Partial Regularity

TH 46 {Morrey, Giaquinta & Giusti, Campanato, Giusti & Miranda} (based on Almgren, De Giorgi)

Let $F : \mathbb{R}^{N \times n} \to \mathbb{R}$ be C^2 and

$$\ell |\lambda|^2 \leqq F''(\xi)[\lambda, \lambda] \leqq L|\lambda|^2 \quad \forall \xi, \lambda \in \mathbb{R}^{N \times n},$$

where $0 < \ell \leqq L < \infty$ are constants.

Suppose $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a minimizer of $\mathcal{F}[v, \Omega] = \int_\Omega F(Dv)$:

$$\mathcal{F}[u, \Omega] \leqq \mathcal{F}[u + \varphi, \Omega] \quad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N).$$

Define

$$\Omega_u := \{ x \in \Omega : u \text{ is } C^1 \text{ near } x \},$$

$$\Sigma_u^1 := \{ x \in \Omega : \lim_{r \to 0+} |(Du)_{x,r}| = \infty \},$$

$$\Sigma_u^2 := \{ x \in \Omega : \overline{\lim_{r \to 0+}} E(x,r) > 0 \},$$

$$\Sigma_u^3 := \{ x \in \Omega : \overline{\lim_{r \to 0+}} |(Du)_{x,r}| = \infty \},$$

$$\Sigma_u^4 := \{ x \in \Omega : \lim_{r \to 0+} E(x,r) > 0 \},$$

where $\underline{\lim} := \liminf$, $\overline{\lim} = \limsup$

and $E(x,r) = \fint_{B(x,r)} |Du - (Du)_{x,r}|^2$ ('excess' or 'mean-oscillation' on $B(x,r)$).

Then Ω_u is open (obvious),

$$\Omega \setminus \Omega_u = \Sigma_u^1 \cup \Sigma_u^2 = \Sigma_u^3 = \Sigma_u^4$$

and $u|\Omega_u$ is $C^{1,\alpha}_{loc}$ for each $\alpha < 1$.

<u>Note</u> All maps are given in terms of their precise representatives.

Remarks on pf

The pf is a perturbation argument and relies on linearization and the fact (Weyl) that there are powerful interior estimates for solutions to linear elliptic systems with constant coefficients.

The approach originates in work on minimal surfaces by <u>Almgren</u> and <u>De Giorgi</u>.

The main difficulty for the implementation of the above linearization strategy is that while the assumptions easily yield estimates

in $W^{1,2}$ what is actually required is
compactness in $W^{1,2}$. We will obtain that
from TH 6 (the difference-quotient method)
and the final result is regularity under
a smallness condition. The smallness condi=
tion can't be garanteed at all points,
hence we only prove 'partial regularity'.
Of course, in view of the exx 37 we
also shouldn't be able to prove more
than that. There is however still a signi=
ficant gap between the size of the
'singular sets' in the exx 37 and the
estimates one gets from measure theory.
The Hausdorff dimension of singular sets
in exx 37 is $n-3$ whereas measure
theory and TH 46 gives $n-2-\sigma$,
where $\sigma > 0$ is small and awkward
the quantify precisely. We won't discuss this
point further here. Instead we give
a rough sketch of the pf:

Let $B(x_0, R) \subset \Omega$ and put

$$\xi_0 = (Du)_{x_0,R} .$$

Let $P(\xi) = F(\xi_0) + F'(\xi_0)[\xi - \xi_0] + \frac{1}{2}F''(\xi_0)[\xi-\xi_0, \xi-\xi_0]$

$$\cancel{= \int_0^1 (1-t) F''(\xi_0 + t(\xi-\xi_0))[\xi-\xi_0, \xi-\xi_0]\,dt}$$

be the 2nd Taylor polynomial for F about ξ_0

Recall: $\dfrac{|F(\xi) - P(\xi)|}{|\xi - \xi_0|^2} \to 0$ as $\xi \to \xi_0$

Let $h \in W^{1,2}_u(B(x_0,R), \mathbb{R}^N)$ be minimizing for

$$\mathcal{P}[v, B(x_0,R)] = \int_{B(x_0,R)} P(Dv).$$

Note: $\mathcal{P}$ is a quadratic strongly convex functional and the Euler-Lagrange system for $\mathcal{P}$ is

$$\operatorname{div} P'(Dh) = 0 \quad \text{in } B(x_0,R),$$

a linear elliptic system with constant coefficients. Hence h is smooth and we have good estimates by Generalized Weyl. We seek to transfer these estimates to u — plausible provided we can show that h is a good approximation of u in $W^{1,2}(B(x_0,R), \mathbb{R}^N)$. This turns out to be the case when the excess $E(x_0,R)$ is small.

$\boxed{\text{Pf of TH 46}}$ We assume $n>2$ (and $\frac{L}{\lambda} \geqslant \left(\frac{n-1}{n-2}\right)^{\frac12} \dots$)

Let $m>0$ and fix $B_{2R}=B(x_0,2R)\subset\Omega$ s.t. $|(Du)_{x_0,R}|<m$. Put $\xi_0 := (Du)_{x_0,R}$

and

$$P(\xi) = F(\xi_0) + F'(\xi_0)[\xi-\xi_0] + \tfrac12 F''(\xi_0)[\xi-\xi_0,\xi-\xi_0]$$

$$\cancel{= \int_0^1 (1-t)F''(\xi_0+t(\xi-\xi_0))[\xi-\xi_0,\xi-\xi_0]\,dt}$$

Then $P''(\xi) = F''(\xi_0)$ and the E-L for $\mathcal{P}(v,B(x_0,R)) = \int_{B(x_0,R)} P(Dv)$ is

$$\int_{B_R} F''(\xi_0)[Dv, D\varphi] = 0 \quad \forall \varphi\in W_0^{1,2}(B_R,\mathbb{R}^N).$$

Hence if $h\in W_u^{1,2}(B_R,\mathbb{R}^N)$ is $\mathcal{P}(\cdot,B_R)$ minimizing then $h\in C^\infty(B_R,\mathbb{R}^N)$ and

$$R^2\sup_{B_{R/2}}|D^2h|^2 \leq c\fint_{B_R}|Dh-(Dh)_{x_0,R}|^2$$
$$\overset{*)}{\leq} c\fint_{B_R}|Du-(Du)_{x_0,R}|^2 = cE(x_0,R)$$

by Generalized Weyl (TH 17)

$\boxed{*):}$ From EL for $\mathcal{J}$: $0 = \int_{B_R} F''(\xi_0)\left[Dh - (Dh)_{x_0,R}, Du - Dh\right]$ and so as $(Dh)_{x_0,R} = (Du)_{x_0,R}$,

$$\int_{B_R} F''(\xi_0)\left[Dh - (Dh)_{x_0,R}, Dh - (Dh)_{x_0,R}\right] =$$

$$\int_{B_R} F''(\xi_0)\left[Dh - (Dh)_{x_0,R}, Du - (Du)_{x_0,R}\right] \leq$$

$$\int_{B_R} F''(\xi_0)\left[Dh - (Dh)_{x_0,R}, Dh - (Dh)_{x_0,R}\right]^{\frac{1}{2}} F''(\xi_0)\left[Du - (Du)_{x_0,R}, Du - (Du)_{x_0,R}\right]^{\frac{1}{2}}$$

$\leq L \int_{B_R} |Dh - (Dh)_{x_0,R}|\,|Du - (Du)_{x_0,R}|$ and so by Cauchy-Schwarz $\int_{B_R} |Dh - (Dh)_{x_0,R}|^2 \leq \left(\frac{L}{\lambda}\right)^2 E(x_0,R)$.

Hence for $0 < r < \frac{R}{2}$ we get

$$(1) \qquad \fint_{B_r} |Dh - (Dh)_{x_0,r}|^2 \leq c\left(\frac{r}{R}\right)^2 E(x_0,R)$$

and changing the constant c if necessary we see that (1) remains valid also for $\frac{R}{2} \leq r < R$.

How well does h approximate u in $W^{1,2}(B(x_0,R), \mathbb{R}^N)$?

We construct a convenient modulus of continuity $\omega = \omega_m$ for F'' on $\{\xi \in \mathbb{R}^{N\times n} : |\xi| \leq m+1\}$:

Put for $t \geqslant 0$,

$$\tilde{\omega}(t) := \frac{1}{2L} \sup\left\{ |F''(\xi)-F''(\eta)| : \begin{array}{l} |\xi|,|\eta| \leq m+1, \\ |\xi-\eta| \leq t \end{array} \right\}.$$

Then $\tilde{\omega} : [0,\infty) \to [0,1]$ is continuous, nondecreasing and $\tilde{\omega}(0) = 0$. For technical reasons that will become clear later we modify $\tilde{\omega}$ as follows: first put

$$\omega_0(t) := \min\{ \max\{\tilde{\omega}(t),1\},1\}$$

so that in addition to the above properties of $\tilde{\omega}(t)$ we also have that $\omega_0(t) = 1 \ \forall t \geq 1$ (and clearly $\omega_0(t) \geqslant \tilde{\omega}(t) \ \forall t \geqslant 0$). Next define for $t \geqslant 0$,

$$\omega(t) := \inf\left\{ c(t) : \begin{array}{l} c \text{ concave} \\ \omega_0 \leq c \text{ on } [0,\infty) \end{array} \right\},$$

ie ω is the concave envelope of ω_0 (= the smallest concave function $c : [0,\infty) \to \mathbb{R}$ s.t. $c \geqslant \omega_0$). Then $\omega = \omega_m : [0,\infty) \to [0,1]$

is continuous, nondecreasing, concave, $\omega(0) = 0$, $\omega(t) = 1$ $\forall t \geq 1$ and

$$|F''(\xi) - F''(\eta)| \leqq 2L\,\omega(|\xi - \eta|)$$

for all $|\xi|, |\eta| \leqq m+1$.

Consequently we have for all ξ:

$$|F(\xi) - P(\xi)| = \left| \int_0^1 (1-t)\Big(F''(\xi_0 + t(\xi - \xi_0)) - F''(\xi_0)\Big)[\xi - \xi_0, \xi - \xi_0]\,dt \right| \leqq \int_0^1 (1-t)\,|F''(\xi_0 + t(\xi-\xi_0)) - F''(\xi_0)|\,dt\,|\xi - \xi_0|^2 .$$

Here

$$|F''(\xi_0 + t(\xi - \xi_0)) - F''(\xi_0)| \leqq \begin{cases} 2L\,\omega(|\xi - \xi_0|) & \text{provided } |\xi - \xi_0| \leqq 1 \\ 2L = 2L\,\omega(|\xi - \xi_0|) & \text{provided } |\xi - \xi_0| > 1, \end{cases}$$

thus:

$$(2) \quad |F(\xi) - P(\xi)| \leqq L\,\omega_m(|\xi - \xi_0|)\,|\xi - \xi_0|^2$$

for <u>all</u> $\xi \in \mathbb{R}^{N \times n}$.

We are now ready for the first estimate of the degree of approximation:

First note

$$F(Du) - F(Dh) - F'(Dh)[Du - Dh]$$

$$\geq \frac{\ell}{2} |Du - Dh|^2 \text{, and hence}$$

$$\frac{\ell}{2} \int_{B_R} |Du - Dh|^2 \leq \int_{B_R} \Big(F(Du) - F(Dh) - F'(Dh)[Du - Dh] \Big)$$

$$= \int_{B_R} \Big(F(Du) - F(Dh) \Big)$$

$$= \int_{B_R} \Big(F(Du) - F(Du) + F(Du) - F(Dh) \Big)$$

$$\overset{u \text{ minimizer}}{\leq} \int_{B_R} \Big(F(Du) - F(Du) + F(Dh) - F(Dh) \Big)$$

$$\overset{(2)}{\leq} L \int_{B_R} \Big(\omega_m(|Du - \xi_0|) |Du - \xi_0|^2 + \omega_m(|Dh - \xi_0|) |Dh - \xi_0|^2 \Big)$$

Would like to use assumption of smallness of excess $E(x_0, 2R)$ here. Recall

TH 6 : $u \in W^{2,2}_{loc}$ and

$$\int_{B_R} |D^2 u|^2 \leq \frac{c}{R^2} \int_{B_{2R}} |Du - (Du)_{B_{2R}}|^2$$

and so by Poincaré-Sobolev's inequality (recall $n>2$)

$$\left\{ \int_{B_R} |Du - \xi_0|^{\frac{2n}{n-2}} \right\}^{\frac{n-2}{n}} \leq c \int_{B_R} |D^2 u|^2$$

$$\leq \frac{c}{R^2} \int_{B_{2R}} |Du - (Du)_{B_{2R}}|^2 ,$$

ie

$$\left(\fint_{B_R} |Du - \xi_0|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq c \fint_{B_{2R}} |Du - (Du)_{B_{2R}}|^2 .$$

We require a similar bound for h.

$\boxed{\text{TH 49}}$ (Generalized Weyl 'up to the bdry')

If $h \in W^{1,2}_u (B_R, \mathbb{R}^N)$ is $\mathcal{F}(\cdot, B_R)$-minimizing, then (as $u \in W^{2,2}$)

$h \in W^{2,2}(B_R, \mathbb{R}^N)$ and

$$\int_{B_R} |D^2 h|^2 \leq c \int_{B_R} |D^2 u|^2 ,$$

where $c = c(n, N, \frac{L}{\ell})$ is a constant.

We omit the pf (it's exactly as pf of TH 40 'Weyl up to the bdry').

Consequently,

$$\left(\fint_{B_R} |Dh - \xi_0|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq c \fint_{B_{2R}} |Du - (Du)_{B_{2R}}|^2 .$$

Hence by Hölder $\left(\left(\frac{n}{n-2}\right)' = \frac{n}{2} \right)$:

$$\frac{\lambda}{2} \fint_{B_R} |Du - Dh|^2 \leq$$

$$L \left\{ \left(\fint_{B_R} \omega(|Du - \xi_0|)^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\fint_{B_R} |Du - \xi_0|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \right.$$

$$\left. + \left(\fint_{B_R} \omega(|Dh - \xi_0|)^{\frac{n}{2}} \right)^{\frac{2}{n}} \left(\fint_{B_R} |Dh - \xi_0|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \right\} \leq$$

$$L \left\{ \left(\fint_{B_R} \omega(|Du - \xi_0|)^{\frac{n}{2}} \right)^{\frac{2}{n}} + \left(\fint_{B_R} \omega(|Dh - \xi_0|)^{\frac{n}{2}} \right)^{\frac{2}{n}} \right\} E(x_0, 2R).$$

Now

$$\left(\fint_{B_R} \omega(|Du - \xi_0|)^{\frac{n}{2}} \right)^{\frac{2}{n}} \overset{\omega \leq 1}{\leq} \left(\fint_{B_R} \omega(|Du - \xi_0|) \right)^{\frac{2}{n}} \overset{\omega \text{ concave}}{\leq} \left(\omega\left(\fint_{B_R} |Du - \xi_0| \right) \right)^{\frac{2}{n}} \overset{\omega \text{ nondecreasing}}{\leq}$$

$$\omega\left(E(x_0, R)^{\frac{1}{2}} \right)^{\frac{2}{n}} \leq \omega\left(2^{\frac{n}{2}} E(x_0, 2R)^{\frac{1}{2}} \right)^{\frac{2}{n}}.$$

Similarly (and using $*$)):

$$\left(\fint_{B_R} \omega(|Dh - \xi_0|)^{\frac{n}{2}} \right)^{\frac{2}{n}} \leq \omega\left(cE(x_0,2R)^{\frac{1}{2}}\right)^{\frac{2}{n}}$$

and therefore:

$$(3) \quad \fint_{B_R} |Du - Dh|^2 \leq c\,\omega\left(cE(x_0,2R)^{\frac{1}{2}}\right)^{\frac{2}{n}} E(x_0,2R)$$

Good approximation when $E(x_0,2R)$ is small:

$$c\,\omega\left(cE(x_0,2R)^{\frac{1}{2}}\right)^{\frac{2}{n}} E(x_0,2R) \ll E(x_0,2R)$$

Define for $0 < r < 2R$,

$$\Phi(r) := \int_{B_r} |Du - (Du)_{B_r}| \quad \left(= |B_r|\, E(x_0,r)\right).$$

For $0 < r < R$ we estimate by use of (1) & (3):

$$\begin{aligned}
\Phi(r) &\leq \int_{B_r} |Du - (Dh)_{B_r}|^2 \\
&\leq 2\int_{B_r} |Du - Dh|^2 + 2\int_{B_r} |Dh - (Dh)_{B_r}|^2 \\
&\overset{(1),(3)}{\leq} c\,\omega\left(cE(x_0,2R)^{\frac{1}{2}}\right)^{\frac{2}{n}} \Phi(2R) + c\left(\frac{r}{R}\right)^{n+2} \Phi(2R) \\
&= c\left[\omega\left(cE(x_0,2R)^{\frac{1}{2}}\right)^{\frac{2}{n}} + \left(\frac{r}{R}\right)^{n+2}\right] \Phi(2R).
\end{aligned}$$

If $c \geq 2^{n+2}$ (and we assume that), then the above inequality trivially holds for $R \leq r < 2R$ too.

Observe that the above yields:

For $B(x_0, R) \subset \Omega$ s.t. $|(Du)_{x_0, \frac{R}{2}}| < m$

we have

$$\circledast \quad \Phi(r) \leq c\left[\omega\left(cE(x_0,R)^{\frac{1}{2}}\right)^{\frac{2}{n}} + \left(\frac{r}{R}\right)^{n+2}\right]\Phi(R)$$

for $0 < r < R$.

For $\delta > 0$ define $\varepsilon_m(\delta) := \sup\{t > 0 : \omega_m(ct^{\frac{1}{2}})^{\frac{2}{n}} < \delta\}$

Then $\varepsilon_m(\delta) > 0$ and $\omega_m(cE(x_0,R)^{\frac{1}{2}})^{\frac{2}{n}} < \delta$ provided $E(x_0, R) < \varepsilon_m(\delta)$.

We would like to iterate $\circledast$.

Let $\delta > 0$. We'll choose it later.

Fix $m > 0$ and consider $\varepsilon = \varepsilon_{m+1}(\delta)$ (note: $m+1$). We have for $B(x_0,R) \subset \Omega$ with

$(**)$ $\quad |(Du)_{x_0,\frac{R}{2}}| < m+1$ and $E(x_0,R) < \varepsilon$

that

$$\Phi(r) \leq c\left[\delta + \left(\tfrac{r}{R}\right)^{n+2}\right]\Phi(R) .$$

for $0<r<R$.

For each $0<\tau<1$ we have with $r=\tau R$ and returning to excess: $E(x_0,r) = |B_r|^{-1}\Phi(r)$

$$E(x_0,\tau R) \leq c\left[\delta\tau^{-n} + \tau^2\right]E(x_0,R)$$

Note $c[\delta\tau^{-n}+\tau^2] = (c\delta\tau^{-n-1} + c\tau)\tau \leq \tau$ when $c\delta\tau^{-n-1} + c\tau \leq 1$ which holds true if

$$c\delta\tau^{-n-1} \leq \tfrac{1}{2} \quad \text{and} \quad c\tau \leq \tfrac{1}{2}.$$

<u>Choose</u> $\quad \tau = \dfrac{1}{2c}$ and $\delta = \dfrac{\tau^{n+1}}{2c} = \left(\dfrac{1}{2c}\right)^{n+2}$.

With these choices we have

$$E(x_0,\tau R) \leq \tau E(x_0,R),$$

provided that $(**)$ holds.

&

Consequently : for $j \in \mathbb{N}$

$$\oplus \quad E(x_0, \tau^j R) \leq \tau E(x_0, \tau^{j-1} R)$$

provided that $B(x_0, \tau^{j-1}R) \subset \Omega$ (clearly true) and

$$\oplus\!\!\oplus \quad |(Du)_{x_0, \tau^{j-1}\frac{R}{2}}| < m+1 \ \& \ E(x_0, \tau^{j-1}R) < \varepsilon.$$

As $\oplus\!\!\oplus$ holds by assumption for $j=1$ we get $\oplus$ for $j=1$.

Suppose that $\oplus\!\!\oplus$ holds for all $j \in \{1, 2, \dots, k\}$.

Then by iteration: $E(x_0, \tau^k R) \leq \tau^k E(x_0, R)$

$$< \tau^k \varepsilon < \varepsilon.$$

Next

$$|(Du)_{x_0, \tau^k \frac{R}{2}}| \leq |(Du)_{x_0, \frac{R}{2}}| + \sum_{j=1}^{k} |(Du)_{x_0, \tau^j \frac{R}{2}} - (Du)_{x_0, \tau^{j-1}\frac{R}{2}}|$$

$$< m + \sum_{j=1}^{k} \fint_{B(x_0, \tau^j \frac{R}{2})} |Du - (Du)_{x_0, \tau^{j-1}\frac{R}{2}}|$$

$$\leq m + \sum_{j=1}^{k} \left(\fint_{B(x_0, \tau^j \frac{R}{2})} |Du - (Du)_{x_0, \tau^{j-1}\frac{R}{2}}|^2 \right)^{\frac{1}{2}}$$

$$\leq m + \sum_{j=1}^{k} \tau^{-\frac{n}{2}} E(x_0, \tau^{j-1}\tfrac{R}{2})^{\frac{1}{2}}$$

$$\leq m + \sum_{j=1}^{k} \left(\tfrac{2}{\tau}\right)^{\frac{n}{2}} E(x_0, \tau^{j-1}R)^{\frac{1}{2}}$$

$$\leq m + \left(\tfrac{2}{\tau}\right)^{\frac{n}{2}} \sum_{j=1}^{k} \tau^{\frac{j-1}{2}} \sqrt{\varepsilon}$$

$$< m + \left(\tfrac{2}{\tau}\right)^{\frac{n}{2}} \frac{\sqrt{\varepsilon}}{1-\sqrt{\tau}} .$$

If we assume that $\left(\frac{2}{\tau}\right)^{\frac{n}{2}} \frac{\sqrt{\varepsilon}}{1-\sqrt{\tau}} \leq 1$,

ie $\cancel{\varepsilon \neq \varepsilon_{m+1}(d)}$

$$\varepsilon \leq \left(\tfrac{\tau}{2}\right)^{n} \left(1-\tau^{\frac{1}{2}}\right)^{2},$$

then $|(Du)_{x_0, \tau^k \frac{R}{2}}| < m+1$.

Define $\varepsilon_m := \min\{ \left(\frac{\tau}{2}\right)^{n}\left(1-\tau^{\frac{1}{2}}\right)^{2}, \varepsilon_{m+1}(d)\}$.

Then $|(Du)_{x_0, \frac{R}{2}\tau^k}| < m+1$ provided $E(x_0,R) < \varepsilon_m$ and we see that (++) holds for $j=k+1$. Consequently:

If $B(x_0,R) \subset \Omega$, $|(Du)_{x_0,\frac{R}{2}}| < m$ and $E(x_0,R) < \varepsilon_m$ (with above choices!), then

$$E(x_0, \tau^j R) \leq \tau^j E(x_0, R) \quad \forall j$$

For $0 < r < R$ take $j \in \mathbb{N}$ s.t. $\tau^j R \leq r < \tau^{j-1} R$ and note that

$$E(x_0, r) \leq \tau^{-n} E(x_0, \tau^{j-1} R) \leq \tau^{j-n-1} E(x_0, R)$$

$$\leq \tau^{-n-1} \left(\frac{r}{R}\right)^{1} E(x_0, R).$$

Hence: If $B(x_0, R) \subset \Omega$, $|(Du)_{x_0, \frac{R}{2}}| < m$ and $E(x_0, R) < \varepsilon_m$, then

$$E(x_0, r) \leq c \frac{r}{R} E(x_0, R)$$

for all $0 < r < R$.

Since the set $\Omega_m := \{ x \in \Omega : \exists R > 0$ s.t. $B(x_0, R) \subset \Omega$, $|(Du)_{x_0, \frac{R}{2}}| < m$ and $E(x_0, R) < \varepsilon_m \}$ is open Campanato's Characterization of Hölder continuity (see L7 & 8) gives that $u \in C^{1, \frac{1}{2}}_{loc}$ on Ω_m, and consequently on $\bigcup_{m \in \mathbb{N}} \Omega_m$.

Once we know that $u \in C^{1,\frac{1}{2}}_{loc}(\bigcup_m \Omega_m)$ then it's easy to check that (by above pf) $u \in C^{1,\alpha}_{loc}(\bigcup_m \Omega_m)$ for all $\alpha < 1$. It follows that $\Omega_u = \bigcup_m \Omega_m$.

Finally, if $x \in \Omega \setminus (\Sigma^1_u \cup \Sigma^2_u)$ so that

$$\underline{\lim}_{r \to 0+} |(Du)_{x,r}| < \infty \ \& \ \overline{\lim}_{r \to 0+} E(x,r) = 0,$$

then for some $r_j \searrow 0$

$$\sup_j |(Du)_{x,r_j}| < \infty \ \& \ E(x, r_j) \longrightarrow 0 \text{ as } r \searrow 0.$$

Put $m = \sup_j |(Du)_{x,r_j}| + 1$ and take $R_0 > 0$ s.t. $E(x,r) < \varepsilon_m$ for $0 < r \leqq R_0$. Take $j_0 \in \mathbb{N}$ s.t. $r_{j_0} \leqq \frac{R_0}{2}$ and put $R = 2r_{j_0}$.

Then $R \leqq R_0$ so

$$|(Du)_{x,\frac{R}{2}}| < m \ \& \ E(x,R) < \varepsilon_m.$$

Consequently, $\Omega_u \supseteq \Omega \setminus (\Sigma^1_u \cup \Sigma^2_u)$ and the opposite inclusion is clear, so $\Omega_u = \Omega \setminus (\Sigma^1_u \cup \Sigma^2_u)$. Similar for Σ^3_u, Σ^4_u. $\square$