Recall from L4:

Given $n, N \in \mathbb{N}$, $n \geqslant 2$, and $0 < \ell \leq L < \infty$.

$\mathbb{B}(\ell, L)$ = set of symmetric bilinear forms
$\mathbb{A} : \mathbb{R}^{N\times n} \times \mathbb{R}^{N\times n} \rightarrow \mathbb{R}$ s.t.
$\ell |\lambda|^2 \leqq \mathbb{A}[\lambda,\lambda] \leqq L|\lambda|^2 \quad \forall \lambda \in \mathbb{R}^{N\times n}$.

- If $\mathbb{A} \in \mathbb{B}(\ell, L)$, then $\frac{1}{\ell}\mathbb{A} \in \mathbb{B}(1, \frac{L}{\ell})$.
- If $\mathbb{A} \in \mathbb{B}(\ell, L)$, then there exists a unique linear map $A : \mathbb{R}^{N\times n} \rightarrow \mathbb{R}^{N\times n}$ s.t.

$$\mathbb{A}[\xi, \eta] = A\xi \cdot \eta \quad \forall \xi, \eta \in \mathbb{R}^{N\times n}.$$

A is real and symmetric (self-adjoint) and its nN eigenvalues belong to $[\ell, L]$.

DEF Let $\mathbb{A} \in \mathbb{B}(\ell, L)$ and $\Omega \subseteq \mathbb{R}^n$ open. $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ is $\mathbb{A}$-harmonic on Ω iff

$$\operatorname{div} ADu = 0 \quad \text{on } \Omega, \text{ ie}$$

$$\int_\Omega \underbrace{\mathbb{A}[Du, D\varphi]}_{= ADu \cdot D\varphi}\, dx = 0 \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^N)$$

- $A = \mathrm{Id}$ corresponds to Laplace's equation (precisely, to N Laplace equations, one for each coordinate function of u).

Weyl's Lemma for $\mathbb{A}$-harmonic maps

If u is $\mathbb{A}$-harmonic on Ω, then $u \in C^\infty(\Omega, \mathbb{R}^N)$ and for $B_{2R} \Subset \Omega$ and $k \in \mathbb{N}$,

$$\sup_{B_R} |D^k u|^2 \leqq \frac{c_k}{R^{2(k-1)}} \fint_{B_{2R}} |Du|^2 \, dx,$$

where $c_k = c_k(n, N, \frac{L}{\ell})$.

If $f \in C^0$ then $\Delta u = f$ doesn't imply that u is twice cont. diff. — it can happen that mixed 2^{nd} order derivatives are unbounded! However, if $f \in C^{k,\alpha}_{loc}$ then $\Delta u = f$ implies that $u \in C^{k+2,\alpha}_{loc}$ (Schauder Estimates).

Schauder Estimates $\sim$ estimates in terms of $[\cdot]_{k,\alpha}$ semi-norms.

DEF Let $k \in \mathbb{N}_0$, $0 < \alpha < 1$, $\Omega \subseteq \mathbb{R}^n$ open

Semi-norms $[\cdot]_{k,\alpha;\Omega}$, $[\cdot]_{0,\alpha;\Omega}$:

$$[u]_{k,\alpha;\Omega} := [D^k u]_{0,\alpha;\Omega} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha}$$

$|\cdot|$ = 'usual norm' on k-linear maps into $\mathbb{R}^N$ (finite-dimensional, so all norms equivalent)

- $C^{k,\alpha}(\Omega, \mathbb{R}^N) := \{ u \in C^k(\Omega, \mathbb{R}^N) : [u]_{k,\alpha;\Omega} < \infty \}$

- $C^{k,\alpha}_{loc}(\Omega, \mathbb{R}^N) := \{ u \in C^k(\Omega, \mathbb{R}^N) : [u]_{k,\alpha;\Omega'} < \infty$ for each $\Omega' \Subset \Omega \}$.

Aim :- Prove Schauder Estimates for

$$\operatorname{div} A D u = f.$$

- Extend to x-dependent coefficients $A = A(x)$. (Our approach follows L. Simon: Calc. Var. & PDE 5 (1997), 391–407)

$\boxed{\text{Corollary 18}}$ (to Weyl's Lemma for A-harmonic maps)

Assume $v \in W^{1,2}_{loc}(\mathbb{R}^n, \mathbb{R}^N)$ is A-harmonic on $\mathbb{R}^n$ and that for some constants $c, q > 0$

$$\sup_{B(0,R)} |Dv| \leqq cR^q$$

for all $R \geqslant 1$. Then v is a polynomial.

$\boxed{\text{Pf:}}$ From Weyl's Lemma we find for each $k \in \mathbb{N}$ a $c_k < \infty$ s.t.

$$\sup_{B(0,R)} |D^k v|^2 \leqq \frac{c_k}{R^{2(k-1)}} \fint_{B(0,2R)} |Dv|^2$$

$$\leqq c_k c^2 R^{2(q+1-k)} .$$

Take $k > q+1$ and let $R \to \infty$ to conclude. $\square$

<u>Remark</u> Corollary 18 is referred to as a 'Liouville-type theorem'.

$\boxed{\text{Lemma 19}}$ Global Schauder Estimate.

Let $n, N \in \mathbb{N}$, $n \geqq 2$, $k \in \mathbb{N}$, $k \geqq 2$, $0 < \alpha < 1$ and $0 < \ell \leqq L < \infty$.

There exists a constant $c = c(n, N, k, \alpha, \frac{L}{\ell})$ s.t. for $A \in B(\ell, L)$, $u \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^N)$ and $f \in C^{k-2,\alpha}(\mathbb{R}^n, \mathbb{R}^N)$ with

$$\int_{\mathbb{R}^n} \left(A[Du, D\varphi] + f\varphi \right) dx = 0 \quad \forall \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^N)$$

it holds that

$$[u]_{k,\alpha;\mathbb{R}^n} \leqq \frac{c}{\ell} [f]_{k-2,\alpha;\mathbb{R}^n}.$$

$\boxed{\text{Pf:}}$ Philosophy 'Liouville $\Rightarrow$ regularity'

It suffices to prove the Lemma for $k=2$. and $A \in B(1, \frac{L}{\ell})$.

Indirect: Assume the Lemma is false.

Then for each $c = j \in \mathbb{N}$ we can find $A_j \in B(1, \frac{L}{\ell})$, $u_j \in C^{2,\alpha}$, $f_j \in C^{0,\alpha}$ with

$$\int_{\mathbb{R}^n} \left(A_j [Du_j, D\varphi] + f_j \varphi \right) = 0 \quad \forall \varphi \in C_c^1$$

but $[u_j]_{2,\alpha} > j\, [f_j]_{0,\alpha}$.

Clearly $[u_j]_{2,\alpha} > 0$ so we can normalize by dividing with $[u_j]_{2,\alpha}$ (linear PDE) and henceforth assume

$$[u_j]_{2,\alpha} = 1 , \quad [f_j]_{0,\alpha} < \frac{1}{j} .$$

Take $x_j \in \mathbb{R}^n$, $h_j \in \mathbb{R}^n \setminus \{0\}$ s.t.

$$(*) \quad \frac{|D^2 u_j(x_j + h_j) - D^2 u_j(x_j)|}{|h_j|^\alpha} > \frac{1}{2} \quad \forall j .$$

Define for $x \in \mathbb{R}^n$,

$$v_j(x) := |h_j|^{-2-\alpha} \Big(u_j(x_j + |h_j| x) - P_j(x_j + |h_j| x) \Big),$$

where

$$P_j(x) = \sum_{|\beta| \leq 2} \frac{D^\beta u_j(x_j)}{\beta!} (x - x_j)^\beta$$

(= the 2^{nd} Taylor polynomial for u_j about x_j)

Check that $[v_j]_{2,\alpha} = [u_j]_{2,\alpha} = 1$.

If $A_j : \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ is the linear map corresponding to $\mathbb{A}_j$, then

$$\operatorname{div} A_j DP_j(x) = (\operatorname{div} A_j Du_j)(x_j) = f_j(x_j)$$

and hence with

$$g_j(x) := |h_j|^{-\alpha} \left(f_j(x_j + |h_j| x) - f_j(x_j) \right), \quad x \in \mathbb{R}^n,$$

we have

$$[g_j]_{0,\alpha} = [f_j]_{0,\alpha} < \frac{1}{j}$$

and

$$\operatorname{div} A_j Dv_j = g_j \quad \text{on } \mathbb{R}^n.$$

Note: $[v_j]_{2,\alpha} = 1$ implies that (v_j), (Dv_j), $(D^2 v_j)$ are locally equi-cont so by Arzela-Ascoli's theorem there exists a subsequence (not relabelled) s.t.

$$v_j \to v, \quad Dv_j \to v^{(1)}, \quad D^2 v_j \to v^{(2)}$$

locally uniformly on $\mathbb{R}^n$. It is not hard to check that $v \in C^2(\mathbb{R}^n, \mathbb{R}^N)$, $D^i v = v^{(i)}$ and $[v]_{2,\alpha} \leq 1$.
Also $v(0) = 0$, $Dv(0) = 0$, $D^2 v(0) = 0$, ~~and~~
Furthermore (for additional subsubseq) we have $A_j \to A \in \mathcal{B}(1, \frac{L}{\ell})$,

$\frac{h_j}{|h_j|} \to h$, $|h| = 1$.

Since $[g_j]_{0,\alpha} \to 0$, v is A-harmonic on $\mathbb{R}^n$ and from ⊛ on p.6

$$|D^2 v(h)| = |D^2 v(h) - D^2 v(0)| \geqq \frac{1}{2}.$$

Since $\sup_{B_R} |Dv| \leqq \sup_{B_R} |D^2 v| \cdot R \leqq R$

the Liouville-type theorem yields that v is a polynomial. In particular,

$$\mathbb{R} \ni t \longmapsto D^2 v(th)$$

must be a <u>non-~~cons~~tant</u> polynomial

($D^2 v(h) \neq 0$ and $D^2 v(0) = 0$), so

because $0 < \alpha < 1$,

$$\sup_{t \in \mathbb{R}\setminus\{0\}} \frac{|D^2 v(th)|}{t^\alpha} = \infty.$$

This contradicts $[v]_{2,\alpha} \leqq 1 < \infty$. □

$\boxed{\text{Lemma 20}}$ Local Schauder Estimate.

Let $n, N \in \mathbb{N}$, $n \geq 2$, $k \in \mathbb{N}$, $k \geq 2$, $0 < \cancel{\tau,}\, \alpha < 1$
and $0 < \ell \leq L < \infty$.

There exists a constant $C = C(n, N, k, \alpha, \cancel{\tau}, \ell, L)$
with the following property.

For $A \in B(\ell, L)$, $u \in C^{k,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ and
$f \in C^{k-2,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ with

$$\int_\Omega \big(A[Du, D\varphi] + f\varphi \big)\, dx = 0 \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^N),$$

it holds that

$$[u]_{k,\alpha; B_{\tau r}} \leq C \left(\frac{1}{\ell} [f]_{k-2,\alpha; B_r} + r^{-k-\alpha} \|u\|_{L^\infty(B_r)} \right)$$

whenever $B_r \Subset \Omega$.

<u>Note</u> $B_r = B(x_0, r)$
and $B_{\tau r} = B(x_0, \tau r)$

$\boxed{\text{Pf:}}$ It suffices to prove the Lemma for $k=2$ and $\mathbb{A} \in \mathcal{B}(1, \frac{L}{\lambda})$.

Fix $B_r = B(x_0, r) \Subset \Omega$ and take cut-off $\rho \in C_c^\infty(B_r)$ satisfying $\mathbb{1}_{B_{\tau r}} \leq \rho \leq \mathbb{1}_{B_r}$ and $|D^j \rho| \leq \left(\frac{2}{(1-\tau)r}\right)^j$ $(j=1,2)$.

Let $$P(x) := \sum_{|\beta| \leq 2} \frac{D^\beta u(x_0)}{\beta!}(x-x_0)^\beta$$

($\equiv$ the 2nd Taylor polynomial for u about x_0)

and put $v = \rho(u-P)$. We have

$\operatorname{div} AD(u-P) = f - f(x_0)$ on Ω (A being the linear map corresponding to $\mathbb{A}$) and so for $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^N)$:

$$\int_{\mathbb{R}^n} \mathbb{A}[Dv, D\varphi] = \int_{\mathbb{R}^n} \mathbb{A}[\rho(Du-DP)+(u-P)\otimes D\rho, D\varphi]$$

$$= \int_{\mathbb{R}^n} \Big(\mathbb{A}[Du-DP, D(\rho\varphi)] + \mathbb{A}[(u-P)\otimes D\rho, D\varphi] - \mathbb{A}[Du-DP, \varphi \otimes D\rho] \Big)$$

$$= -\int_{\mathbb{R}^n} \Big(\rho(f-f(x_0)) + \operatorname{div} A(\underset{(u-P)}{u} \otimes D\rho) + A\underset{(Du-DP)}{Du}\, D\rho \Big) \cdot \varphi$$

so from the Global Schauder Estimate

(since $v \in C^{2,\alpha}$, $\rho(f-f(x_0)) \in C^{0,\alpha}$!)

$$[v]_{2,\alpha;\mathbb{R}^n} \leq G\frac{1}{l}[\rho(f-f(x_0)) + \operatorname{div} A(u-P)\otimes D\rho$$

$$+ A(Du-DP)D\rho]_{0,\alpha;\mathbb{R}^n}$$

$$\leq \frac{c}{l}\Big([\rho(f-f(x_0))]_{0,\alpha} + [\operatorname{div}(A(u-P)\otimes D\rho)]_{0,\alpha}$$

$$+ [A(Du-DP)D\rho]_{0,\alpha}\Big),$$

where $G = G(n, N, \alpha, \frac{L}{l})$.

Here

$$[\operatorname{div}(A(u-P)\otimes D\rho)]_{0,\alpha} \leq c\frac{L}{l}\Big([|Du-DP||D\rho|]_{0,\alpha}$$

$$+ [|u-P|D^2\rho]_{0,\alpha} \leq$$

$$c\frac{L}{l}\Big([Du-DP]_{0,\alpha;B_r}\|D\rho\|_{L^\infty} + \|Du-DP\|_{L^\infty(B_r)}[D\rho]_{0,\alpha}$$

$$+ [u-P]_{0,\alpha;B_r}\|D^2\rho\|_{L^\infty} + \|u-P\|_{L^\infty(B_r)}[D^2\rho]_{0,\alpha}\Big)$$

$$\leq \tilde{G}\frac{1}{(1-\tau)^{2+\alpha}}\cdot\frac{1}{r^\alpha}\|D^2(u-P)\|_{L^\infty(B_r)}.$$

Likewise $[A(Du-DP)D\rho]_{0,\alpha} \leq$

$$C\frac{L}{\ell}\frac{1}{(1-\tau)^{2+\alpha}}\frac{1}{r^\alpha}\|D^2(u-P)\|_{L^\infty(B_r)},$$

$$[\rho(f-f(x_0))]_{0,\alpha} \leq [\rho]_{0,\alpha}\|f-f(x_0)\|_{L^\infty(B_r)}$$

$$+\|\rho\|_{L^\infty}[f-f(x_0)]_{0,\alpha;B_r}$$

$$\leq \frac{C}{(1-\tau)^\alpha}[f]_{0,\alpha;B_r}.$$

Thus

$$[u-P]_{2,\alpha;B_{\tau r}} \leq \frac{C_1}{\ell}\left([f]_{0,\alpha;B_r}+r^{-\alpha}\|D^2(u-P)\|_{L^\infty(B_r)}\right)$$

$$\|$$

$$[u]_{2,\alpha;B_{\tau r}} \quad (+)$$

$\boxed{\text{Auxiliary Lemma 1}}$ Interpolation Inequality.

Let $g \in C^{2+\alpha}(B_r,\mathbb{R}^N)$.
Then for each $\varepsilon>0$ there exists $C_\varepsilon<\infty$ s.t.

$$r^{-\alpha}\|D^2g\|_{L^\infty(B_r)} \leq \varepsilon[g]_{2,\alpha;B_r}+C_\varepsilon r^{-2-\alpha}\|g\|_{L^\infty(B_r)}$$

$\boxed{\text{Pf:}}$ Note that

$$\mathbb{P}_2 = \{ p : \mathbb{R}^n \to \mathbb{R}^N : \ p \text{ polynomial of degree at most } 2 \}$$

is finite dimensional.

$p \mapsto |D^2 p|$ is a seminorm on $\mathbb{P}_2$

$p \mapsto \|p\|_{L^\infty(B(0,1))}$ is a norm on $\mathbb{P}_2$

so $\exists\, c < \infty$ s.t.

$$|D^2 p| \leq c \|p\|_{L^\infty(B(0,1))} .$$

Take $p \in \mathbb{P}$ and put $\cancel{\tilde{p}(x) = \varrho^2 p(\frac{x-x_0}{\varrho})}$.

$$\tilde{p}(x) = \varrho^{-2} p(x_0 + \varrho x)$$

Then $D^2 \tilde{p} = D^2 p$, $\|\tilde{p}\|_{L^\infty(B(0,1))} = \varrho^{-2} \|p\|_{L^\infty(B(x_0,\varrho))}$

to get $|D^2 p| \leq c \varrho^{-2} \|p\|_{L^\infty(B(x_0,\varrho))}$.

Note: c independent of ϱ, x_0.
Use this for pf of
→ Interpolation Inequality

Let P_y denote the 2^{nd} Taylor polynomial for g about $y \in B_r = B(x_0, r)$:

$$P_y(x) = \sum_{|\beta| \leq 2} \frac{D^\beta g(y)}{\beta!} (x-y)^\beta . \qquad (\in \mathbb{P}_2)$$

Note

$$|g(x) - P_y(x)| \leq c[g]_{2,\alpha;B_r} |x-y|^{2+\alpha}$$

ie if $x \in B_\varsigma(y) \subset B_r$ then $\|P_y\|_{L^\infty(B_\varsigma)} \leq \|g\|_{L^\infty(B_\varsigma)} + c\varsigma^{2+\alpha}[g]_{2,\alpha;B_r}$

Note $t \mapsto \|D^2 g\|_{L^\infty(B_t)}$ is cont.

so we may select $t \in (0,r)$ s.t.

$$\|D^2 g\|_{L^\infty(B_t)} > \frac{1}{2}\|D^2 g\|_{L^\infty(B_r)} .$$

Take $y \in \overline{B}_t$ s.t. $\|D^2 g\|_{L^\infty(B_t)} = |D^2 P_y| \leq$

$c\varsigma^{-2}\|P_y\|_{L^\infty(B_\varsigma)}$ for $\varsigma \in (0, r-t]$

$$\leq c\left(\varsigma^{-2}\|g\|_{L^\infty(B_r)} + \varsigma^\alpha [g]_{2,\alpha;B_r}\right)$$

Thus

$$\|D^2 g\|_{L^\infty(B_r)} \leq c\varsigma^{-2}\|g\|_{L^\infty(B_r)} + c\varsigma^\alpha [g]_{2,\alpha;B_r} .$$

Take $\varsigma \in (0, r-t)$ s.t. $c\varsigma^\alpha \leq \varepsilon r^\alpha$

to conclude. $\square$

Recall (+) from p.12 :

$$[u]_{2,\alpha;B_{\tau r}} \leq \frac{C}{\ell}\left([f]_{0,\alpha;B_r} + r^{-\alpha} \|D^2(u-P)\|_{L^\infty(B_r)} \right)$$

Interpolation Ineq.

$$\leq \frac{C}{\ell}\Big([f]_{0,\alpha;B_r} + \frac{\ell}{C}\varepsilon [u-P]_{2,\alpha;B_r}$$

$$+ C_{\frac{\ell}{C}\varepsilon}\, r^{-2-\alpha} \|u-P\|_{L^\infty(B_r)} \Big)$$

$$\leq \varepsilon [u]_{2,\alpha;B_r} + C_\varepsilon \|u\|_{L^\infty(B_r)} r^{-2-\alpha} + C' [f]_{0,\alpha;B_r}$$

and hence

$$(*) \quad r^{2+\alpha}[u]_{2,\alpha;B_{\tau r}} \leq \varepsilon r^{2+\alpha}[u]_{2,\alpha;B_r} + C_\varepsilon \|u\|_{L^\infty(B_r)} + C r^{2+\alpha}[f]_{0,\alpha;B_r}.$$

Note : C, C_ε depends on $n, N, \alpha, \tau, \ell, L$ too.

Auxiliary Lemma 2

Let S be a monoton subadditive function on the convex subsets of a ball $B \subset \mathbb{R}^n$: if $A, A_1, \dots, A_k \subset B$ are convex and $A \subset \bigcup_{j=1}^{k} A_j$, then $S(A) \leq \sum_{j=1}^{k} S(A_j)$.

Let $\tau_0 \in (0, \frac{1}{2}]$ and $k > 0$, $E > 0$.

Assume that for some $\varepsilon > 0$,

$$r^k S(B_{\tau_0 r}) \leq \varepsilon r^k S(B_r) + E$$

for all balls $B_r \subset B$.

There exists $\varepsilon_0 = \varepsilon_0(\tau_0, n, k) \in (0,1)$ s.t. if $\varepsilon \leq \varepsilon_0$, then for any ball $B_r \subset B$ and any $\tau \in (0,1)$ we have

$$r^k S(B_{\tau r}) \leq C_\tau E,$$

where $C_\tau = C_\tau(n, \tau_0, k) < \infty$.

Pf of Auxiliary Lemma 2:

Put $$Q := \sup_{B_r \subset B} r^k S(B_{\tau_0 r}).$$

By monotonicity of S, $Q < \infty$.

Let $B_{\tau_0^{-1} r} \subset B$. There exists a constant $C_0 = C_0(\tau_0, n) \in \mathbb{N}$ s.t. it's possible to find $y_1, \dots, y_m \in B_r$ with $m \leq C_0$ and so
$$B_r \subset \bigcup_{j=1}^{m} B(y_j, \tau_0^2 r).$$

Since $\tau_0 \leq \frac{1}{2}$, $B(y_j, r) \subset B$, so by properties of S
$$r^k S(B_r) \leq r^k \sum_{j=1}^{m} S(B(y_j, \tau_0^2 r)) \leq$$
$$\sum_{j=1}^{m} \left(\varepsilon r^k S(B(y_j, \tau_0 r)) + E \right) \leq \varepsilon m Q + m E$$
$$\leq \varepsilon C_0 Q + C_0 E.$$

This holds true for all balls with $B_{\tau_0^{-1} r} \subset B$, hence
$$Q \leq \varepsilon C_0 Q + C_0 E$$
and we conclude with $\varepsilon_0 = \frac{1}{2C_0}$. $\square$

Return to ⊛ on p.15 :

$$r^{2+\alpha}[u]_{2,\alpha;B_{\tau r}} \leq \varepsilon r^{2+\alpha}[u]_{2,\alpha;B_r} + C_\varepsilon \|u\|_{L^\infty(B_r)} + C r^{2+\alpha}[f]_{0,\alpha;B_r}$$

Fix $B_R \subset\subset \Omega$ and put

$$E = C_\varepsilon \|u\|_{L^\infty(B_R)} + C R^{2+\alpha}[f]_{0,\alpha;B_R} ,$$

so that with $S(B_r) = [u]_{2,\alpha;B_r}$ we get

$$r^{2+\alpha} S(B_{\tau r}) \leq \varepsilon r^{2+\alpha} S(B_r) + E$$

for all $B_r \subseteq B_R$. If $\varepsilon \leq \varepsilon_0(\alpha, n)$ we conclude by the auxiliary lemma □

$\boxed{\text{TH 21}}$ Local Schauder Estimate (final version, strong solutions)

Let $n, N \in \mathbb{N}$, $n \geq 2$, $k \in \mathbb{N}$, $k \geq 2$, $0 < \alpha, \tau < 1$ and $0 < \ell \leq L < \infty$.

There exists a constant $C = C(n, N, k, \alpha, \tau, \frac{L}{\ell})$ with the following property.

For $\mathbb{A} \in B(\ell, L)$, $f \in C^{k-2,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ and $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ with

$$\int_\Omega \left(\mathbb{A}[Du, D\varphi] + f\varphi \right) dx = 0 \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^N)$$

we have $u \in C^{k,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ and for any ball $B_r \Subset \Omega$,

$$[u]_{k,\alpha;B_{\tau r}} \leq C\left(r^{-k-\alpha} \|u\|_{L^\infty(B_r)} + \frac{1}{\ell}[f]_{k-2,\alpha;B_r} \right).$$

$\boxed{\text{Pf:}}$ Note that $\frac{1}{\ell}\mathbb{A} \in B(1, \frac{L}{\ell})$ and

$$\int_\Omega \left(\tfrac{1}{\ell}\mathbb{A}[Du, D\varphi] + \tfrac{1}{\ell} f\varphi \right) dx = 0 \quad \forall \varphi \in C^1_c.$$

Let $(\phi_\varepsilon)_{\varepsilon > 0}$ be a smooth mollifier and put $u_\varepsilon = \phi_\varepsilon * u$, $f_\varepsilon = \phi_\varepsilon * f$ on $\Omega_\varepsilon = \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon\}$.

Then $u_\varepsilon \in C^{k+2,\alpha}_{loc}(\Omega_\varepsilon, \mathbb{R}^N)$, $f_\varepsilon \in C^{k,\alpha}_{bc}(\Omega_\varepsilon, \mathbb{R}^N)$ and

$$0 = \int_\Omega \left(\frac{1}{\ell} A[Du_\varepsilon, D\varphi] + f_\varepsilon \varphi \right) dx, \quad \forall \varphi \in C^1_c(\Omega_\varepsilon, \mathbb{R}^N).$$

Consequently, given $B_r \Subset \Omega$ take $\tilde{\varepsilon} > 0$ so small that $B_r \Subset \Omega_{\tilde{\varepsilon}}$ to get from Lemma 20,

$$[u_\varepsilon]_{k,\alpha;B_{\tau r}} \leq C \left(\frac{1}{\ell} [f_\varepsilon]_{k-2,\alpha;B_r} + r^{-k-\alpha} \|u_\varepsilon\|_{L^\infty(B_r)} \right)$$

for $\varepsilon \in (0, \tilde{\varepsilon}]$. Note $C = C(n, N, k, \alpha, \tau, 1, \frac{L}{\ell})$

and

$$[f_\varepsilon]_{k-2,\alpha;B_r} \leq [f]_{k-2,\alpha;B_{r+\varepsilon}},$$

$$\|u_\varepsilon\|_{L^\infty(B_r)} \leq \|u\|_{L^\infty(B_{r+\varepsilon})},$$

$$[u]_{k,\alpha;B_{\tau r}} \leq \liminf_{\varepsilon \to 0} [u_\varepsilon]_{k,\alpha;B_{\tau r}}$$

to conclude. $\square$

$\boxed{\text{TH 22}}$ (x-dependent coefficients)

Let $n, N \in \mathbb{N}$, $n \geq 2$, $k \in \mathbb{N}$, $0 < \alpha, \tau < 1$

and $0 < \ell \leq L < \infty$.

Assume $A : \Omega \to \mathcal{L}^2(\mathbb{R}^{N\times n})$ (space of bilinear real forms on $\mathbb{R}^{N\times n}$)

is $C^{k-1,\alpha}_{loc}$ and $A(x) \in B(\ell, L)$ for all $x \in \Omega$.

If $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ and

$$\int_\Omega A(x)[Du, D\varphi]\, dx = 0 \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^N),$$

then $u \in C^{k,\alpha}_{loc}(\Omega, \mathbb{R}^N)$ and for any

ball $B_r \Subset \Omega$,

$$[u]_{k,\alpha; B_{\tau r}} \leq C\, r^{-k-\alpha} \|u\|_{L^\infty(B_r)}$$

where $C = C(n, N, k, \tau, \alpha, \frac{L}{\ell}, [A]_{k-1,\alpha;B_r}) < \infty$.

Pf. by perturbation method (often referred to as 'pf by freezing the coefficients').