"De Giorgi – Nash – Moser theory"

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$\boxed{\text{TH 28}}$ Let $a \in L^p(\Omega)^+$ for some $p > \frac{n}{2}$ and let $Q > 1$.

If $u \in W^{1,2}_{loc}(\Omega)$ is a Q-minimizer for

$$\int_\Omega \left(|Du|^2 + a(x) \right) dx,$$

then u is locally Hölder continuous in Ω.

<u>NOTE</u> $N = 1$ so $u : \Omega \to \mathbb{R}$, result fails when $N > 1$.

We shall focus on the following:

$\boxed{\text{TH 29}}$ DeGiorgi's Theorem (1957)

Assume $A: \Omega \to \mathbb{R}^{n\times n}$ is measurable, $A(x) = A(x)^T$ and $\ell|\lambda|^2 \leq A(x)\lambda\cdot\lambda \leq L|\lambda|^2$ for all $\lambda \in \mathbb{R}^n$ for a.e. $x \in \Omega$.

If $u \in W^{1,2}_{loc}(\Omega)$ and $\operatorname{div} A(x) Du = 0$ in Ω, ie

$$\int_\Omega A(x) Du \cdot D\varphi \, dx = 0 \quad \forall \varphi \in C^1_c(\Omega),$$

then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha = \alpha(n, \frac{L}{\ell}) \in (0,1)$.

$\boxed{30}$

<u>Consequences:</u> 'Minimizers of regular variational problems are as regular as the data allow' (solution to Hilbert's 19th problem in scalar case).

If $F : \mathbb{R}^n \to \mathbb{R}$ is C^k $(k \geq 2)$ and

$$\ell|\lambda|^2 \leq F''(\xi)[\lambda,\lambda] \leq L|\lambda|^2$$

for all $\xi, \lambda \in \mathbb{R}^n$, then if $u \in W^{1,2}_{loc}(\Omega)$ is minimizing $\mathcal{F}(v,\Omega) = \int_\Omega F(Dv)$,

then $u \in C^{k-1,\alpha}_{loc}(\Omega)$ for all $\alpha < 1$.

Pf: We know that $u \in W^{2,2}_{loc}(\Omega)$ and that for each $1 \leq s \leq n$,

$$\int_\Omega F''(Du)[D(D_s u), D\varphi] = 0 \quad \forall \varphi \in C^1_c(\Omega).$$

Hence by DeGiorgi's theorem $D_s u \in C^{0,\tilde{\alpha}}_{loc}(\Omega)$ for some $\tilde{\alpha} = \tilde{\alpha}(n, \frac{L}{\ell}) \in (0,1)$.

Since $F \in C^2$, $F''(Du) \in C^0(\Omega, \mathbb{R}^{n\times n})$ and so by TH 24, $D(D_s u) \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^n)$ for each $\lambda < n$, $1 \leq s \leq n$. Hence by Morrey's embedding theorem $D_s u \in C^{0,\alpha}_{loc}(\Omega)$ for all $\alpha < 1$ (since $\alpha = \frac{2+\lambda-n}{2} \nearrow 1$ as $\lambda \nearrow n$), and so $u \in C^{1,\alpha}_{loc}(\Omega)$ for all $\alpha < 1$.

If $F \in C^k$, $k > 2$, then we can bootstrap:

From above $F''(Du) \in C^{0,\alpha}_{loc}$, $\alpha < 1$, and hence by Schauder estimates (TH 22), $D_s u \in C^{1,\alpha}_{loc}$, $1 \leq s \leq n$, and so $u \in C^{2,\alpha}_{loc}$, $\alpha < 1$.

Assume $k > j+2$, $j \geq 1$ and $0 < \alpha < 1$. If $u \in C^{j,\alpha}_{loc}$ then $F''(Du) \in C^{j-1,\alpha}_{loc}$ and so by Schauder Estimates, $D_s u \in C^{j+1,\alpha}_{loc}$, $1 \leq s \leq n$, hence $u \in C^{j+2,\alpha}_{loc}$. $\square$

<u>Note</u> The above argument used $N=1$ only because it is required in DeGiorgi's theorem (to get from $W^{1,2}_{loc}$ to $C^{1,\tilde{\alpha}}_{loc}$).

The remaining part of the argument that's based on Schauder Estimates works for any $N \geqslant 1$. Hence if we <u>for some reason</u> know the solution is $C^{1,\hat{\alpha}}_{loc}$ for an $\hat{\alpha} > 0$, then it's as regular as the data will allow (ie, $F \in C^k \Rightarrow u \in C^{k-1,\alpha}_{loc}$, $\alpha < 1$).

DeGiorgi's proof, that we only follow partly, relies on Caccioppoli inequalities on sub- and super-level sets of the solution, and careful iteration schemes.

$\boxed{\text{Lemma 31}}$ Under the conditions of DeGiorgi's theorem: If $B_R = B(x_0, R) \Subset \Omega$, $k \in \mathbb{R}$ and $0 < r < R$, then

(i) $$\int_{B_r} |D(u-k)^+|^2 \leqslant \frac{c}{(R-r)^2} \int_{B_R} |(u-k)^+|^2,$$

(ii) $$\int_{B_r} |D(u-k)^-|^2 \leqslant \frac{c}{(R-r)^2} \int_{B_R} |(u-k)^-|^2.$$

$\boxed{\text{Pf:}}$ Let $\rho \in C^1_c(B_R)$, $\mathbb{1}_{B_r} \leq \rho \leq \mathbb{1}_{B_R}$ and $|D\rho| \leq \frac{2}{R-r}$. Put $\varphi = \rho^2 (u-k)^+$.

Then $\varphi \in W^{1,2}(\Omega)$ with $\operatorname{spt}\varphi \subseteq \overline{B}_R \Subset \Omega$ and so we can use φ as test function:

$$0 = \int_\Omega A(x)\, Du \cdot D\varphi \, dx$$

$$= \int_{B_R} \left(A(x) Du \cdot D(u-k)^+ \rho^2 + A(x) Du \cdot D\rho \cdot 2\rho (u-k)^+ \right)$$

Since $D(u-k)^+ = \begin{cases} Du & \text{a.e. in } \{u>k\} \\ 0 & \text{a.e. in } \{u \leq k\} \end{cases}$

we get

$$\ell \int_{B_R} \rho^2 |D(u-k)^+|^2 \leq \int_{B_R} A(x) Du \cdot D(u-k)^+ \rho^2 ,$$

$$\int_{B_R} 2\rho A(x) D(u-k)^+ \cdot D\rho (u-k)^+ = \int_{B_R} 2\rho A(x) Du \cdot D\rho \, (u-k)^+ ,$$

hence

$$\ell \int_{B_R} \rho^2 |D(u-k)^+|^2 \leq -2 \int_{B_R} \rho A(x) D(u-k)^+ \cdot D\rho (u-k)^+$$

$$\leq 2 \int_{B_R} \rho \left(A(x) D(u-k)^+ \cdot D(u-k)^+ \right)^{\frac{1}{2}} \left(A(x) D\rho \cdot D\rho \right)^{\frac{1}{2}} (u-k)^+$$

$$\leq 2L \int_{B_R} \rho |D(u-k)^+| \, |D\rho| \, |(u-k)^+| \leq$$

$$2L\left(\int_{B_R}\rho^2|D(u-k)^+|^2\right)^{\frac{1}{2}}\left(\int_{B_R}|D\rho|^2|(u-k)^+|^2\right)^{\frac{1}{2}}$$

ie

$$\int_{B_R}\rho^2|D(u-k)^+|^2 \leq 4\left(\frac{L}{\ell}\right)^2\int_{B_R}|D\rho|^2|(u-k)^+|^2.$$

Invoking properties of ρ (i) follows with $c = 16\left(\frac{L}{\ell}\right)^2$. The proof of (ii) is similar. □

Note If (i) and (ii) of Lemma 31 hold, then we say that u is of DeGiorgi class on Ω (written: $u \in DG(\Omega)$). It can be shown that $DG(\Omega) \subset C^{0,\gamma}_{loc}(\Omega)$ (see E. Giusti's book 'Direct methods in the Calculus of Variations').

$\boxed{\text{Lemma 32}}$ If $u \in DG(\Omega)$, then $u \in L^\infty_{loc}(\Omega)$ and for $B_R \Subset \Omega$,

$$\sup_{B_{\frac{R}{2}}}|u| \leq c\left(\fint_{B_R}|u|^2\right)^{\frac{1}{2}}.$$

Notation: $A(k,R) = \{x \in B_R : u(x) > k\}$

$B(k,R) = \{x \in B_R : u(x) < k\}$.

In terms of super-level $A(k,R)$ and sublevel sets $B(k,R)$ the conditions (i), (ii) in the definition of $DG(\Omega)$ become

$$(i') \quad \int_{A(k,r)} |Du|^2 \leq \frac{c}{(R-r)^2} \int_{A(k,R)} |u-k|^2 ,$$

$$(ii') \quad \int_{B(k,r)} |Du|^2 \leq \frac{c}{(R-r)^2} \int_{B(k,R)} |u-k|^2 .$$

$\boxed{\text{Pf:}}$ WLOG $R=1$. Let $\frac{1}{2} \leq s < t \leq 1$ and take $\rho \in C_c^\infty(B_{\frac{s+t}{2}})$ with $\mathbb{1}_{B_s} \leq \rho \leq \mathbb{1}_{B_{\frac{s+t}{2}}}$ and $|D\rho| \leq \frac{4}{t-s}$. Put $v = \rho(u-k)^+$.

Then $v \in W_0^{1,2}(B_{\frac{s+t}{2}})$, $v = \begin{cases} u-k & \text{on } A(k,s) \\ 0 & \text{off } A(k,\frac{s+t}{2}) \end{cases}$

and so by Sobolev's inequality ($2^* = \frac{2n}{n-2}$, $n>2$)

$$\left(\int |v|^{2^*}\right)^{\frac{2}{2^*}} \leq c \int |Dv|^2,$$

ie

$$\int_{A(k,s)} |u-k|^2 \leq \int_{A(k,\frac{s+t}{2})} |v|^2 \overset{\text{Hölder}}{\leq} \left(\int_{A(k,\frac{s+t}{2})} |v|^{2^*}\right)^{\frac{2}{2^*}} |A(k,\tfrac{s+t}{2})|^{1-\frac{2}{2^*}}$$

$$\leq c\,|A(k,t)|^{\frac{2}{n}} \int_{A(k,\frac{s+t}{2})} |Dv|^2 \leq$$

$$2c\,|A(k,t)|^{\frac{2}{n}} \int_{A(k,\frac{s+t}{2})} \left(|Du|^2 + \frac{1}{(t-s)^2}|u-k|^2\right)$$

$$\overset{(i')}{\leq} \frac{c}{(t-s)^2} \int_{A(k,t)} |u-k|^2 \cdot |A(k,t)|^{\frac{2}{n}}.$$

If $h<k$ then $A(k,t) \subseteq A(h,t)$, hence

$$|A(k,t)|\,(k-h)^2 \leq \int_{A(h,t)} (u-h)^2$$

and
$$\int_{A(k,t)} |u-k|^2 \leqq \int_{A(h,t)} |u-k|^2 \leqq \int_{A(h,t)} |u-h|^2 .$$

Thus
$$\int_{A(k,s)} |u-k|^2 \leqq \frac{C}{(t-s)^2} \int_{A(h,t)} |u-h|^2 \left\{ \frac{1}{(k-h)^2} \int_{A(h,t)} |u-h|^2 \right\}^{\frac{2}{n}}$$

(*)
$$= \frac{C}{(t-s)^2} \frac{1}{(k-h)^{\frac{4}{n}}} \left(\int_{A(h,t)} |u-h|^2 \right)^{1+\frac{2}{n}}$$

for all $\frac{1}{2} \leqq s < t \leqq 1$ and $h < k$.

Fix $h_0 \in \mathbb{R}$ (determined later).

Put
$$k_j = 2h_0 (1 - 2^{-j-1})$$
$$s_j = \frac{1}{2}(1 + 2^{-j})$$
$$\Phi_j = h_0^{-2} \int_{A(k_j, s_j)} |u-k_j|^2 .$$

(*) with $s = s_{j+1}$, $t = s_j$, $k = k_{j+1}$, $h = k_j$ yields:
$$\Phi_{j+1} \leqq \frac{C}{(s_{j+1}-s_j)^2} \frac{1}{(k_{j+1}-k_j)^{\frac{4}{n}}} \Phi_j^{1+\frac{2}{n}}$$
$$\leqq C h_0^{-\frac{4}{n}} 2^{4j} \cdot \Phi_j^{1+\frac{2}{n}} .$$

Assume $h_0 \geq 1$. Then

$$(+) \quad \boxed{\Phi_{j+1} \leq C\, 2^{4j}\, \Phi_j^{1+\frac{2}{n}} \qquad \forall j}$$

Auxiliary Lemma

Let $A, \varepsilon > 0$ and $B > 1$. If (x_j) is a sequence of positive reals satisfying

$$x_{j+1} \leq A \cdot B^j\, x_j^{1+\varepsilon} \qquad \forall j$$

and if $x_0 \leq A^{-\frac{1}{\varepsilon}} B^{-\frac{1}{\varepsilon^2}}$, then $x_j \leq B^{-\frac{j}{\varepsilon}} x_0 \ \forall j$.

In particular, $x_j \longrightarrow 0$.

Pf: By induction on $j \in \mathbb{N}_0$:

$x_1 \leq A \cdot B^0 \cdot x_0^{1+\varepsilon} \leq A\,(A^{-\frac{1}{\varepsilon}} B^{-\frac{1}{\varepsilon^2}})^{\varepsilon}\, x_0 = B^{-\frac{1}{\varepsilon}} x_0$.

If $x_j \leq B^{-\frac{j}{\varepsilon}} x_0$, then $x_{j+1} \leq AB^j (B^{-\frac{j}{\varepsilon}} x_0)^{1+\varepsilon}$

$\leq AB^j B^{-\frac{j}{\varepsilon}-j} (A^{-\frac{1}{\varepsilon}} B^{-\frac{1}{\varepsilon^2}})^{\varepsilon} x_0 = B^{-\frac{j+1}{\varepsilon}} x_0$. $\square$

In $(+)$: $A = C$, $B = 16$, $\varepsilon = \frac{2}{n}$

$$\Phi_0 = h_0^{-2} \int_{A(h_0,1)} |u - h_0|^2 \leq h_0^{-2} \int_B |u|^2$$

so we should take $h_0 = 1 + \tilde{c}\left(\int_B |u|^2\right)^{\frac{1}{2}}$
for a suitable $\tilde{c} > 0$. Then $\Phi_j \longrightarrow 0$
and therefore as $k_j \to 2h_0$, $s_j \to \frac{1}{2}$. we
obtain

$$\int_{A(2h_0, \frac{1}{2})} |u - 2h_0|^2 = 0,$$

ie $u \leqq 2h_0$ on $B_{\frac{1}{2}}$.

Similarly we can show that $u \geqslant -2h_0$ on
$B_{\frac{1}{2}}$, and thus $u \in L^\infty(B_{\frac{1}{2}})$ and

$$\sup_{B_{\frac{1}{2}}} |u| \leqq 2 + 2\tilde{c}\left(\int_B |u|^2\right)^{\frac{1}{2}}.$$

Since tu $(t>0)$ is a solution too we
get

$$\sup_{B_{\frac{1}{2}}} |u| \leqq \frac{2}{t} + 2\tilde{c}\left(\int_B |u|^2\right)^{\frac{1}{2}} \quad \forall t > 0$$

and hence the conclusion. $\square$

33 Oscillation Lemma

Assume that $u \in W^{1,2}_{loc}(B_4)$ is a bounded weak solution, $\operatorname{div} A(x) Du = 0$ in B_4.

If

(i) $|\{x \in B_1 : u \leq 0\}| \geq \frac{1}{2}|B_1|$

then

(ii) $\sup_{B_1} u^+ \leq c\,|\{x \in B_2 : u > 0\}|^{\alpha} \sup_{B_4} u^+$,

where $c = c(n, \frac{L}{\ell}) > 0$ and $\alpha = \alpha(n, \frac{L}{\ell}) > 0$.

<u>Note</u> The Oscillation Lemma gives us Hölder continuity.