Quasiminimizers

We recall (and slightly generalize) our definition (DEF 7):

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain,

$G : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ Carathéodory

($G = G(x, \xi)$ is measurable in x for fixed ξ and continuous in ξ for fixed x.)

satisfying $\quad 0 \leqq G(x, \xi) \leqq L(|\xi|^2 + a(x))$

for all $x \in \Omega$, $\xi \in \mathbb{R}^{N \times n}$, where $0 \leqq a \in L^1(\Omega)$.

Put $\quad g[u, \Omega'] = \int_{\Omega'} G(x, Du)\, dx$, $u \in W^{1,p}(\Omega, \mathbb{R}^N)$,

where $\Omega' \subseteqq \Omega$ is an open subset.

DEF 7'

Let $Q \geqq 1$. Then $w \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a Q-minimizer for $g[\cdot, \Omega]$ iff

$$g[w, \Omega'] \leqq Q\, g[w + \varphi, \Omega']$$

for all $\varphi \in W_0^{1,2}(\Omega', \mathbb{R}^N)$ and $\Omega' \Subset \Omega$.

Proposition 8 Let $Q \geqq 1$.

Then $u \in W^{1,2}(\Omega)$ is a Q-minimizer for $\int_\Omega (1 + |Dv|^2)\, dx$ iff there exists a convex Carathéodory integrand

$$G : \Omega \times \mathbb{R}^n \to \mathbb{R}$$

satisfying $|\xi|^2 + 1 \leqq G(x, \xi) \leqq Q(|\xi|^2 + 1)$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$, ~~where $c = c(Q)$.~~

Pf (Sketch only): $\Uparrow$ Easy.

$\Downarrow$ For each $x \in \Omega$ put

$$\tilde{G}(x, \xi) = \begin{cases} 1 + |Du(x)|^2 & \text{if } \xi = Du(x) \\ Q(1 + |\xi|^2) & \text{if } \xi \neq Du(x). \end{cases}$$

Define $G(x, \cdot)$ = convex envelope of $\tilde{G}(x, \cdot)$. □

(Details of $\Downarrow$ will involve a relaxation theorem as can for instance be found in Ekeland & Temam or Dacorogna's books.)

EX 9 Bounded weak solutions to PDE.

Assume

$A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

are Carathéodory (measurable in $x \in \Omega$, continuous in $(y, \xi) \in \mathbb{R} \times \mathbb{R}^n$) and satisfy

$$A(x,y,\xi).\xi \geqslant |\xi|^2 - a_1(x) \quad (18)$$

$$|A(x,y,\xi)| \leqslant L|\xi| + a_2(x) \quad (19)$$

$$|B(x,y,\xi)| \leqslant H|\xi|^2 + a_3(x) \quad (20)$$

for all $x \in \Omega$, $y \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, where $a_1, a_2, a_3 \geqslant 0$ and $a_1, a_3 \in L^1(\Omega)$, $a_2 \in L^2(\Omega)$.

<u>NOTE</u> When (20) is satisfied together with (18), (19) we speak of 'natural growth conditions'.

Consider the PDE in <u>divergence form</u>:

$$(21) \quad -\operatorname{div} A(x,u,Du) = B(x,u,Du) \text{ in } \Omega$$

DEF 10 $u \in W^{1,2}(\Omega)$ is a weak solution to the PDE (21) iff

$$\int_\Omega A(x,u,Du).D\varphi\, dx = \int_\Omega B(x,u,Du)\varphi\, dx$$

holds for all $\varphi \in W_0^{1,2}(\Omega)$.

If $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a weak solution to (21), then it is a Q-minimizer for $\int_\Omega (|Du|^2 + a)\, dx$, where

$$a := a_1 + a_2^2 + a_3 \qquad \in L^1(\Omega)^+$$

and

$$Q = 2\left(1 + L^2 (1 + 2H\|u\|_{L^\infty})^2 e^{4H\|u\|_{L^\infty}}\right).$$

Pf. Let $v \in W^{1,2}(\Omega)$, $K := spt(u-v) \Subset \Omega$ and assume $\|v\|_{L^\infty} \leq M := \|u\|_{L^\infty}$.

Put $\varphi := (u-v)^+ e^{H(u-v)}$.

Then $\varphi \in W_0^{1,2}(\Omega)$ and so

BACKGROUND RESULT: If $w_1, w_2 \in W^{1,p}(\Omega)$, then $\min(w_1,w_2), \max(w_1,w_2) \in W^{1,p}(\Omega)$.

writing

$$A = A(x, u, Du)$$

$$B = B(x, u, Du)$$

and $S = \operatorname{spt} \varphi = \operatorname{spt} (u-v)^+$,

$$\int_S A \cdot (Du - Dv)(1 + H(u-v)) e^{H(u-v)} dx =$$

$$\int_S B(u-v) e^{H(u-v)} dx \; ,$$ ie

$$\int_S A \cdot Du \, (1 + H(u-v)) \, e^{H(u-v)} dx =$$

$$\int_S A \cdot Dv \, (1 + H(u-v)) e^{H(u-v)} dx + \int_S B(u-v) e^{H(u-v)} dx .$$

Invoke conditions (18), (19), (20) :

$$\int_S (|Du|^2 - a_1)(1 + H(u-v)) e^{H(u-v)} dx \leq$$

$$\int_S (L|Du| + a_2)|Dv|(1 + H(u-v)) e^{H(u-v)} dx$$

$$+ \int_S (\mathcal{H} \cdot |Du|^2 + a_3)(u-v) e^{H(u-v)} dx$$

Note : $u - v \geqslant 0$ on S

Rearranging terms:

$$\int_S |Du|^2 \left(1 + H(u-v)\right) e^{H(u-v)}\, dx \leq$$

$$\int_S a_1 \left(1 + H(u-v)\right) e^{H(u-v)}\, dx$$

$$+ \int_S (L|Du| + a_2)|Dv| \left(1 + H(u-v)\right) e^{H(u-v)}\, dx$$

$$+ \int_S (H \cdot |Du|^2 + a_3)(u-v)\, e^{H(u-v)}\, dx.$$

Because $|u|, |v| \leq M$ a.e.

$$\int_S |Du|^2 \left(1 + H(u-v)\right) e^{H(u-v)}\, dx \leq$$

$$(1+2HM)e^{2HM} \int_S \left(a_1 + (L)|Du| + a_2)|Dv| + a_3\right) dx$$

$$+ \int_S |Du|^2 H(u-v)\, e^{H(u-v)}\, dx$$

ie

$$\int_S |Du|^2\, dx \leq \int_S |Du|^2 e^{H(u-v)}\, dx \leq$$

$$(1+2HM)e^{2HM} \int_S (a_1 + a_3)\, dx$$

$$+ \int_S \left(|Du| \cdot L(1+2HM)e^{2HM} |Dv| + (1+2HM)e^{2HM} a_2 |Dv|\right) dx$$

Use

$$|Du| \cdot L(1+2HM)e^{2HM}|Dv| \leq$$

$$\frac{1}{2}|Du|^2 + \frac{1}{2}L^2(1+2HM)^2 e^{4HM}|Dv|^2 \quad \text{etc}$$

and rearrange to get

$$\int_S |Du|^2 dx \leq c\int_S (|Dv|^2 + a_1 + a_2^2 + a_3)dx$$

with $c = L^2(1+2HM)^2 e^{4HM}$.

Hence with $a := a_1 + a_2^2 + a_3$,

$$\int_S (|Du|^2 + a)dx \leq (c+1)\int_S (|Dv|^2 + a)dx.$$

Recall: $S = \operatorname{spt}(u-v)^+$.

Next take $\varphi = (v-u)^+ e^{H(v-u)}$ to get

$$\int_{\operatorname{spt}(v-u)^+} (|Du|^2 + a)dx \leq (c+1)\int_{\operatorname{spt}(v-u)^+} (|Dv|^2+a)dx$$

Add inequalities to get $(K := \operatorname{spt}(u-v))$

$$\int_K (|Du|^2 + a)dx \leq 2(c+1)\int_K (|Dv|^2+a)dx$$

BACKGROUND RESULT: If $f \in W^{1,1}$, then $Df = 0$ a.e. on $\{f=c\}$.

General case: $v \in W^{1,2}(\Omega)$, $K := \operatorname{spt}(u-v) \Subset \Omega$.

Put
$$w(x) := \begin{cases} M & \text{if } v(x) > M \\ v(x) & \text{if } |v(x)| \leq M \\ -M & \text{if } v(x) < -M \end{cases}$$

ie $\quad w = \min\{M, \max\{v, -M\}\}$.

Then $w \in W^{1,2}(\Omega)$ and as $\|u\|_{L^\infty} = M$, $\operatorname{spt}(u-w) \subseteq K$, and $|Dw| \leq |Dv|$ a.e.

so

$$\int_K (|Du|^2 + a)\,dx \leq 2(C+1) \int_K (|Dw|^2 + a)\,dx$$
$$\leq 2(C+1) \int_K (|Dv|^2 + a)\,dx. \square$$

EX 11 Boundedness of the weak solution in EX 10 is essential:

Let $u(x) = \sqrt{2} \log \log \frac{1}{|x|}$; $|x| < \frac{1}{e}$ $(x \in \mathbb{R}^2)$.

Then u is an extremal for

$$v \mapsto \int_{B(0,\frac{1}{e})} \left(1 + \frac{1}{1 + e^{v} (\log|x|)^{-\sqrt{2}}}\right) |Dv|^2\,dx.$$

(scalar case!)

We shall show later that a ~~bounded~~ Q-minimizer for $\int_\Omega (|Du|^2 + a)\,dx$ must be bounded.

DEF. 12 $f : \Omega \to \mathbb{R}^N$ is α-Hölder continuous $(0 < \alpha < 1)$ iff $\exists\, c < \infty$ s.t.

$$(*) \quad |f(x) - f(y)| \leq c|x-y|^\alpha \quad \forall x, y \in \Omega.$$

$(*)$ is also called an α-Hölder condition on Ω (= an L^∞-Nikolskii condition of order α)

When $\alpha = 1$ we often refer to $(*)$ as a Lipschitz condition on Ω and say that f is Lipschitz continuous.

f is locally α-Hölder continuous on Ω iff f is α-Hölder continuous on each $\Omega' \Subset \Omega$.

$$C^{0,\alpha}(\Omega) = \{ f : f \ \alpha\text{-Hölder on } \Omega \}$$

$$[f]_{0,\alpha;\Omega} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is a seminorm on $C^{0,\alpha}(\Omega)$

(and $\|f\|_{L^\infty(\Omega)} + [f]_{0,\alpha;\Omega}$ is a norm making $C^{0,\alpha}(\Omega)$ a Banach space).

$C^{0,\alpha}_{loc}(\Omega) = \{ f : f \text{ locally } \alpha\text{-Hölder on } \Omega \}$

EX 13 $C^{0,\alpha}(\Omega)$, $\|\cdot\|_{L^\infty(\Omega)} + [\cdot]_{0,\alpha;\Omega}$ is a nonseparable Banach space and smooth functions are <u>not</u> dense.

On $(-1,1)$: $f(x) = |x|^\alpha$ is α-Hölder since $\left| |x|^\alpha - |y|^\alpha \right| \leq |x-y|^\alpha \;\forall x,y$.

If $0 < \alpha < \beta \leq 1$ and $\varphi \in C^{0,\beta}(-1,1)$ with $\varphi(0) = 0$, then for $x \neq 0$ and small $r > 0$,

$$\sup_{|x|<r} \frac{|f(x) - \varphi(x)|}{|x|^\alpha} \geqslant 1 - [\varphi]_{0,\beta;(-1,1)} r^{\beta-\alpha} > 0.$$

$\therefore$ $C^{0,\beta}$ <u>not</u> dense in $C^{0,\alpha}$ when $0<\alpha<\beta\leq 1$

EX 14 Limits to regularity of Q-minimizers.

Let $n \geq 2$, $0 < \alpha < 1$.

Then $u(x) = x_1 |x|^{-\alpha}$, $x \in B(0,1) \subset \mathbb{R}^n$,

is a bounded weak solution to PDE

$$(*) \quad \int_{B(0,1)} A_{ij}(x) D_j u D_i \varphi \, dx = 0 \quad \forall \varphi \in W_0^{1,2}(B(0,1)),$$

where

$$A_{ij}(x) = \delta_{ij} + \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)} \frac{x_i x_j}{|x|^2}.$$

Note $|A_{ij}| \leq 1 + \sigma$ and $(*)$ is Euler-Lagrange equation for convex functional

$$v \longmapsto \int_{B(0,1)} \left(|Dv|^2 + \sigma \left(\frac{x}{|x|} \cdot Dv \right)^2 \right) dx$$

with

$$\sigma := \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)} > 0.$$

The integrand $G(x,\xi) = |\xi|^2 + \sigma \left(\frac{x}{|x|} \cdot \xi\right)^2$ satisfies

$$|\xi|^2 \leq G(x,\xi) \leq (1+\sigma)|\xi|^2$$

and so u is a Q-minimizer for $\mathrm{Dir}[\cdot, B(0,1)]$ with $Q = 1 + \sigma$.

Note $Q \to 1^+$ as $\alpha \to 0+$ but

$$D_2 u = -\alpha \frac{x_1 x_2}{|x|^{\alpha+2}}$$ is unbounded near 0

(though u is $(1-\alpha)$-Hölder continuous).

Check that u is a solution to ⊛:

$$\int_{B(0,1)} A Du \cdot D\varphi = 0 \qquad \forall \varphi \in W_0^{1,2}(B(0,1))$$

$$A = (A_{ij}), \quad A_{ij} = \delta_{ij} + \sigma \frac{x_i x_j}{|x|^2}$$

$$\sigma = \frac{\alpha(n-\alpha)}{(1-\alpha)(n-1-\alpha)}$$

$$u(x) = x_1 |x|^{-\alpha} \in C^2(B(0,1))$$

so calculation yields $\operatorname{div} A Du = 0$ in $B(0,1) \setminus \{0\}$. Hence ⊛ holds for all $\varphi \in C_c^\infty(B(0,1) \setminus \{0\})$.

General supported test functions:

Let $\varphi \in C_c^\infty(B(0,1))$. For $0<\delta<\frac{1}{2}$ take cut-off function $\rho \in C^1(B(0,1))$ s.t.

$\rho = 0$ on $B(0,\delta)$

$\rho = 1$ on $B(0,1)\setminus B(0,2\delta)$, $0 \leq \rho \leq 1$

and $|D\rho| \leq \frac{2}{\delta}$.

Now $\rho\varphi \in C_c^1(B\setminus\{0\})$ so

$$0 = \int_{B(0,1)} ADu \cdot D(\rho\varphi)\,dx =$$

$$\int_{B(0,1)} \rho ADu \cdot D\varphi\,dx + \int_{B(0,1)} \varphi ADu \cdot D\rho\,dx.$$

Here

$$\left|\int_{B(0,1)} \varphi ADu \cdot D\rho\,dx\right| \leq \quad \text{Hölder } \frac{1}{p}+\frac{1}{p'}=1$$

$$c\|\varphi\|_{L^\infty} \int_{B(0,2\delta)\setminus B(0,\delta)} |Du|\frac{1}{\delta} \leq$$

$$c\|\varphi\|_{L^\infty} \frac{1}{\delta} |B(0,2\delta)\setminus B(0,\delta)|^{\frac{1}{p'}} \left(\int_{B(0,1)} |Du|^p\right)^{\frac{1}{p}} \leq$$

$$\tilde{c}\|\varphi\|_{L^\infty} \delta^{\frac{n}{p'}-1} \left(\int_{B(0,1)} |Du|^p\right)^{\frac{1}{p}} \longrightarrow 0 \text{ as } \delta \to 0^+$$

since $\frac{n}{p'}-1>0$ for $n\geq 2$, $p>2$. □

Hölder continuity, Schauder Estimates

DEF 15 For $k \in \mathbb{N}_0$, $0 < \alpha < 1$

$$[u]_{k,\alpha;\Omega} := [D^k u]_{0,\alpha;\Omega},$$

$$[D^k u]_{0,\alpha;\Omega} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha}.$$

<u>Note</u> Applying the mean value theorem componentwise it follows that

$$C^{j,1}_{loc} \subseteq C^{k,\alpha}_{loc} \quad \text{for } j < k.$$

But as before the $C^{j,1}_{loc}$ - functions are not dense in $C^{k,\alpha}_{loc}$ with seminorms $[\cdot]_{k,\alpha;\Omega'}$, $\Omega' \Subset \Omega$.

EX 16 Motivating example ... Let $B = B(0,1) \subset \mathbb{R}^2$.

If $f \in C^0(B)$ and $u \in W^{1,2}(B)$ is a weak solution to $\Delta u = f$ in B, then it's generally <u>not</u> true that $u \in C^2(B)$ ($\frac{\partial^2 u}{\partial x_1 \partial x_2}$ can be unbounded!) However if $f \in C^{0,\alpha}_{loc}$ then $u \in C^{2,\alpha}_{loc}$ (Schauder Estimates)

Let $\mathcal{B}(\ell, L)$ be the collection of all symmetric bilinear forms $A : \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying

$$\ell |\lambda|^2 \leq A[\lambda,\lambda] \leq L|\lambda|^2$$

for all $\lambda \in \mathbb{R}^{N\times n}$ $\quad (0 < \ell \leq L < \infty)$.

TH 17 (Generalization of Weyl's Lemma to A-harmonic maps)

If $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is A-harmonic in Ω, that is

$$\int_\Omega A[Du, D\varphi]\,dx = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N),$$

then $u \in C^\infty(\Omega, \mathbb{R}^N)$ and for each $k \in \mathbb{N}$ there exists a constant $c = c(k,n,N)$ s.t.

$$\sup_{B_R} |D^k u|^2 \leq c\left(\frac{L}{\ell}\right)^2 \frac{1}{R^{2k-2}} ⨍_{B_{2R}} |Du - (Du)_{B_{2R}}|^2\,dx$$

when $B_{2R} \subset \Omega$.