Automatic convexity of rank–1 convex functions ⋆

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Abstract

We announce new structural properties of 1–homogeneous rank–1 convex integrands, and discuss some of their consequences.

Résumé

Convexité automatique de fonctions convexes de rang 1. Nous présentons de nouvelles propriétés structurelles de fonctions convexes de rang 1 et 1–homogènes, ainsi que certaines conséquences.

Questions about sharp integral estimates for derivatives of mappings can often be recast as questions about certain semiconvexity properties of associated integrands (we refer the reader to [Dac89] for a survey of the relevant convexity notions and their roles in the calculus of variations). Particularly fascinating examples of the utility of this viewpoint are presented in [I02], where the fact that rank–1 convexity is a *manageable and necessary* condition for quasiconvexity leads to a long list of tempting conjectures, all of which – if proven – would have significant impact on the foundations of Geometric Function Theory in higher dimensions. The obstacle to success is that rank–1 convexity in general does not imply quasiconvexity. This negative result, known as Morrey's conjecture [Mo52], was established in [Sv92]. It does, however, not exclude the possibility that some of these semiconvexity notions agree within more restricted classes of integrands having natural homogeneity properties. A very interesting case being the positively 1–homogeneous integrands. Their semiconvexity properties correspond to L^1–estimates, and are therefore difficult to establish using interpolation or other harmonic analysis tools.

The purpose of this note is to announce the results of [KK10] about new structural properties of such integrands. In particular it is shown (Theorem 1) that a positively 1–homogeneous and rank–1 convex integrand must be convex at 0 and at all rank–1 matrices. This class of integrands has been investigated several times previously, see e.g. [DM07] or the older work [M92], where it was shown they are not necessarily convex at rank–2 matrices (and hence our result is sharp). The surprising automatically improved convexity at all matrices of rank at most one remained, however, unnoticed.

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The result can be viewed as a generalization of Ornstein's L^1–non–inequality (see Theorem 2), and in particular the approach allows also a streamlined and very elementary proof of the original Ornstein result. The link between an Ornstein type result, concerning the failure of the L^1–version of Korn's inequality, and semiconvexity properties of the associated integrand – though expressed in a dual formulation – was observed already in [CFM05]. There it was utilized in an ad-hoc construction which required a very sophisticated refinement in [CFMM05], where it was transferred from an essentially two–dimensional situation into three dimensions. Our arguments handle these situations with ease, see Theorem 3 below.

Due to concentration effects on rank–1 matrices, see [A93], our result seems taylored to simplify, and, in fact, was motivated by the characterization of BV Gradient–Young measures given in [KR10] (see [KK10] for more details).

The key result is best stated in abstract terms, and we pause to introduce the requisite terminology. Let V be a finite dimensional real vector space and $\mathcal{D}$ a balanced cone that spans V (so $tx \in \mathcal{D}$ for all $x \in \mathcal{D}$, $t \in \mathbb{R}$, and $\mathcal{D}$ contains a basis for V). A real–valued function $F\colon \mathrm{V} \to \mathbb{R}$ is $\mathcal{D}$–convex [MP98] provided its restrictions to lines in directions of $\mathcal{D}$ are convex: the functions $\mathbb{R} \ni t \mapsto F(x+ty)$ are convex for all $x \in \mathrm{V}$ and all $y \in \mathcal{D}$. The function F is positively 1–homogeneous provided $F(tx) = tF(x)$ for all $t > 0$ and all $x \in \mathrm{V}$. Finally we say that F has linear growth at infinity if there exist a norm $\|\cdot\|$ on V and a constant $c > 0$ such that $|F(x)| \leq c(\|x\| + 1)$ holds for all $x \in \mathrm{V}$.

Theorem 1 *Let* V *be a finite dimensional real vector space and let* $\mathcal{D}$ *be a balanced cone that spans* V*. If* $F\colon \mathrm{V} \to \mathbb{R}$ *is* $\mathcal{D}$*–convex, of linear growth at infinity, and positively* 1*-homogeneous, then* F *is convex at each point of* $\mathcal{D}$ *(so by* 1*–homogeneity, for each* $x_0 \in \mathcal{D}$ *there exists a linear function* $\ell\colon \mathrm{V} \to \mathbb{R}$ *satisfying* $\ell(x_0) = F(x_0)$ *and* $F \geq \ell$*).*

We remark that the conclusion remains unchanged if the function is only defined on an open convex cone in V. The prototypical examples to have in mind for $\mathcal{D}$ are the rank–one cone when $\mathrm{V} = \mathbb{R}^{N\times n}$, the space of first derivatives or, see below, when V is the space of k^{th} order derivatives of maps from $\mathbb{R}^n$ to $\mathbb{R}^N$. The full proof is presented in [KK10]. However, if we additionally assume that F is differentiable at $x_0 \in \mathcal{D} \setminus \{0\}$, then the proof is very easy:

Proof of Theorem 1 under additional differentiability assumption. Assume that F is differentiable at $x_0 \in \mathcal{D} \setminus \{0\}$. Fix a finitely supported probability measure μ on V with centre of mass at x_0. We must show $I := \int_{\mathrm{V}} (F - F(x_0))\,\mathrm{d}\mu \geq 0$. Let $A\colon \mathrm{V} \to \mathbb{R}$ be a linear function with $F(x_0) = A(x_0)$, so that by homogeneity also $F = A$ on the half–line $\{tx_0 : t > 0\}$. Clearly, I is unchanged if we replace F by $F - A$, hence we may assume that $F = 0$ on the half–line $\{tx_0 : t > 0\}$. Now the key is to observe that $F(x) \geq F(x + x_0)$ for all x. Indeed, this is seen to be a consequence of $\mathcal{D}$–convexity and linear growth as follows. First, linear growth and the fact that $\mathcal{D}$ spans V gives Lipschitz continuity in a standard way (see, e.g. [BKK00] and [KK10] for details): for a constant L and a norm $\|\cdot\|$, $|F(x) - F(y)| \leq L\|x - y\|$ for all x, $y \in \mathrm{V}$. Next, for $x \in \mathrm{V}$, $\lambda \in (0, 1)$, using convexity in the x_0–direction, and then Lipschitz continuity yield

$$
\begin{aligned}
F(x + x_0) &\leq (1 - \lambda)F(x) + \lambda F(x + \frac{1}{\lambda}x_0) \leq (1 - \lambda)F(x) + \lambda(F(\frac{1}{\lambda}x_0) + L\|x\|) \\
&= (1 - \lambda)F(x) + \lambda L\|x\|,
\end{aligned}
$$

and the observation follows upon letting $\lambda \searrow 0$. To conclude the proof rewrite $I = \int_{\mathrm{V}} F(tx)/t\,\mathrm{d}\mu$ for $t > 0$, and hence, by the above observation and since $F(x_0) = 0$,

$$
I \geq \int_{\mathrm{V}} \frac{F(tx + x_0) - F(x_0)}{t}\,\mathrm{d}\mu(x) \overset{t\to 0}{\to} \int_{\mathrm{V}} F'(x_0)[x]\,\mathrm{d}\mu(x) = F'(x_0)[x_0] = F(x_0) = 0.
$$

Finally, by homogeneity it follows that F is convex at all points of the half–line $\{tx_0 : t \geq 0\}$. *QED*

A remarkable result of Ornstein [Or62] states that given a set of linearly independent linear homogeneous constant–coefficient differential operators in n variables of order k, say B, $Q_1, \ldots, Q_m$, and any number $K > 0$, there is a C^∞ smooth function f vanishing outside the unit cube such that $\int |Bf| > K$ and $\int |Q_j f| < 1$ for all $1 \le j \le m$.

This result convincingly manifest the fact that estimates for differential operators, usually based on Fourier multipliers and Calderon-Zygmund operators, can be obtained for all L^p, $p \in (1, \infty)$ by interpolation and (more directly) even for the weak-L^1 spaces but fail to extend to the limit case $p = 1$. Ornstein used his result to answer a question by L. Schwarz by constructing a distribution in the plane that was not a measure but whose first order partial derivatives were distributions of order -1. He then gave a very technical and rather concise proof of his statement for general dimensions n and degree k. Whereas the more transparent first part of his paper finally received the recognition it deserved (e.g. for proving non-solvability of $\operatorname{div}\Phi = f \in \mathrm{L}^\infty$ with $\Phi \in \mathrm{W}^{1,\infty}$ see the nice duality argument in [MM98] and [BB03]), its higher order version was by the same authors not used in a very similar situation ([BB07]).

Our main result not only gives a simple and convincing proof of all these so–called L^1–non–inequalities, but also persists if we admit vector–valued maps and certain nonlinear differential expressions. Let us state a special version. We denote by $\mathrm{C}^\infty_c(\mathbb{R}^n, \mathbb{R}^N)$ the space of compactly supported C^∞ maps from $\mathbb{R}^n$ into $\mathbb{R}^N$, whose k^{th} derivative $D^k f(x)$ are for every $x \in \mathbb{R}^n$ in $L^k_s(\mathbb{R}^n, \mathbb{R}^N)$, the space of symmetric k–linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^N$.

Theorem 2 *Let $P\colon L^k_s(\mathbb{R}^n, \mathbb{R}^N) \to \mathbb{R}$ be a continuous and 1–homogeneous function (i.e., $P(t\xi) = |t| P(\xi)$ for $t \in \mathbb{R}$ and all ξ). Then $\int_{\mathbb{R}^n} P(D^k f(x))\,\mathrm{d}x \ge 0$ for all $f \in \mathrm{C}^\infty_c(\mathbb{R}^n, \mathbb{R}^N)$, if and only if $P(\xi) \ge 0$ for all $\xi \in L^k_s(\mathbb{R}^n, \mathbb{R}^N)$.*

Proof. Only one implication needs comment. By assumption

$$Q(\xi) := \inf_{\varphi \in \mathrm{C}^\infty_c((0,1)^n, \mathbb{R}^N)} \int_{(0,1)^n} P(\xi + D^k \varphi(x))\,\mathrm{d}x$$

equals 0 at $\xi = 0$. It is then easily checked that Q is real–valued and 1–homogeneous. By a standard argument (see [KK10] for details) Q is $\mathcal{D}$–convex, where the *rank–1 cone*, $\mathcal{D} := \{ b \otimes \bigotimes^k a : \; a \in \mathbb{R}^n, b \in \mathbb{R}^N \}$, is a balanced spanning cone. From Theorem 1 we deduce that Q is convex at 0, and as $P \ge Q$ with $P(0) = Q(0) = 0$ also P is convex at 0. The conclusion, $P \ge 0$, now follows from the 1–homogeneity.

QED

To see that Theorem 2 implies Ornstein's non–inequality, including a natural vector–valued version, note that $Bf = \tilde{B}(D^k f)$, $Q_j f = \tilde{Q}_j(D^k f)$ for all $f \in \mathrm{C}^\infty_c(\mathbb{R}^n, \mathbb{R}^N)$, where $\tilde{B}$, $\tilde{Q}_j\colon L^k_s(\mathbb{R}^n, \mathbb{R}^N) \to \mathbb{R}^\ell$ are linear. Now Ornstein's non–inequality amounts to equivalence of the statements:

(i) There exists a linear $C\colon \mathbb{R}^{\ell m} \to \mathbb{R}^\ell$ such that for all $\xi \in L^k_s(\mathbb{R}^n, \mathbb{R}^N)$, $\tilde{B}(\xi) = C(\tilde{Q}_1(\xi), \ldots, \tilde{Q}_m(\xi))$.

(ii) There exists $c > 0$ such that for all $f \in \mathrm{C}^\infty_c(\mathbb{R}^n, \mathbb{R}^N)$, $\|Bf\|_{\mathrm{L}^1} \le c \sum_{j=1}^m \|Q_j f\|_{\mathrm{L}^1}$.

We assume that (ii) holds and deduce (i): Define $P(\xi) := c \sum_{j=1}^m |\tilde{Q}_j(\xi)| - |\tilde{B}(\xi)|$ for $\xi \in L^k_s(\mathbb{R}^n, \mathbb{R}^N)$, and note that P is a continuous, 1–homogeneous function to which Theorem 2 applies. Accordingly, P is a nonnegative function, and hence the inclusion $\ker \tilde{B} \supset \bigcap \ker \tilde{Q}_j$ must hold for the kernels. A standard linear algebra argument allows us to conclude (i).

It is well–known that the distributional Hessian of a real–valued convex function is a (matrix–valued) measure. The natural question arises if this is valid also for the semi-convexity notions important in the vectorial calculus of variations. In [CFMM05] a fairly complicated construction was introduced to show that this is not true for rank–1 convex functions defined on symmetric 2×2 matrices.

Applying the ideas outlined above to the negative of the euclidean norm on the open cone of strictly rank–1 convex second gradients we show in combination with [Ki03] (see [KK10]):

Theorem 3 *Let $n > 1$ be an integer. There exists a rank–1 convex function $F\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ whose distributional Hessian F'' is not a bounded measure in any open nonempty subset O of $\mathbb{R}^{n\times n}_{\mathrm{sym}}$:*

$$\sup \int_O F \frac{\partial^2 \Phi}{\partial x_{ij} \partial x_{i'j'}} = \infty,$$

where the supremum is over all $\Phi \in \mathrm{C}^\infty_c(O)$ with $\sup|\Phi| \le 1$ and $i, j, i', j' \in \{1, \ldots, n\}$.

References

[A93] G. Alberti, *Rank one property for derivatives of functions with bounded variation*, Proc. Roy. Soc. Edinburgh Sect. A **123**(2) (1993), 239–274.

[BKK00] J.M. Ball, B. Kirchheim, and J. Kristensen, *Regularity of quasiconvex envelopes*, Calc. Var. Partial Differential Equations **11** (2000), 333–359.

[BB03] J. Bourgain and H. Brezis, *On the equation* $\operatorname{div} Y = f$ *and application to control of phases*, J. Amer. Math. Soc. **16**(2) (2003), 393–426.

[BB07] J. Bourgain and H. Brezis, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. (JEMS) **9**(2) (2007), 277–315.

[CFM05] S. Conti, D. Faraco, and F. Maggi, *A new approach to counterexamples to L^1 estimates: Korn's inequality, geometric rigidity, and regularity for gradients of separately convex functions*, Arch. Ration. Mech. Anal. **175**(2) (2005), 287–300.

[CFMM05] S. Conti, D. Faraco, F. Maggi and S. Müller, *Rank-one convex functions on 2×2 symmetric matrices and laminates on rank-three lines*, Calc. Var. Partial Differential Equations **24**(4) (2005), 479–493.

[Dac89] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Applied Mathematical Sciences, vol. 78, Springer-Verlag, 1989.

[DM07] B. Dacorogna and P. Maréchal, *The role of perspective functions in convexity, polyconvexity, rank-one convexity and separate convexity*, J. Convex Anal. **15**(2) (2008), 271–284.

[I02] T. Iwaniec, *Nonlinear Cauchy-Riemann operators in $\mathbb{R}^n$*, Trans. Amer. Math. Soc. **354** (2002), 1961–1995.

[Ki03] B. Kirchheim, *Rigidity and geometry of microstructures*, Lecture Notes MPI Mathematics in the Sciences Leipzig **16**, 2003.

[KK10] B. Kirchheim and J. Kristensen, *On rank one convex functions that are homogeneous of degree one*, in preparation.

[KR10] J. Kristensen and F. Rindler, *Characterization of Generalized Gradient Young Measures Generated by Sequences in* $\mathrm{W}^{1,1}$ *and* BV, Arch. Ration. Mech. Anal. **197** (2010), 539–598.

[MM98] C.T. McMullen, *Lipschitz maps and nets in Euclidean space*, Geom. Funct. Anal. **8**(2) (1998), 304–314.

[MP98] J. Matoušek and P. Plecháč, *On functional separately convex hulls*, Discrete Comput. Geom., **19** (1998), 105–130.

[Mo52] C.B. Morrey, Jr, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.

[M92] S. Müller, *On quasiconvex functions which are homogeneous of degree* 1, Indiana Univ. Math. J. **41** (1992), 295–301.

[Or62] D. Ornstein, *A non-inequality for differential operators in the L^1-norm*, Arch. Ration. Mech. Anal. **11** (1962), 40–49.

[Sv92] V. Šverák, *Rank-one convexity does not imply quasiconvexity*, Proc. Roy. Soc. Edinburgh Sect. A **120**(1–2) (1992), 185–189.