$\boxed{\text{DEF 23}}$ Morrey space $L^{p,\lambda}$

Let $1 \leq p < \infty$, $0 \leq \lambda \leq n$, $\Omega \subset \mathbb{R}^n$.

$f \in L^{p,\lambda}_{loc}(\Omega, \mathbb{R}^d)$ iff for $\Omega' \Subset \Omega'' \Subset \Omega$

$$\sup_{\substack{x \in \Omega' \\ 0 < r < \operatorname{dist}(\Omega', \partial\Omega'')}} \left\{ r^{-\lambda} \int_{B(x,r)} |f|^p \right\} < \infty$$

<u>Note</u> $$L^p_{loc} = L^{p,0}_{loc} \subsetneqq \underset{0<\lambda<n}{L^{p,\lambda}_{loc}} \subsetneqq L^{p,n}_{loc} = L^\infty_{loc}$$

$\boxed{\text{TH 24}}$ Assume $\mathbb{A} \in C^0(\overline{\Omega}, \mathcal{L}^2(\mathbb{R}^{N \times n}))$

and $\mathbb{A}(x) \in B(\ell, L)$ for all x. If $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ satisfies

$$\int_\Omega \mathbb{A}(x)[Du, D\varphi] = 0 \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^N),$$

then $Du \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{N \times n})$ for all $\lambda < n$.

Furthermore, if $B_R \Subset \Omega$ then

$$r^{-\lambda} \int_{B_r} |Du|^2 \leq C_\lambda R^{-\lambda} \int_{B_R} |Du|^2 \quad \text{for } 0 < r < R.$$

NOTE: Inhomogeneous version valid too.

$\boxed{\text{Pf:}}$ Suppose $B_R = B(x_0, R)$.

Let $h \in W^{1,2}_u(B_R, \mathbb{R}^N)$ be $A(x_0)$-harmonic.

Then by the generalized Weyl's Lemma

(i) $\sup_{B_{R/2}} |Dh|^2 \leq c ⨍_{B_R} |Dh|^2 \leq \tilde{c} ⨍_{B_R} |Du|^2$,

(ii) $\sup_{B_{R/2}} |D^2h|^2 \leq \frac{c}{R^2} ⨍_{B_R} |Dh|^2 \leq \frac{\tilde{c}}{R^2} ⨍_{B_R} |Du|^2$,

where $c = c(n, N, \frac{L}{\ell})$.

For $0 < r < R/2$ we get from (ii):

(iii) $⨍_{B_r} |Dh - (Dh)_{B_r}|^2 \leq \sup_{B_{R/2}} |D^2h|^2 \cdot r^2 \leq c\left(\frac{r}{R}\right)^2 ⨍_{B_R} |Du|^2$.

Use $\varphi = u - h \in W^{1,2}_0(B_R, \mathbb{R}^N)$:

$$0 = \int_{B_R} A(x)[Du, D(u-h)]$$

$$= \int_{B_R} A(x_0)[Du - Dh, Du - Dh] + \int_{B_R} (A(x) - A(x_0))[Du, Du - Dh].$$

A uniformly cont. so $\exists$ modulus of cont.

$\omega = \omega(t)$ (nonnegative, nondecreasing, $\omega(t) \searrow 0$ as $t \searrow 0$)

s.t. $|A(x)-A(x_0)| \le \omega(|x-x_0|) \overset{x\in B_R}{\le} \omega(R)$,

hence

$$\ell \int_{B_R} |Du-Dh|^2 \le \omega(R) \int_{B_R} |Du|\,|Du-Dh|,$$

and so

$$\text{(iv)} \qquad \int_{B_R} |Du-Dh|^2 \le \frac{\omega(R)^2}{\ell^2} \int_{B_R} |Du|^2 .$$

Now if $0<r<R/2$ we get

$$\int_{B_r} |Du|^2 \le 2\int_{B_r} |Du-Dh|^2 + 2\int_{B_r} |Dh|^2$$

$$\overset{\text{(iv),(i)}}{\le} 2\,\frac{\omega(R)^2}{\ell^2} \int_{B_R} |Du|^2 + 2c\left(\frac{r}{R}\right)^n \int_{B_R} |Du|^2$$

$$\text{(v)} \qquad = c\left(\omega(R)^2 + \left(\frac{r}{R}\right)^n\right) \int_{B_R} |Du|^2 .$$

If $R/2 \le r < R$ then $\left(\frac{r}{R}\right)^n \ge 2^{-n}$ so (v) also holds in this case provided $c \ge 2^n$.

We conclude by use of the following... $\square$

25. Iteration Lemma

Let $\Phi : (0, R_0] \to [0,\infty)$ be nondecreasing and assume

$$\Phi(r) \leqq A\left[\varepsilon + \left(\frac{r}{R}\right)^\alpha\right]\Phi(R) + BR^\beta$$

for all $0<r<R\leqq R_0$, where $A, B, \alpha, \beta \geqq 0$ are constants and $\alpha > \beta$.

There exists $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta) > 0$ s.t. if $\varepsilon \leqq \varepsilon_0$, then

$$\Phi(r) \leqq c\left[\left(\frac{r}{R}\right)^\beta \Phi(R) + Br^\beta\right]$$

for all $0<r<R\leqq R_0$, where $c = c(\alpha,\beta,A)$ is a constant.

$\boxed{\text{Pf:}}$ Let $0<\tau<1$, $0<R\leqq R_0$. Then

$$\Phi(\tau R) \leqq A(\varepsilon + \tau^\alpha)\Phi(R) + BR^\beta$$

$$= A\tau^\alpha(\varepsilon\tau^{-\alpha} + 1)\Phi(R) + BR^\beta.$$

Take τ s.t.

$$2A\tau^\alpha = \tau^\gamma,$$

where $\gamma \in (\beta, \alpha)$. Put $\varepsilon_0 := \frac{1}{2}\tau^\alpha < \tau^\alpha$.

Then $\Phi(\tau R) \leqq \tau^\gamma \Phi(R) + BR^\beta$ and so by iteration $\Phi(\tau^{k+1}R) \leq \tau^{\gamma}\Phi(\tau^{k}R) + B\tau^{k\beta}R^{\beta}$

$$\leq \tau^{(k+1)\gamma}\Phi(R) + B\tau^{k\beta}R^{\beta}\sum_{j=0}^{k}\tau^{j(\gamma-\beta)}$$

$$\leq \tau^{(k+1)\gamma}\Phi(R) + B\,\frac{\tau^{k\beta}R^{\beta}}{1-\tau^{\gamma-\beta}}$$

$$\leq C\tau^{(k+1)\beta}\left(\Phi(R) + BR^{\beta}\right),$$

where $C := \dfrac{\tau^{-\beta}}{1-\tau^{\gamma-\beta}} = \dfrac{1}{\tau^{\beta}-\tau^{\gamma}}$.

Given $r \in (0,R)$ choose $k \in \mathbb{N}$ s.t.

$\tau^{k+1}R < r \leq \tau^{k}R$ to get the conclusion. $\square$

26. Campanato's integral characterization of Hölder continuity.

We start by defining the precise representative for $f \in L^{1}_{loc}(\Omega, \mathbb{R}^{d})$.

Let us say that $x \in \Omega$ is a good point for f iff

$$\lim_{r \to 0+} f_{x,r} \text{ exists in } \mathbb{R}^{d}.$$

Notation $f_{x,r} = f_{B(x,r)} := \dfrac{1}{\mathcal{L}^{n}(B(x,r))}\displaystyle\int_{B(x,r)} f(y)\,dy.$

Because $f_{x,r} = g_{x,r}$ whenever $f = g$ a.e. it follows that the notion of good point only depends on the equivalence class and not on the particular representative used to calculate the integral average. It follows from Lebesgue's differentiation theorem that $\mathcal{L}^n$-almost all points $x \in \Omega$ are good points for f.

The precise representative for f is the map $\tilde{f} : \Omega \to \mathbb{R}^d$ which is defined at all points by the formula

$$\tilde{f}(x) := \begin{cases} \lim\limits_{r \to 0+} f_{x,r} & \text{when } x \text{ is good for } f \\ 0 & \text{otherwise.} \end{cases}$$

We usually omit the tilde and write f also for the precise representative of f.

$\boxed{\text{TH}}$ (Campanato 1963, Meyers 1964.)

Let $f \in L^p(\Omega, \mathbb{R}^d)$, $1 \leq p < \infty$ and $0 < \alpha \leq 1$.

Assume

$$⨍_{B(x,r)} |f - f_{x,r}|^p \leq m^p r^{\alpha p}$$

for all $B(x,r) \subset \Omega$.

Then the precise representative $f \in C^{0,\alpha}_{loc}$ and moreover

$$|f(x) - f(y)| \leq c\, m\, |x-y|^{\alpha}$$

for $x, y \in B(x_0, \frac{r}{4})$ whenever $B(x_0, r) \subset \Omega$.

Here $c = c(n, \alpha, p)$.

$\boxed{\text{Pf:}}$ Let $x \in \Omega$ be good for f. If $B(x, \frac{R}{2}) \subset B(x_0, R)$ and $B(x_0, 2R) \subset \Omega$, then

$$
\begin{aligned}
|f_{x,\frac{R}{2}} - f_{x_0,R}| &\leq ⨍_{B(x,\frac{R}{2})} |f - f_{x_0,R}| \\
&\leq \left(⨍_{B(x,\frac{R}{2})} |f - f_{x_0,R}|^p \right)^{\frac{1}{p}} \\
&\leq 2^{\frac{n}{p}} \left(⨍_{B(x_0,R)} |f - f_{x_0,R}|^p \right)^{\frac{1}{p}} \\
&\leq 2^{\frac{n}{p}} m R^{\alpha}.
\end{aligned}
$$

Iteration yields:

$$
\begin{aligned}
|f_{x,2^{-j}r} - f_{x,r}| &\leq \sum_{i=1}^{j} |f_{x,2^{-i}r} - f_{x,2^{-i+1}r}| \\
&\leq \sum_{i=1}^{j} 2^{\frac{n}{p}} m (2^{-i+1} r)^{\alpha} \\
&\leq 2^{\frac{n}{p}} \frac{m r^{\alpha}}{1 - 2^{-\alpha}},
\end{aligned}
$$

and thus: $|f_{x,2^{-j}r} - f_{x,r}| \leq c_1 m r^\alpha \quad \forall j$

where $c_1 := \dfrac{2^{\frac{n}{p}}}{1-2^{-\alpha}}$. Let $j \to \infty$, we

get: $|f(x) - f_{x,r}| \leq c_1 m r^\alpha$

when $B(x,2r) \subset \Omega$ and f is the precise representative.

Hence if $B(x,\frac{r}{2}) \subset B(x_0,r)$ and $B(x_0,2r) \subset \Omega$, then

$$|f(x) - f_{x_0,r}| \leq |f(x) - f_{x,\frac{r}{2}}| + |f_{x,\frac{r}{2}} - f_{x_0,r}|$$
$$\leq c_1 m \left(\tfrac{r}{2}\right)^\alpha + 2^{\frac{n}{p}} m r^\alpha =: c_2 m r^\alpha .$$

Let $B(x,R) \subset \Omega$. If $x_1, x_2 \in B(x,\frac{R}{4})$ are good, put $\bar{x} = \frac{1}{2}(x_1+x_2)$ and $r = |x_1 - x_2|$. Then $B(x_i,\frac{r}{2}) \subset B(\bar{x},r) \subset B(x,R)$ and so by the above we infer

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f_{\bar{x},r}| + |f_{\bar{x},r} - f(x_2)|$$
$$\leq 2c_2 m r^\alpha,$$

that is, (with $c := 2c_2$):

(*) $|f(x_1) - f(x_2)| \leq cm|x_1 - x_2|^\alpha$

when $x_1, x_2 \in B(x,\frac{R}{4})$ are good.

We conclude by showing that all points

in $B(x,\frac{R}{4})$ are good: fix any $y\in B(x,\frac{R}{4})$ and for $B(y,r)\subset B(y,s)\subset B(x,\frac{R}{4})$ select a good $\bar{x}\in B(y,\frac{r}{4})$. Then

$$|f_{y,s}-f_{y,r}| \leq |f_{y,s}-f(\bar{x})| + |f(\bar{x})-f_{y,r}|$$
$$\leq cms^{\alpha} + cmr^{\alpha},$$

that is, $(f_{y,r})_{r>0}$ is Cauchy in $\mathbb{R}^d$. $\square$

27. Morrey's embedding theorem

Let $1\leq p\leq n$, $0<\lambda<n$.

Then $L^{p,\lambda}_{loc} \hookrightarrow C^{0,\alpha}_{loc}$

provided $\alpha := \frac{p+\lambda-n}{p} > 0$.

Moreover, if $r^{-\lambda}\int_{B(x,r)} |Du|^p \leq m$ for all $B(x,r)\subset\Omega$, then

$$\fint_{B(x,r)} |u-u_{x,r}|^p \leq cm\cdot r^{p+\lambda-n}$$

for all $B(x,r)\subset\Omega$.

Pf: By Poincaré's inequality

$$\fint_{B(x,r)} |u-u_{x,r}|^p \leq cr^p \fint_{B(x,r)} |Du|^p$$

and hence the theorem follows from Campanato's result. $\square$

We now turn to the proof of TH 22.
In fact we'll ~~state~~ state it's inhomogeneous version first:

$\boxed{\text{TH } 22'}$ Let $k \in \mathbb{N}$, $0 < \alpha < 1$, $0 < \ell \leq L < \infty$,
$\mathbb{A} \in C^{k-1,\alpha}_{loc}(\Omega, \mathcal{L}^2(\mathbb{R}^{N\times n}))$ with $\mathbb{A}(x) \in B(\ell, L)$ for all x and $V \in C^{k-1,\alpha}_{loc}(\Omega, \mathbb{R}^{N\times n})$.
If $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ and

$$\int_\Omega \left(\mathbb{A}(x)[Du, D\varphi] + V.D\varphi \right) dx = 0 \quad \forall \varphi \in C^1_c(\Omega, \mathbb{R}^N),$$

then $u \in C^{k,\alpha}_{loc}(\Omega, \mathbb{R}^N)$.

$\boxed{\text{Pf:}}$ We give details only for the homogeneous case and $k = 1$.

Fix $B_R = B(x_0, R) \Subset \Omega$.

Let $h \in W^{1,2}_u(B_R, \mathbb{R}^N)$ be $\mathbb{A}(x_0)$-harmonic. By the generalized Weyl's Lemma, $h \in C^\infty$ and

$$\sup_{B_{R/2}} |Dh|^2 \leq \tilde{c} \fint_{B_R} |Dh|^2 \leq c \fint_{B_R} |Du|^2 ,$$

$$\sup_{B_{R/2}} |D^2 h|^2 \leq \frac{\tilde{c}}{R^2} \fint_{B_R} |Dh - (Dh)_{B_R}|^2 \leq \frac{c}{R^2} \fint_{B_R} |Du - (Du)_{B_R}|^2 .$$

Hence for $0 < r < \frac{R}{2}$,

$$\fint_{B_r} |Dh - (Dh)_{B_r}|^2 \leq \sup_{B_{R/2}} |D^2 h|^2 \cdot r^2 \leq c \left(\frac{r}{R}\right)^2 \fint_{B_R} |Du - (Du)_{B_R}|^2 ,$$

or

$$\text{(i)} \quad \int_{B_r} |Dh - (Dh)_{B_r}|^2 \leq c \left(\frac{r}{R}\right)^{n+2} \int_{B_R} |Du - (Du)_{B_R}|^2 .$$

How well does h approximate u in B_R?

Test PDE by $\varphi = u - h \in W_0^{1,2}(B_R, \mathbb{R}^N)$:

$$\begin{aligned} 0 &= \int_{B_R} \mathbb{A}(x) [Du, D(u-h)] \\ &= \int_{B_R} \mathbb{A}(x_0) [Du - Dh, Du - Dh] \\ &\quad + \int_{B_R} (\mathbb{A}(x) - \mathbb{A}(x_0)) [Du, Du - Dh] \\ &\geq \ell \int_{B_R} |Du - Dh|^2 - [\mathbb{A}]_{0,\alpha;B_R} R^\alpha \int_{B_R} |Du| |Du - Dh| \end{aligned}$$

hence

$$\int_{B_R} |Du - Dh|^2 \leq \left(\frac{[A]_{0,\alpha;B_R}}{\ell}\right)^2 R^{2\alpha} \int_{B_R} |Du|^2 .$$

Recall from TH 24 that $Du \in L^{2,\lambda}_{loc}$ for all $\lambda < n$. Therefore

$$\text{(ii)} \quad \int_{B_R} |Du - Dh|^2 \leq G \cdot R^{2\alpha + \lambda} ,$$

$$G := \left(\frac{[A]_{0,\alpha;B_R}}{\ell}\right)^2 R^{-\lambda} \int_{B_R} |Du|^2 .$$

Now for $0 < r < \frac{R}{2}$ we get from (i), (ii):

$$\int_{B_r} |Du - (Du)_{B_r}|^2 \leq \int_{B_r} |Du - (Dh)_{B_r}|^2 \leq$$

$$2\int_{B_r} |Du - Dh|^2 + 2\int_{B_r} |Dh - (Dh)_{B_r}|^2 \leq$$

$$2G R^{2\alpha+\lambda} + 2c\left(\frac{r}{R}\right)^{n+2} \int_{B_R} |Du - (Du)_{B_R}|^2 ,$$

that is,

$$\text{(iii)} \quad \Phi(r) \leq A\left(\frac{r}{R}\right)^{n+2} \Phi(R) + B \cdot R^{\lambda + 2\alpha}$$

for $0 < r < \frac{R}{2}$ and $A := 2c$, $B := 2G$.

If $\frac{R}{2} \leq r < R$ then $\left(\frac{r}{R}\right)^{n+2} \geq 2^{-n-2}$ so (iii) also holds in this range provided we take $A \geqslant 2^{n+2}$. Now use the Iteration Lemma*) to conclude that

$$\text{(iv)} \quad \Phi(r) \leq C\left(\left(\tfrac{r}{R}\right)^{2\alpha+\lambda} \Phi(R) + B r^{2\alpha+\lambda} \right)$$

and hence $Du \in C^{0,\alpha'}_{loc}$ for each $\alpha' < \alpha$ by Campanato.

*) Φ is nondecreasing:
for any $g \in L^2$, $\int_A |g - g_A|^2 \leq \int_A |g - \xi|^2 \ \forall \xi$
so when $0 < r < s$,
$\Phi(r) \leq \int_{B_r} |Du - (Du)_{B_s}|^2 \leq \Phi(s)$.

In particular, $Du \in L^\infty_{loc}$ so that we can actually take $\lambda = n$ in (ii) and hence get (iv) with $\lambda = n$. Campanato then gives the conclusion $Du \in C^{0,\alpha}_{loc}$.

- The ~~inh~~ homogeneous case is exactly as above but ~~with~~ additional bookkeeping
- In the case $k > 1$ we take a multi-index β of length $k-1$ and proceed as

follows: for $\varphi \in C_c^\infty$, also $D^\beta \varphi \in C_c^\infty$

so

$$0 = \int_\Omega \left(A(x)[Du, D^\beta D\varphi] + V . D^\beta D\varphi \right)$$

$$= (-1)^{k-1} \int_\Omega \left(D^\beta(A(x)Du) . D\varphi + D^\beta V . D\varphi \right)$$

so $$0 = \int_\Omega \Big(A(x)[D(D^\beta u), D\varphi]$$

$$+ (W + D^\beta V) . D\varphi \Big)$$

lower order terms in u

Now apply $k=1$ version and Interpola=tion Inequalities (a la Auxiliary Lemma 1) $\square$