We require some preliminary results that happen to be of independent interest too.

$\boxed{\text{TH } 40}$ (Weyl's Lemma 'up to the boundary')

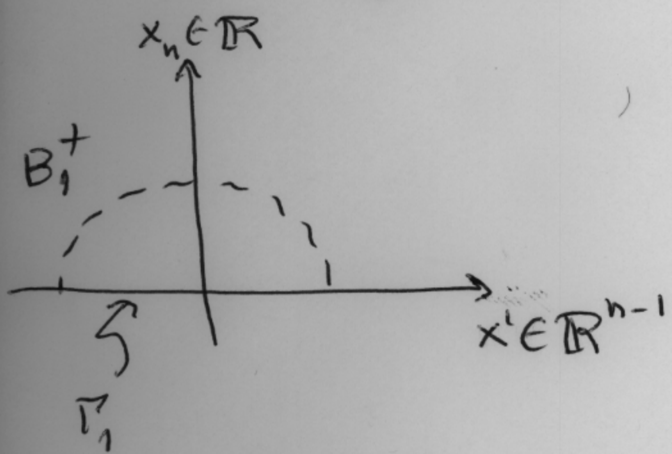
Assume $g \in C^\infty(\overline{B}(x_0, R), \mathbb{R}^N)$.

closed ball of centre x_0 and radius r

If $v \in W^{1,2}_g(B(x_0, R), \mathbb{R}^N)$ is harmonic then $v \in C^\infty(\overline{B}(x_0, R), \mathbb{R}^N)$.

$\boxed{\text{Sketch of pf:}}$ The usual pf goes via a technique known as 'flattening of the boundary'. We skip the details of this and consider the simpler situation:

$$\Gamma_r := \{x = (x', 0) \in \mathbb{R}^n : |x'| < r\}$$
$$B_r^+ := \{x \in \mathbb{R}^n : |x| < r,\ x_n > 0\}$$



$g \in C^\infty(\overline{B_1^+})$, $v \in W^{1,2}(B_1^+)$

$$\begin{cases} \Delta v = 0 & \text{in } B_1^+ \\ v = g & \text{on } \Gamma_1 \end{cases}$$

For $\rho \in C_c^1(B_R)$, $0 < r < R < 1$, $\mathbb{1}_{B_r} \leq \rho \leq \mathbb{1}_{B_R}$ and $|D\rho| \leq \frac{2}{R-r}$, let for $1 \leq s \leq n-1$,

$$\varphi = \Delta_{s,-h}(\rho^2 \Delta_{s,h}(v-g)), \quad |h| < 1-R.$$

Then $\varphi \in W_0^{1,2}(B_1^+)$ and so

$$0 = \int_{B_1^+} Dv \cdot D\varphi \text{ hence}$$

$$0 = \int_{B_1^+} \Big(\rho^2 \Delta_{s,h} Dv \cdot (\Delta_{s,h} Dv - \Delta_{s,h} Dg)$$
$$+ \Delta_{s,h} Dv \cdot \Delta_{s,h}(v-g) D(\rho^2) \Big)$$

and so

$$\int_{B_1^+} \rho^2 |\Delta_{s,h} Dv|^2 = \int_{B_1^+} \rho^2 \Delta_{s,h} Dv \cdot \Delta_{s,h} Dg$$

$$\mp \int_{B_1^+} 2\rho \Delta_{s,h} Dv \cdot D\rho \, \Delta_{s,h}(v-g)$$

ie

$$\int_{B_1^+} \rho^2 |\Delta_{s,h} Dv|^2 \leq 2\int_{B_1^+} \rho^2 |\Delta_{s,h} Dg|^2$$

$$+ 8\int_{B_1^+} |D\rho|^2 |\Delta_{s,h}(v-g)|^2$$

$$\leq c\int_{B_R^+} |D_s Dg|^2 h^2$$

$$+ \frac{c}{(R-r)^2}\int_{B_R^+} |D_s v - D_s g|^2 h^2$$

and therefore $D_s Dv \in L^2(B_r^+)$ with the Caccioppoli estimate

$$\int_{B_r^+} |D_s Dv|^2 \leq c\int_{B_R^+} |D_s Dg|^2$$

$$+ \frac{c}{(R-r)^2}\int_{B_R^+} |D_s v - D_s g|^2.$$

Hence $D_1^2 v, \dots, D_{n-1}^2 v \in L^2(B_r^+)$ for $0<r<1$ and then $D_n^2 v = -(D_1^2 v + \dots + D_{n-1}^2 v) \in L^2(B_r^+)$ for $0<r<1$ too. Thus $v \in W^{2,2}(B_r^+)$ for $0<r<1$.

Now note that (integration by parts) also $D_s v \in W^{1,2}(B_1^+)$ is harmonic on B_1^+ and that $D_s v = D_s g$ on Γ_1 for each $1 \le s \le n-1$ and we may therefore proceed similarly to the pf of Weyl's Lemma to conclude. $\square$

Let $B \subset \mathbb{R}^n$ be an open ball. Recall that $f \in C^1(\partial B)$ iff there exists $F \in C^1(\mathbb{R}^n)$ s.t. $F|\partial B = f$. Furthermore, for $x \in \partial B$,

$$\underbrace{D_t f(x)}_{\text{tangential gradient}} = \underbrace{\pi_x}_{\substack{\text{orthogonal projection}\\ \text{of } \mathbb{R}^n \text{ onto tangent}\\ \text{space } \mathrm{Tan}(\partial B, x) \text{ at } x}} (DF(x))$$

[When $B = B(0,1)$: $\mathrm{Tan}(\partial B, x) = \{x\}^\perp$,
$\pi_x(\xi) = \xi - (\xi \cdot x)x$.]

$\boxed{\text{DEF 41}}$ The Sobolev space $W^{1,p}(\partial B)$ is the completion of $C^1(\partial B)$ in the norm

$$\|f\|_{W^{1,p}(\partial B)} = \left(\int_{\partial B} \left(|f|^p + |D_t f|^p \right) d\mathcal{H}^{n-1} \right)^{\frac{1}{p}}.$$

$\boxed{\text{Lemma 42}}$ Let $1<p<\infty$ and $f \in W^{1,p}(B)$.

Then for a.e. $r \in (0,R)$ the pointwise restriction $f|\partial B_r$ coincides with the functional analytic two-sided trace $\operatorname{Tr}_{\partial B_r}[f]$ of f on ∂B_r and belongs to $W^{1,p}(\partial B_r)$.

Furthermore, for such $r \in (0,R)$,

$$|D_t(f|\partial B_r)| \leq |Df| \quad \text{a.e. on } \partial B_r.$$

<u>Note</u> All functions in terms of precise representatives in Lemma 42.

$\boxed{\text{Pf:}}$ WLOG $B = B(0,1)$.

Put $f_j(x) = \fint_{B(x,\frac{1}{j})} f \,, \quad |x| < 1 - \frac{1}{j}$.

Then $f_j \in C^1(B_{1-\frac{1}{j}})$, $f_j(x) \to f(x)$ and $Df_j(x) \to Df(x)$ for a.e. $x \in B$. Also, $f_j \to f$ strongly in $W^{1,p}_{loc}(B)$. By Fubini we have for a.e. $r \in (0,1)$, as $j \to \infty$

(1) $f_j(x) \to f(x)$, $Df_j(x) \to Df(x)$

pointwise in $\mathcal{H}^{n-1}$ - a.e. $x \in \partial B_r$.

By Fatou for $0<R<1$,

$$\int_0^R \underline{\lim} \int_{\partial B_r} \{ |f-f_j|^p + |Df-Df_j|^p \}$$

$$\leq \underline{\lim} \int_{B_R} (|f-f_j|^p + |Df+Df_j|^p) = 0,$$

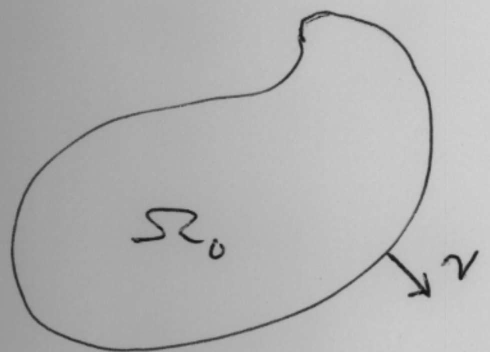
and so for a.e. $r \in (0,1)$ we have for a subsequence

(2) $f_j \to f$, $Df_j \to Df$ strongly in $L^p(\partial B_r)$.

Note (1) & (2) hold for a.e. $r \in (0,1)$ (and some subseq possibly depending on r). By (2) $(f_j|_{\partial B_r})$ is Cauchy in $W^{1,p}(\partial B_r)$ and so $f|_{\partial B_r} \in W^{1,p}(\partial B_r)$. By (1) it then follows that $|D_t(f|_{\partial B_r})| \leq |Df|$ a.e. on ∂B_r.

The rest now follows from the Trace Theorem (we omit the details). $\square$

Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded C^1 domain and let ν denote the outward unit normal on $\partial\Omega_0$



If $w \in C^1(\overline{\Omega}_0, \mathbb{R}^n)$, then the divergence theorem yields:

$$\int_{\Omega_0} \operatorname{div} w = \int_{\partial\Omega_0} w \cdot \nu \, d\mathcal{H}^{n-1}.$$

In particular, if $u \in C^2(\overline{\Omega}_0)$ and we take $w = Du$

$$\circledast \quad \int_{\Omega_0} \Delta u = \int_{\partial\Omega_0} Du \cdot \nu \, d\mathcal{H}^{n-1} = \int_{\partial\Omega_0} D_\nu u \, d\mathcal{H}^{n-1}$$

results.

$\boxed{\text{Lemma 43}}$ (Mean-value Property)

Assume $u \in C^2(\Omega)$ is subharmonic in Ω ($\Delta u \geqq 0$ in Ω). If $B(x_0, R) \Subset \Omega$, then

$$(0,R) \ni r \longmapsto \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} u \, d\mathcal{H}^{n-1}$$

is nondecreasing (and $\geqq u(x_0)$), where

$\omega_n := |B(0,1)|$.

<u>Note</u> Solid Mean-Value Property

$$(0,R) \ni r \mapsto ⨍_{B(x_0,r)} u \, dx$$

is nondecreasing (and $\geq u(x_0)$).

$\boxed{\text{Pf:}}$ For $0<r<R$ we get from ⊛ with $\Omega_0 = B(x_0,r)$:

$$0 \leq \int_{\partial B(x_0,r)} D_n u \, d\mathcal{H}^{n-1}$$

$$= \int_{\partial B(x_0,r)} Du(x) \cdot \frac{x-x_0}{r} \, d\mathcal{H}^{n-1}(x)$$

$$= r^{n-1} \int_{|y|=1} Du(x_0+ry) \cdot y \, d\mathcal{H}^{n-1}(y)$$

$$= r^{n-1} \int_{|y|=1} \frac{\partial}{\partial r} u(x_0+ry) \, d\mathcal{H}^{n-1}(y)$$

$$= r^{n-1} \frac{\partial}{\partial r} \int_{|y|=1} u(x_0+ry) \, d\mathcal{H}^{n-1}(y)$$

$$= r^{n-1} \frac{d}{dr} \left\{ r^{1-n} \int_{\partial B(x_0,r)} u \, d\mathcal{H}^{n-1} \right\}. \quad \square$$

$\boxed{\text{Lemma 44}}$ If u is harmonic, then $|Du|^2$ is subharmonic.

Pf: Simply check that $\Delta |Du|^2 = 2|D^2u|^2 \geq 0$. $\square$

<u>Note</u> It can be shown that in fact $|Du|^p$ is subharmonic for $p \geq \frac{n-2}{n-1}$ for $n>2$ and $\log |Du|$ is subharmonic for $n=2$.

Lemma 45 Let $v : B(x_0,R) \to \mathbb{R}^N$ be harmonic. Then for $0<r<R$,

$$\int_{\partial B(x_0,r)} |D_t v|^2 = \int_{\partial B(x_0,r)} |D_n v|^2 + \frac{n-2}{r} \int_{B(x_0,r)} |Dv|^2 .$$

Pf: WLOG $x_0 = 0$, $R=1$, $N=1$.

Put $w(x) := (x \cdot Dv(x))\, Dv(x)$, $x \in B$.

Then (by Weyl) $w \in C^\infty$ and so by divergence theorem

$$\int_{B_r} \operatorname{div} w = \int_{\partial B_r} w \cdot \frac{x}{r}$$

for $0<r<1$. By calculation

$$\operatorname{div} w = |Dv|^2 + \tfrac{1}{2} x \cdot D(|Dv|^2),$$

and for $|x| = r$, $w \cdot \frac{x}{r} = (x \cdot Dv(x))^2 \frac{1}{r} = |D_n v(x)|^2 r$.

Hence

$$\int_{B_r} \left(|Dv|^2 + \tfrac{1}{2} x \cdot D(|Dv|^2) \right) = r \int_{\partial B_r} |D_n v|^2 .$$

Here

$$\int_{B_r} \tfrac{1}{2} x \cdot D(|Dv|^2) = \tfrac{1}{2} \int_0^r \int_{\partial B_s} x \cdot D(|Dv|^2)$$

$$= \tfrac{1}{2} \int_0^r \int_{\partial B_1} (sx) \cdot D(|Dv|^2)(sx)\, s^{n-1}$$

$$= \tfrac{1}{2} \int_0^r \int_{\partial B_1} x \cdot D(|Dv|^2)(sx) \cdot s^n$$

$$= \tfrac{1}{2} \int_0^r \int_{\partial B_1} \frac{\partial}{\partial s} \left\{ |Dv|^2(sx) \right\} s^n$$

Fubini

$$= \tfrac{1}{2} \int_{\partial B_1} \int_0^r s^n \frac{\partial}{\partial s} \left\{ |Dv|^2(sx) \right\} ds \, d\mathcal{H}^{n-1}$$

parts

$$= \tfrac{1}{2} \int_{\partial B_1} \left\{ \left[s^n |Dv|^2(sx) \right]_{s=0}^{s=r} - \int_0^r n s^{n-1} |Dv|^2(sx)\, ds \right\}$$

$$= \tfrac{1}{2} \int_{\partial B_1} |Dv|^2(rx) r^n - \tfrac{1}{2} \int_{\partial B_1} \int_0^r n s^{n-1} |Dv|^2(sx)$$

$$= \frac{r}{2} \int_{\partial B_r} |Dv|^2 - \frac{n}{2} \int_{B_r} |Dv|^2 ,$$

and so

$$\left(1 - \frac{n}{2}\right) \int_{B_r} |Dv|^2 + \frac{r}{2} \int_{\partial B_r} |Dv|^2 = r \int_{\partial B_r} |D_n v|^2 .$$

Using Pythagoras, $|Dv|^2 = |D_n v|^2 + |D_t v|^2$,
we get

$$\int_{\partial B_r} |D_t v|^2 = \frac{2}{r}\left\{ r \int_{\partial B_r} |D_n v|^2 + \frac{n-2}{2} \int_{B_r} |Dv|^2 \right\}$$

$$- \int_{\partial B_r} |D_n v|^2$$

$$= \int_{\partial B_r} |D_n v|^2 + \frac{n-2}{r} \int_{B_r} |Dv|^2 . \quad \square$$

$\boxed{\text{Pf of TH 39:}}$

Given : $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ is a (spherical -) Q-minimizer for $Dir[\cdot, \Omega]$,
ie for $B_r = B(x_0, r) \Subset \Omega$ we have

$$\int_{B_r} |Du|^2 \leq Q \int_{B_r} |Dv|^2,$$

where $v \in W^{1,2}_u(B_r, \mathbb{R}^N)$ is harmonic.

Put $D(r) := \int_{B_r} |Du|^2$.

Note $D(r) \geq 0$ is nondecreasing and AC
with $D'(r) = \int_{\partial B_r} |Du|^2$ a.e. r.

For $0 < r < \text{dist}(x_0, \partial\Omega)$ we select r
s.t. $u|\partial B_r \in W^{1,2}(\partial B_r, \mathbb{R}^N)$ (Lemma 42)

Let $(\phi_\varepsilon)_{\varepsilon>0}$ be a standard smooth mollifier.

For $0<\varepsilon<\operatorname{dist}(x_0,\partial\Omega)-r$ let $u_\varepsilon=\phi_\varepsilon * u$.
Then $u_\varepsilon \in C^\infty(\overline{B}_r,\mathbb{R}^N)$, $u_\varepsilon \to u$ strongly in $W^{1,2}(B_r,\mathbb{R}^N)$ and $u_\varepsilon|\partial B_r \to u|\partial B_r$ strongly in $W^{1,2}(\partial B_r,\mathbb{R}^N)$. Let $v_\varepsilon \in W^{1,2}_{u_\varepsilon}(B_r,\mathbb{R}^N)$ be harmonic. Then by Weyl 'up to the bdry' $v_\varepsilon \in C^\infty(\overline{B}_r,\mathbb{R}^N)$ and by Lemma 45

$$\int_{\partial B_s} |D_t v_\varepsilon|^2 = \int_{\partial B_s} |D_n v_\varepsilon|^2 + \frac{n-2}{s}\int_{B_s} |Dv_\varepsilon|^2$$

for $0<s<r$. Let $s \nearrow r$ to get (by smoothness)

$$\int_{\partial B_r} |D_t v_\varepsilon|^2 = \int_{\partial B_r} |D_n v_\varepsilon|^2 + \frac{n-2}{r}\int_{B_r} |Dv_\varepsilon|^2 .$$

Now since $v_\varepsilon = u_\varepsilon$ on ∂B_r we also have $D_t v_\varepsilon = D_t u_\varepsilon$ on ∂B_r and so as $\varepsilon \searrow 0$

$$\int_{\partial B_r} |D_t v_\varepsilon|^2 = \int_{\partial B_r} |D_t u_\varepsilon|^2 \to \int_{\partial B_r} |D_t u|^2 .$$

If $v \in W^{1,2}_u(B_r,\mathbb{R}^N)$ is harmonic, then $v_\varepsilon \to v$ strongly in $W^{1,2}(B_r,\mathbb{R}^N)$; hence

$$\int_{B_r} |Dv_\varepsilon|^2 \longrightarrow \int_{B_r} |Dv|^2 .$$

Thus:

$$\lim_{\varepsilon \to 0} \int_{\partial B_r} |D_n v_\varepsilon|^2 = \int_{\partial B_r} |D_t u|^2 - \frac{n-2}{r} \int_{B_r} |Dv|^2 .$$

Now $\int_{B_r} |Dv_\varepsilon|^2 = \int_0^r \int_{\partial B_s} |Dv_\varepsilon|^2 =$

$$\int_0^r s^{n-1} \left\{ s^{1-n} \int_{\partial B_s} |Dv_\varepsilon|^2 \right\} \leq \int_0^r s^{n-1} r^{1-n} \int_{\partial B_r} |Dv_\varepsilon|^2$$

$= \frac{r}{n} \int_{\partial B_r} |Dv_\varepsilon|^2$ and so as $\varepsilon \to 0$

$$\int_{B_r} |Dv|^2 \leq \frac{r}{n} \left\{ \int_{\partial B_r} |D_t u|^2 + \lim_{\varepsilon \to 0} \int_{\partial B_r} |D_n v_\varepsilon|^2 \right\}$$

$$= \frac{2r}{n} \int_{\partial B_r} |D_t u|^2 - \frac{n-2}{n} \int_{B_r} |Dv|^2 .$$

Thus: $D(r) \leq Q \int_{B_r} |Dv|^2$

$$\leq \frac{2Qr}{n} \int_{\partial B_r} |D_t u|^2 - \frac{n-2}{n} Q \int_{B_r} |Dv|^2$$

$$\leq \frac{2Qr}{n} \int_{\partial B_r} |Du|^2 - \frac{n-2}{n} D(r)$$

$$= \frac{2Qr}{n} D'(r) - \frac{n-2}{n} D(r),$$

ie

$$\boxed{D(r) \leq \frac{2Qr}{n} D'(r)}$$

for a.e. $r \in (0, \operatorname{dist}(x_0, \partial\Omega))$.

Hence for a.e. r with $D(r) > 0$,

$$\frac{n-1}{Q} \cdot \frac{1}{r} \leq \frac{D'(r)}{D(r)},$$

and so for $0 < r < R < \operatorname{dist}(x_0, \partial\Omega)$ we get by FTC

$$\frac{n-1}{Q} \log \frac{R}{r} \leq \log \frac{D(R)}{D(r)},$$

ie $\quad r^{-\frac{n-1}{Q}} D(r) \leq R^{-\frac{n-1}{Q}} D(R)$. $\square$