<u>Material from L9 & 10</u>

Oscillation Lemma $\Rightarrow$ Hölder cont.

$\boxed{\text{Pf :}}$

Given $u \in (W^{1,2} \cap L^\infty)(B_4)$ satisfying

$$\int_{B_4} A(x) Du \cdot D\varphi \, dx = 0 \quad \forall \varphi \in W_0^{1,2}(B_4).$$

If $\circledast$ (i) $|\{x \in B_1 : u \leq 0\}| \geq \frac{1}{2}|B_1|$

then (ii) $\sup_{B_1} u^+ \leq c\,|\{x \in B_2 : u > 0\}|^\alpha \sup_{B_4} u^+$.

Note $t(u-k)$ is also a bounded solution for all $t, k \in \mathbb{R}$. Hence we may assume WLOG that

$$\sup_{B_4} u = 1, \quad \inf_{B_4} u = -1.$$

Then (i) holds for u or $-u$.
WLOG it holds for u.

Now (i) also holds for $u-1+\varepsilon$, $0<\varepsilon<1$
so by Oscillation Lemma:

$$\sup_{B_1} (u-1+\varepsilon)^+ \leq c\,|\{x \in B_2 : u-1+\varepsilon > 0\}|^\alpha \sup_{B_4} (u-1+\varepsilon)^+$$

ie

$$\circledast \quad \sup_{B_1} u \leqq 1-\varepsilon + \varepsilon C \, |\{x \in B_2 : u > 1-\varepsilon\}|^{\alpha}$$

Put $A_t := \{x \in B_2 : u > t\}$.

Seek estimate of $|A_{1-\varepsilon}|$ for small $\varepsilon > 0$.

For $\delta > 1$, let $\varphi = \dfrac{\rho^2}{\delta - u}$, where

$\rho \in W_0^{1,\infty}(B_4)$. Then $\varphi \in W_0^{1,2}(B_4)$ so

we can test pde with φ:

$$0 = \int_{B_4} A Du \cdot \left(\frac{Du}{(\delta-u)^2} \rho^2 + \frac{2\rho D\rho}{\delta - u} \right)$$

hence
$$\ell \int_{B_4} \frac{|Du|^2}{(\delta-u)^2} \rho^2 \leqq L \int_{B_4} 2\rho \frac{|Du|}{\delta - u} |D\rho|$$

ie
$$\int_{B_2} \frac{|Du|^2}{(\delta-u)^2} \leqq C \quad (\text{const. indep. of } \delta > 1)$$

Take $\delta \searrow 1$:
$$\int_{B_2} \frac{|Du|^2}{(1-u)^2} \leqq C.$$

Now if $w = \max(-\log(1-u), 0)$, then

$w \in W^{1,2}(B_2)$ and by (i)

$$|\{x \in B_2 : w = 0\}| \geqslant |\{x \in B_2 : u \leqq 0\}| \geqslant \tfrac{1}{2}|B_1|$$

hence by a version of Poincaré's inequality (Th. 3.16 p. 102 in E. Giusti) we get

$\fint_{B_2} w^2 \leq c \fint_{B_2} |Dw|^2$ for an absolute constant c. Hence

$$\left(\fint_{B_2} w\right)^2 \leq c \fint_{B_2} |Dw|^2 \leq c \fint_{B_2} \frac{|Du|^2}{(1-u)^2} \leq \tilde{C}$$

and so

$$\tilde{C} \geq \fint_{B_2} w \geq \int_{A_{1-\varepsilon}} -\log(1-u) \geq |A_{1-\varepsilon}| \cdot \log \frac{1}{\varepsilon}.$$

Insert this in $\circledast$:

$$\sup_{B_1} u \leq 1 - \varepsilon + \varepsilon C \left(\frac{\tilde{C}}{\log \frac{1}{\varepsilon}}\right)^\alpha$$

$$= 1 - \underbrace{\left(1 - \frac{C_1}{(\log \frac{1}{\varepsilon})^\alpha}\right)}_{>0 \text{ for small } \varepsilon > 0} \varepsilon .$$

Hence we can take $\varepsilon > 0$ small and obtain $\sup_{B_1} u \leq 1 - \theta$ for some $0 < \theta < 1$.

Recall $\sup_{B_4} u = 1 = -\inf_{B_4} u$, so the oscillation of u is reduced by a constant factor on passing from B_4 to B_1:

$$\operatorname{osc}_{B_1} u = \sup_{B_1} u - \inf_{B_1} u \leq 1 - \theta - (-1) = 2 - \theta$$

$$= \frac{2-\theta}{2} \operatorname{osc}_{B_4} u .$$

By scaling we deduce that

$$\operatorname*{osc}_{B_{\frac{1}{4}r}} u \leq \frac{2-\theta}{2} \operatorname*{osc}_{B_r} u$$

and by iteration

$$\operatorname*{osc}_{B_{4^{-j}r}} u \leq \left(\frac{2-\theta}{2}\right)^j \operatorname*{osc}_{B_r} u .$$

This is Hölder continuity.