

# Compactness versus Regularity in the Calculus of Variations

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## Abstract

In this note we take the view that compactness in  $L^p$  can be seen quantitatively on a scale of fractional Sobolev type spaces. To accommodate this viewpoint one must work on a scale of spaces, where the degree of differentiability is measured, not by a power function, but by an arbitrary function that decays to zero with its argument. In this context we provide new  $L^p$  compactness criteria that were motivated by recent regularity results for minimizers of quasiconvex integrals. We also show how rigidity results for approximate solutions to certain differential inclusions follow from the Riesz–Kolmogorov compactness criteria.

**Key words:** *Weak convergence methods, Compactness criteria, Differential inclusion*

## Introduction

Two of the main questions in the Calculus of Variations and in the theory of Partial Differential Equations are the regularity and compactness properties of minimizers and solutions. These two issues that might appear, at a first glance, of different nature are in fact intimately related and in a sense even equivalent. In this note we highlight this relation, and illustrate it by investigating some concrete examples. In particular, we will discuss some differential inclusions where the relation is especially neat. Furthermore, we will emphasize that the criteria for compactness can be seen as a weak version of regularity. In fact we will provide a new compactness criterion which is inspired by the regularity theory for minimizers of quasiconvex functionals.

In recent years there has been a growing interest in understanding the compactness properties of approximate solutions to differential inclusions of the basic type

$$Df \in K, \tag{1}$$

where  $K$  is a given closed set of  $k$  times  $n$  matrices,  $\mathbb{R}^{k \times n}$ , and  $f: \Omega \rightarrow \mathbb{R}^k$  is a Sobolev mapping. Here  $\Omega \subset \mathbb{R}^n$  is a bounded open set, which, as it turns out, is essentially irrelevant for

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the compactness issues discussed here. Precisely, for a given exponent  $p \in [1, \infty)$ , one would like to characterize those sets  $K$  with the following rigidity property: whenever a sequence  $(f_j)$  in  $W_{\text{loc}}^{1,p}(\Omega)$  satisfies the conditions

- (i)  $f_j \rightharpoonup f$  weakly in  $W_{\text{loc}}^{1,p}$ , and
- (ii)  $\text{dist}_K(Df_j) \rightarrow 0$  strongly in  $L^p(\Omega)$ ,

then it follows that  $f_j \rightarrow f$  strongly in  $W_{\text{loc}}^{1,p}$ . Here we denoted  $\text{dist}_K(\xi) := \inf\{|\xi - \eta| : \eta \in K\}$ , where  $|\cdot|$  is the usual euclidean norm on  $\mathbb{R}^{k \times n}$ . We refer, somewhat imprecisely, to the maps  $f_j$  satisfying the conditions (i)–(ii) as *approximate solutions* to (1), and note that if the set  $K$  has the above rigidity property, then limits  $f$  of approximate solutions  $f_j$  will be *exact solutions* of (1):  $Df \in K$  a.e. in  $\Omega$ .

Questions of this type appear naturally in the compensated compactness theory [28] and in connection with variational models for microstructures [3, 4, 20], and they have been considered in detail for example in [2, 7, 15, 27, 23, 24, 25]. For compact sets  $K \subset \mathbb{R}^{2 \times 2}$  a geometric characterization (absence of rank-one connections and  $T4$  configurations) was recently obtained in [8].

On the other hand, it is known that a lack of compactness for approximate solutions to the inclusion (1) can sometimes be utilized to build very irregular exact solutions. This is best seen in the convex integration method of Müller and Šverák initiated in [21] and further developed for example in [15, 16, 22, 26]. However, as the  $T4$  configuration shows, it is possible to have no compactness for approximate solutions even though exact solutions must all be affine, see [5].

In the spirit of this note we will investigate whether one can reverse this philosophy. That is, whether one can prove *first* some regularity for approximate solutions and from that conclude compactness. This is indeed the case, for example, if  $K$  is contained in a affine subspace without rank-one connections such as the conformal matrices in  $\mathbb{R}^{2 \times 2}$ . This philosophy is implicitly used in [25] to deal with the so-called multiple incompatible wells corresponding to the set  $K = \cup_{i=1}^n SO(2)H_i$  where  $H_i$  are 2 times 2 matrices. The key fact here is that there exists a constant  $\mathcal{K}$ , such that for  $A, B \in K$ ,

$$\mathcal{K} \det(A - B) \geq \|A - B\|^2,$$

that is,  $A - B$  is  $\mathcal{K}$ -quasiconformal. Hence solutions to the differential inclusion are locally Hölder continuous with exponent  $1/\mathcal{K}$ , and a similar argument provides compactness for approximate solutions.

In general, however, one can not expect such strong pointwise regularity results but instead merely a control on the oscillation of the function in an integral average sense. Classically, results in this direction amounts to versions of Riesz–Kolmogorov type theorems, which relate compactness with uniform control on the modulus of continuity – obviously these conditions are related to so-called Nikolskii conditions used to define classes of fractional Sobolev functions. In fact, the spaces are closely related to the generalized Lipschitz spaces defined in [14]. In the first section of the paper we will review a version of this theorem which is appropriate for our

purposes. We will also provide a new compactness criterion, which is related to the conditions often used in the definitions of Sobolev–Slobodetskii functions. As it can be formulated in terms of the *excess* it is possible that it is more amenable for applications in regularity theory. In fact, the criterion was inspired by a Carleson condition for the excess that plays a role in the regularity theory for minimizers of strongly quasiconvex integrals in [17]. We will provide a proof based on generalized Young measures which might be of its own interest.

In section 2 we will turn back to differential inclusions to show how this point of view provides compactness for sets  $K$  such that  $\mathcal{L}(A - B) > 0$  for all distinct  $A, B \in K$ , where  $\mathcal{L}$  is a null Lagrangian, that is, a linear combination of subdeterminants. This was proved by Šverák in [24] using the weak continuity of null Lagrangians. Here it will be a corollary of the compactness criteria.

## 1 Compactness Criteria

In this section we provide two compactness criteria in which the extra regularity is measured in two different manners. We start by stating a version of the Riesz-Kolmogorov theorem for domains in  $\mathbb{R}^n$ . There is a substantial literature on the theme of compactness, and a nice review can be found in [12]. Let  $\Omega$  be an open set in  $\mathbb{R}^n$  and  $\Omega' \Subset \Omega$  a compactly contained open subset. Let  $f \in L^p(\Omega, \mathbb{R}^k)$ . For  $t \leq \text{dist}(\Omega', \Omega)$  we define the  $L^p$  modulus of continuity  $\omega_{f, \Omega'}(t)$  as follows:

$$\omega_{f, \Omega'}(t) = \sup_{|y| \leq t} \left( \int_{\Omega'} |f(x+y) - f(x)|^p dx \right)^{\frac{1}{p}}.$$

We will write simply  $\omega_f$  when no confusion concerning the domain is possible. Notice that when  $\omega_f(t) \leq t^\epsilon$ , then  $f$  belongs to  $B_p^{\epsilon, \infty}(\Omega', \mathbb{R}^k)$ . This space, within the scale of Besov spaces, is often called a Nikolskii space. In the more general situation where  $\omega_f(t) \leq v(t)$ , for some positive increasing function  $v(t)$  that tends to 0 at 0 it seems natural to say that  $f$  belongs to  $B_p^{v(\cdot), \infty}(\Omega', \mathbb{R}^k)$  (with the obvious definition). Indeed such spaces have been defined and investigated in [14].

**Theorem 1.** *Let  $\Omega' \Subset \Omega$  and  $\mathfrak{F} \subset L^p(\Omega, \mathbb{R}^k)$ . Then  $\mathfrak{F}$  is precompact in  $L^p(\Omega', \mathbb{R}^k)$  if and only if the following two conditions hold:*

(C1) *There exists a constant  $M < \infty$  such that  $\int_{\Omega'} |f|^p dx \leq M$  for all  $f \in \mathfrak{F}$ .*

(C2) *There exists a nondecreasing function  $v: [0, \infty) \rightarrow [0, \infty)$  with  $\lim_{t \searrow 0} v(t) = 0$  such that*

$$\omega_f(t) \leq v(t)$$

*for  $t \leq \text{dist}(\Omega', \Omega)$  and all  $f \in \mathfrak{F}$ .*

*Proof.* As mentioned already there are many proofs. Perhaps the more direct is the following: To prove that (C1) and (C2) guarantee compactness consider a sequence  $(f_j)$  and a standard convolution kernel  $\rho$ . Then declare  $\rho_k = k^n \rho(kx)$  and  $f_j^k = \rho_k * f_j$ . On one hand, the equicontinuity condition (C2) yields that the convergence of  $f_j^k$  to  $f$  in  $L^p$  as  $k$  tends to infinity is uniform

in  $j$ . On the other hand, for fixed  $k$ ,  $(f_j^k)$  satisfies the conditions in the classical Ascoli–Arzelá theorem and thus has a converging subsequence in  $L^\infty$ . Together with a diagonal argument this provides us with a converging subsequence of  $(f_j)$  in  $L^p(\Omega')$ .

The necessity of (C1) and (C2) follows easily from total boundedness: For each  $\epsilon$  we can by assumption cover  $\mathfrak{F}$  by a finite  $\epsilon$  net, that is, there exists a finite collection  $\{f_j\}_{j=1}^{N(\epsilon)}$  such that  $\mathfrak{F} \subset \bigcup_{j=1}^{N(\epsilon)} B(f_j, \epsilon)$ . The uniform bound in the  $L^p$  norm follows from  $\epsilon = 1$ . Since for  $f \in B(f_j, \epsilon)$  it holds that  $\omega(f)(t) \leq \omega(f_j)(t) + \epsilon$  we get for  $t$  such that  $\sup_{1 \leq j \leq N(\epsilon)} \omega(f_j)(t) \leq \epsilon$  that

$$\sup_{f \in \mathfrak{F}} \omega_f(t) \leq 2\epsilon,$$

and we find that (C2) holds. □

**Remark 1.** *We stated the theorem above to describe compactness in  $\Omega' \Subset \Omega$  because this is enough for the application in the next section. However the control on the modulus of continuity yields also equiintegrability and hence convergence in  $\Omega$  under mild assumptions. A quick proof is valid for domains with porous boundaries. Let  $Q(x, r)$  be a cube centered at  $x$  and with radius  $r$ . We assume the existence of a constant  $\alpha > 0$  such that for  $x \in \Omega$  and  $r < \text{diam}(\Omega)$  there exists  $y \in Q(x, r) \cap \Omega$  such that  $Q(y, \alpha r) \subset Q(x, r) \cap \Omega$  (this is valid for Lipschitz domains, uniform domains and others)*

*Let  $Q_m = Q(0, 2^{-m}) \subset \Omega$ ,  $2^{-j} < \text{diam}(\Omega)$  and  $\tilde{Q} \subset Q_j \cap \Omega$  cube of side  $\alpha 2^{-j}$  given by porosity. Clearly,  $\tilde{Q}$  contains at least  $[\alpha 2^{m-j}]$  cubes  $\{Q_i\}$  of side  $2^{-m}$ . For each  $Q_i$  we find  $|y_i| \leq 2^{-j}$  such that  $Q_i = Q_m + y_i$ . Equicontinuity (C2) implies that for each  $i$*

$$\int_{Q_m} |f|^p \leq \int_{Q_i} |f|^p + v(2^{-j})$$

*Therefore if we add this inequality for each  $i$*

$$\begin{aligned} [\alpha 2^{m-j}] \int_{Q_m} |f|^p &\leq \sum \int_{Q_i} |f|^p + [\alpha 2^{m-j}] v(2^{-j}) \\ &\leq \int_{Q_j \cap \Omega} |f|^p + [\alpha 2^{m-j}] v(2^{-j}) + \leq M + v(2^{-j}) \end{aligned}$$

*and thus,*

$$\int_{Q_m} |f|^p \leq \frac{M}{[\alpha 2^{m-j}]} + v(2^{-j})$$

*Choose now  $m = 4j$ , let  $j \rightarrow \infty$  and equiintegrability follows.*

A crucial step in proving porosity of the singular set for minimizers  $u \in W^{1,\infty}(\Omega, \mathbb{R}^k)$  of a strongly quasiconvex integral in [17] involved a Carleson type condition for the quadratic excess defined as

$$E(x, r) := \int_{B(x,r)} |Du - (Du)_{x,r}|^2.$$

For a minimizer  $u$  of a strongly quasiconvex integral the condition reads

$$\int_{B(x_0, R)} \int_0^R E(x, r) \frac{dr}{r} \leq cE(x_0, 2R) \quad (2)$$

whenever  $B(x_0, 2R) \subset \Omega$ , where  $c$  is an absolute constant. We show here how this kind of condition can be seen as a special case in a family of compactness conditions.

For global versions of compactness results the following condition (which is implied by porosity) on the set  $\Omega$  is natural (see [11]). A bounded open subset  $\Omega$  of  $\mathbb{R}^n$  is said to satisfy the  $\alpha$ -density condition, for some constant  $\alpha \in (0, 1)$ , provided for all  $x \in \overline{\Omega}$  and all  $r \in (0, \text{diam}\Omega)$  we have

$$\mathcal{L}^n(\Omega(x, r)) \geq \alpha \mathcal{L}^n(B(x, r)),$$

where  $\Omega(x, r) := \Omega \cap B(x, r)$  denotes the relative ball. For mappings  $f: \Omega \rightarrow \mathbb{R}^k$  of class  $L^p$  we write for  $x \in \overline{\Omega}$  and  $r \in (0, \text{diam}(\Omega))$

$$E_f(x, r) = \int_{\Omega(x, r)} |f - f_{x, r}|^p \quad \text{and} \quad f_{x, r} := \int_{\Omega(x, r)} f,$$

where a bar on an integral as usual signifies an integral average. For comparison we recall from [6] that the Sobolev–Slobodetskii spaces  $B_p^{\epsilon, p}(\mathbb{R}^n)$  can be characterized as those  $f \in L^p$  for which

$$\int_{\mathbb{R}^n} \int_0^\infty E_f(x, r) \frac{dr}{r^{1+\epsilon p}} dx < \infty.$$

**Lemma 1.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open subset satisfying the  $\alpha$ -density condition, and let  $1 \leq p < \infty$  be an exponent. Then a norm-bounded family  $\mathfrak{F} \subset L^p(\Omega, \mathbb{R}^k)$  is precompact in  $L^p(\Omega, \mathbb{R}^k)$  if and only if there exist numbers  $m < \infty$ ,  $\delta \in (0, \text{diam}(\Omega))$ , and an increasing, strictly positive function  $\omega: [0, \infty) \rightarrow (0, \infty)$  satisfying*

$$\int_0^1 \frac{dt}{\omega(t)} = \infty \quad (3)$$

and

$$\int_{\Omega} \int_0^\delta E_f(x, r) \frac{dr}{\omega(r)} dx \leq m \quad (4)$$

for all  $f \in \mathfrak{F}$ .

*Proof.* We give the details for the case  $p = 1$  only and leave the other cases to the interested reader.

*Sufficiency:* Observe that  $\mathfrak{F}$  is precompact if we can show that any norm-bounded sequence  $(f_j)$  satisfying (4) admits an  $L^1$ -strongly converging subsequence. By standard results there exists a subsequence (for convenience still denoted by  $f_j$ ) that generates a generalized Young measure  $(\nu_x, \lambda, \nu_x^\infty)$ , see [1] and [18]. We recall that  $\nu_x$ ,  $x \in \Omega$ , are probability measures on  $\mathbb{R}^k$  (the usual oscillation Young measure),  $\lambda$  is a nonnegative finite measure on  $\overline{\Omega}$  (the concentration

measure, giving location and magnitude of concentration in  $\overline{\Omega}$ , and  $\nu_x^\infty$ ,  $x \in \overline{\Omega}$ , probability measures on the unit sphere  $\mathbb{S}^{k-1} \subset \mathbb{R}^k$  (the concentration–angle measures, giving the direction of concentration at  $x$ ). In particular it follows that

$$f_j \xrightarrow{*} f \quad \text{in} \quad C^0(\overline{\Omega}, \mathbb{R}^k)^*,$$

where  $f = \bar{\nu}_x \mathcal{L}^n + \bar{\nu}_x^\infty \lambda$ , and a bar over a probability measure denotes its center of mass.

For each  $x \in \Omega$ , the set  $R_x := \{r \in (0, \text{dist}(x, \partial\Omega)) : \lambda(\partial B(x, r)) > 0\}$  is at most countable, and for fixed  $x \in \Omega$  and  $r \in (0, \text{dist}(x, \partial\Omega)) \setminus R_x$  we have

$$\lim_{j \rightarrow \infty} (f_j)_{x,r} = f_{x,r} := \frac{f(\Omega(x, r))}{\mathcal{L}^n(\Omega(x, r))}$$

and

$$\liminf_{j \rightarrow \infty} E_{f_j}(x, r) \geq \int_{B(x,r)} \int_{\mathbb{R}^k} |\xi - f_{x,r}| \, d\nu_y(\xi) \, dy.$$

Hence by Fatou's lemma and (4) we deduce for each  $\Omega' \Subset \Omega$  with  $\text{dist}(\Omega', \partial\Omega) \leq \delta$ ,

$$\int_{\Omega'} \int_0^{\text{dist}(\Omega', \partial\Omega)} \int_{B(x,r)} \int_{\mathbb{R}^k} |\xi - f_{x,r}| \, d\nu_y(\xi) \, dy \frac{dr}{\omega(r)} \, dx \leq m. \quad (5)$$

Let  $f = f^{ac} dx + f^s$  denote the Lebesgue–decomposition of  $f$  with respect to Lebesgue measure. By Lebesgue's differentiation theorem, as  $r \searrow 0$ ,

$$\int_{B(x,r)} \int_{\mathbb{R}^k} |\xi - f_{x,r}| \, d\nu_y(\xi) \, dy \rightarrow \int_{\mathbb{R}^k} |\xi - f^{ac}(x)| \, d\nu_x(\xi)$$

for  $\mathcal{L}^n$ –almost all  $x$ . By Egorov's theorem the convergence is also  $\mathcal{L}^n$ –almost uniform, hence if the set

$$E := \{x \in \Omega : \int_{\mathbb{R}^k} |\xi - f^{ac}(x)| \, d\nu_x(\xi) > 0\}$$

has positive  $\mathcal{L}^n$ –measure we easily reach a contradiction from (5) and (3): Indeed, then there exist  $\Omega' \Subset \Omega$ , a Borel set  $E' \subset \Omega'$  with  $\mathcal{L}^n(E') > 0$ , numbers  $\varepsilon > 0$  and  $r_0 > 0$  such that

$$\int_{B(x,r)} \int_{\mathbb{R}^k} |\xi - f_{x,r}| \, d\nu_y(\xi) \, dy > \varepsilon$$

for all  $x \in E'$  and  $r \leq r_0$ . Now from (5) we deduce that

$$m \geq \varepsilon \mathcal{L}^n(E') \int_0^{r_0} \frac{dr}{\omega(r)},$$

which is impossible by (3). Thus the set  $E$  is null as asserted, and hence  $\nu_x = \delta_{f^{ac}(x)}$  on  $\Omega$ , that is,  $(f_j)$  does not oscillate, but converges in  $\mathcal{L}^n$ –measure to  $f^{ac}$ .

Next we turn to the concentration measure  $\lambda$ . Let  $x \in \Omega$ ,  $r \in (0, \delta)$ ,  $\varepsilon \in (0, 1)$  and put

$$\Omega(x, r)_\varepsilon := \{y \in \Omega(x, r) : \text{dist}(y, \partial\Omega(x, r)) > \varepsilon\}.$$

We start with the estimate

$$\begin{aligned} E_{f_j}(x, r) &\geq \frac{1}{\mathcal{L}^n(\Omega(x, r))} \int_{\Omega(x, r)_\varepsilon} |f_j - (f_j)_{x, r}| \\ &\geq \frac{1}{2\mathcal{L}^n(\Omega(x, r))} \int_{\Omega(x, r)_\varepsilon} |f_j - (f_j)_{\Omega(x, r)_\varepsilon}|. \end{aligned}$$

Now the set  $R_{x, r} := \{\varepsilon \in (0, 1) : \lambda(\partial\Omega(x, r)_\varepsilon) > 0\}$  is at most countable, and for  $\varepsilon \in (0, 1) \setminus R_{x, r}$  we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\Omega(x, r)_\varepsilon} |f_j - (f_j)_{\Omega(x, r)_\varepsilon}| &= \int_{\Omega(x, r)_\varepsilon} |f^{ac} - f_{\Omega(x, r)_\varepsilon}| + \lambda(\Omega(x, r)_\varepsilon) \\ &\geq \lambda(\Omega(x, r)_\varepsilon). \end{aligned}$$

Therefore also

$$\liminf_{j \rightarrow \infty} E_{f_j}(x, r) \geq \frac{\lambda(\Omega(x, r)_\varepsilon)}{2\mathcal{L}^n(\Omega(x, r))}$$

for all  $\varepsilon \in (0, 1) \setminus R_{x, r}$ , and so by standard properties of measures,

$$\liminf_{j \rightarrow \infty} E_{f_j}(x, r) \geq \frac{\lambda(\Omega(x, r))}{2\mathcal{L}^n(\Omega(x, r))} = \frac{1}{2}\lambda_{x, r}.$$

Combining this with Fatou's lemma and (4) we arrive at

$$\int_{\Omega} \int_0^\delta \lambda_{x, r} \frac{dr}{\omega(r)} dx \leq 2m \quad (6)$$

We now use Fubini to rewrite

$$\begin{aligned} \int_{\Omega} \lambda_{x, r} dx &= \int_{\overline{\Omega}} \int_{\Omega} \frac{1}{\mathcal{L}^n(\Omega(x, r))} \mathbf{1}_{\Omega(y, r)}(x) dx d\lambda(y) \\ &\geq \int_{\overline{\Omega}} \int_{\Omega} \frac{1}{\mathcal{L}^n(B(y, r))} \mathbf{1}_{\Omega(y, r)}(x) dx d\lambda(y) \\ &= \int_{\overline{\Omega}} \frac{\mathcal{L}^n(\Omega(y, r))}{\mathcal{L}^n(B(y, r))} d\lambda(y). \end{aligned}$$

Observe that for each  $y \in \overline{\Omega}$  and  $r \in (0, \text{diam}\Omega)$  we have according to the  $\alpha$ -density condition,

$$\frac{\mathcal{L}^n(\Omega(y, r))}{\mathcal{L}^n(B(y, r))} \geq \alpha$$

and so

$$\int_{\Omega} \lambda_{x, r} dx \geq \alpha \lambda(\overline{\Omega}). \quad (7)$$

From (5) and (3) follows then easily that  $\lambda(\overline{\Omega}) = 0$ , that is,  $(f_j)$  does not concentrate, but is equi-integrable. Consequently, we can identify  $f = f^{ac}$ , and  $f_j \rightarrow f$  strongly in  $L^1(\Omega, \mathbb{R}^k)$ . The proof of sufficiency is complete.

*Necessity:* Assuming that the family  $\mathfrak{F}$  is precompact in  $L^1$ , define for  $r > 0$ ,

$$\theta(r) := \sup_{f \in \mathfrak{F}} \int_{\Omega} E_f(x, r) dx.$$

We assert that

$$\lim_{r \searrow 0} \theta(r) = 0.$$

Assume for a moment that the assertion holds true. We can also assume that for a constant  $m$ ,  $m \geq \theta(r) > 0$  for all  $r > 0$ . Let  $\mu$  be the concave envelope of  $\theta$ . It is easy to check that  $\mu(r) \rightarrow 0$  as  $r \searrow 0$ , and because concave (real-valued) functions are absolutely continuous with non-increasing derivatives we can find  $\varepsilon, \delta > 0$  such that  $\mu'(r) > \varepsilon$  for a.e.  $r \in (0, \delta)$ , and such that  $\mu$  is increasing on  $(0, \delta)$ . We can therefore define  $w : (0, \infty) \rightarrow (0, \infty)$  by

$$w(r) := \begin{cases} \frac{\mu(r)}{\mu'(r)} & \text{for } 0 < r < \delta \\ \sup_{r < \delta} \left( \frac{\mu(r)}{\mu'(r)} \right) & \text{for } r \geq \delta. \end{cases}$$

Now it is easy to check that with this choice,  $w$  is positive, increasing and both (3) and (4) hold. Hence it only remains to show that the assertion is true. As a first step let us note that for each  $f \in L^1(\Omega, \mathbb{R}^k)$ ,

$$\int_{\Omega} E_f(x, r) dx \rightarrow 0 \text{ as } r \searrow 0. \quad (8)$$

Indeed, for  $\varepsilon > 0$  take  $g \in C_c^0(\Omega, \mathbb{R}^k)$  such that  $\|f - g\|_{L^1} < \varepsilon$ . Now invoking the  $\alpha$ -density condition and Fubini we estimate

$$\int_{\Omega} E_f(x, r) dx \leq \frac{2}{\alpha} \|f - g\|_{L^1} + \int_{\Omega} E_g(x, r) dx,$$

hence

$$\limsup_{r \searrow 0} \int_{\Omega} E_f(x, r) dx \leq \frac{2\varepsilon}{\alpha},$$

and (8) follows. Returning to the assertion for  $\mathfrak{F}$ . If it was false, then we could find a sequence  $(f_k)$  in  $\mathfrak{F}$  and an  $\varepsilon > 0$  such that

$$\int_{\Omega} E_{f_k}(x, 2^{-k}) dx \geq \varepsilon$$

for all  $k$ . By precompactness of  $\mathfrak{F}$ , passing to a subsequence if necessary, we can assume that  $f_k \rightarrow f$  in  $L^1(\Omega, \mathbb{R}^k)$ . Again, invoking the  $\alpha$ -density condition and Fubini we estimate

$$\varepsilon \leq \int_{\Omega} E_{f_k}(x, 2^{-k}) dx \leq \frac{2}{\alpha} \|f_k - f\|_{L^1} + \int_{\Omega} E_f(x, 2^{-k}) dx,$$

contradicting (8). □

## 2 Differential inclusions

In this section we give an alternative proof of a theorem of Šverák which says that if  $K$  is a compact set of matrices such that there exists an  $m$ -homogeneous null Lagrangian  $\mathcal{L}$  satisfying  $\mathcal{L}(A - B) > 0$  for all distinct  $A, B \in K$ , then approximate solutions are precompact. We remind the reader that throughout this note  $\Omega$  is a fixed bounded and open subset of  $\mathbb{R}^n$ , and that an  $m$ -homogeneous null Lagrangian is a homogeneous polynomial of degree  $m$  of the form  $\mathcal{L}(Df) = \sum_{I,J} \alpha_{I,J} \frac{\partial f^I}{\partial x_J}$ , where  $I = (i_1, \dots, i_m)$ ,  $J = (j_1, \dots, j_m)$  are  $m$ -tuples of numbers from  $\{1, \dots, k\}$ ,  $\{1, \dots, n\}$ , respectively,  $2 \leq m \leq \min(k, n)$ , and  $\alpha_{I,J} \in \mathbb{R}$  are constants. The terms  $\frac{\partial f^I}{\partial x_J}$  are short-hand for the minors

$$\frac{\partial(f^{i_1}, \dots, f^{i_m})}{\partial(x_{j_1}, \dots, x_{j_m})}.$$

**Theorem 2.** *Let  $1 \leq p < \infty$ ,  $K \subset \mathbb{R}^{k \times n}$  be compact and  $\mathcal{L}: \mathbb{R}^{k \times n} \rightarrow \mathbb{R}$  be an  $m$ -homogeneous null Lagrangian. Suppose that for all  $A, B \in K$  with  $A \neq B$ ,  $\mathcal{L}(A - B) > 0$ . Then a sequence  $(f_j)$  in  $W^{1,p}(\Omega, \mathbb{R}^k)$  of approximate solutions (Conditions (i) and (ii) in the introduction) to the inclusion*

$$Df \in K$$

*is precompact in  $W_{\text{loc}}^{1,p}(\Omega, \mathbb{R}^k)$*

Our strategy is to find a uniform bound on the integral modulus of continuity of approximate solutions and then use the results of section 1. To that end we start by quantifying the degenerate condition:  $\mathcal{L}(A - B) > 0$  for all distinct  $A, B \in K$ .

**Lemma 2.1.** *There exists an increasing convex function  $\psi_K: [0, \infty) \rightarrow [0, \infty)$  with  $\psi_K(0) = 0$ ,  $0 < \psi_K(t) \leq t$  for  $t > 0$  such that*

$$\mathcal{L}(A - B) \geq \psi_K(|A - B|^m) \tag{9}$$

for all  $A, B \in K$ .

*Proof.* Let for  $t \in [0, \delta]$ , where  $\delta := (\text{diam}K)^m$ ,

$$\tilde{\varphi}(t) := \inf\{\mathcal{L}(A - B) : A, B \in K \text{ and } |A - B|^m \geq t\}.$$

Put  $\varphi(t) := \min\{\tilde{\varphi}(t), t\}$  on  $[0, \delta]$  and  $\varphi(t) = t$  for  $t > \delta$ . Then  $\varphi: [0, \infty) \rightarrow [0, \infty)$  is increasing,  $\varphi(0) = 0$  and  $t \geq \varphi(t) > 0$  for  $t > 0$  by our assumptions on  $K$  and  $\mathcal{L}$ . By construction,  $\mathcal{L}(A - B) \geq \varphi(|A - B|^m)$  for all  $A, B \in K$ . We now define  $\psi_K$  to be the convex envelope of  $\varphi$ :

$$\psi_K(t) := \sup\{c(t) : c \leq \varphi, \text{ and } c: [0, \infty) \rightarrow \mathbb{R} \text{ is convex}\}.$$

Note that  $\psi_K(0) = 0$  and  $t \geq \psi_K(t) \geq 0$  on  $[0, \infty)$ . Let  $0 < s \leq \delta$  and define

$$c(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{s}{2} \\ \frac{\varphi(s/2)}{\delta - s/2} (t - \frac{s}{2}) & \text{if } t > \frac{s}{2}. \end{cases}$$

Then as  $\varphi(s/2) > 0$  it follows that  $c: [0, \infty) \rightarrow [0, \infty)$  is convex. Because  $\varphi(t) \leq t$  we also have that the slope  $\varphi(s/2)/(\delta - s/2) \leq 1$ , and hence that  $c \leq \varphi$  on  $[0, \infty)$ . Consequently,  $c \leq \psi_K$ , and so in particular,  $0 < c(s) \leq \psi_K(s)$ . Finally, the fact that  $\psi_K$  is increasing follows from  $\psi_K(0) = 0$ ,  $\psi_K \geq 0$  and the convexity.  $\square$

We will use often that for  $0 < \epsilon < 1$  it holds that

$$\psi_K(\epsilon t) \leq \psi_K(\epsilon t + (1 - \epsilon)0) \leq \epsilon \psi_K(t). \quad (10)$$

The next lemma contains an inequality, which is the key ingredient in our argument. It affords a control on the integral modulus of continuity of approximate solutions to the differential inclusion.

**Lemma 2.2.** *Let  $\Omega' \Subset \Omega'' \Subset \Omega$ . There exists a continuous function  $v_K: [0, \infty) \rightarrow \mathbb{R}$ ,  $v_K(0) = 0$  and a constant  $C = C(K, \Omega', \Omega'', \Omega)$  such that for every  $f \in W_{\text{loc}}^{1,m}(\Omega, \mathbb{R}^k)$  and  $y \in \mathbb{R}^n$  such that  $y + \Omega' \subset \Omega''$  it holds that*

$$\psi_K\left(\int_{\Omega'} |Df(x+y) - Df(x)|^m dx\right) \leq C\left(v_K(|y|) + \int_{\Omega} \text{dist}_K^m(Df) dx\right) \quad (11)$$

*Proof.* Since  $K$  is compact we can, as in [9, 29], reduce the situation to the case where  $f$  is Lipschitz. There it is proved that for  $K$  compact and  $f \in W^{1,m}(\Omega, \mathbb{R}^k)$ , truncating  $f$  in terms of level sets of the maximal function of its gradient, we can find an  $L$ -Lipschitz map  $v$  with  $L = L(K)$  such that

$$\int_{\Omega} |Df - Dv|^m dx \leq C \int_{\Omega} \text{dist}_K^m(Df) dx.$$

Thus, since  $\psi_K(t) \leq t$  it suffices to prove (11) when  $f$  is an  $L$ -Lipschitz map, where  $L = L(K)$ . Next, by the measurable selection principle we can find a projection  $P_f: \Omega \rightarrow K$  such that

$$|Df(x) - P_f(x)| = \text{dist}_K(Df(x)).$$

Let, as in the statement of the lemma,  $\Omega' \Subset \Omega'' \Subset \Omega$ ,  $y$  such that  $y + \Omega'' \subset \Omega$  and  $f_y(x) = (f_y^1(x), \dots, f_y^k(x)) := f(x+y) - f(x)$ . Let  $\eta \in C_0^\infty(\Omega'')$  be a cut-off function such that  $\eta = 1$  on  $\Omega'$ ,  $0 \leq \eta \leq 1$  and  $|D\eta| \leq 2\text{dist}(\Omega', \partial\Omega'')^{-1}$ . For a fixed  $y$  as above and almost all  $x \in \Omega'$ , we get by monotonicity and convexity,

$$\begin{aligned} \psi_K\left(3^{-m}|Df_y(x)|^m\right) &\leq 3^{-1}\psi_K\left(|P_f(x+y) - P_f(x)|^m\right) + 3^{-1}\psi_K\left(\text{dist}_K^m(Df(x+y))\right) \\ &\quad + 3^{-1}\psi_K\left(\text{dist}_K^m(Df(x))\right) \\ &\stackrel{(9)}{\leq} 3^{-1}\mathcal{L}\left(P_f(x+y) - P_f(x)\right) + 3^{-1}\psi_K\left(\text{dist}_K^m(Df(x+y))\right) \\ &\quad + 3^{-1}\psi_K\left(\text{dist}_K^m(Df(x))\right). \end{aligned}$$

Multiply by  $\eta(x)^m$  and integrate over  $x \in \Omega$  to get

$$\begin{aligned}
\int_{\Omega} \eta(x)^m \psi_K \left( 3^{-m} |Df_y(x)|^m \right) dx &\leq 3^{-1} \int_{\Omega} \eta(x)^m \mathcal{L} \left( P_f(x+y) - P_f(x) \right) dx \\
&\quad + 3^{-1} \int_{\Omega} \eta(x)^m \psi_K \left( \text{dist}_K^m(Df(x+y)) \right) dx \\
&\quad + 3^{-1} \int_{\Omega} \eta(x)^m \psi_K \left( \text{dist}_K^m(Df(x)) \right) dx \\
&\leq 3^{-1} \int_{\Omega} \eta(x)^m \mathcal{L} \left( P_f(x+y) - P_f(x) \right) dx \\
&\quad + \int_{\Omega} \text{dist}_K^m(Df) dx,
\end{aligned}$$

where the last inequality follows from  $\psi_K(t) \leq t$  and the choice of  $\eta, y$ . Using that  $f$  is  $L$ -Lipschitz,  $\mathcal{L}$  is locally Lipschitz, and  $|Df - P_f| = \text{dist}_K(Df)$  we estimate

$$\int_{\Omega} \eta(x)^m \mathcal{L} \left( P_f(x+y) - P_f(x) \right) dx \leq \int_{\Omega} \left( \eta^m \mathcal{L}(Df_y) + C'_K \text{dist}_K^m(Df) \right) dx,$$

and hence, using (10),

$$\begin{aligned}
\int_{\Omega} \psi_K \left( \eta(x)^m 3^{-m} |Df_y(x)|^m \right) dx &\leq \int_{\Omega} \eta(x)^m \psi_K \left( 3^{-m} |Df_y(x)|^m \right) dx \\
&\leq 3^{-1} \int_{\Omega} \eta^m \mathcal{L}(Df_y) dx \\
&\quad + C_K \int_{\Omega} \text{dist}_K^m(Df) dx,
\end{aligned} \tag{12}$$

where  $C_K = 3^{-1}C'_K + 1$ . Now we estimate the term  $\int_{\Omega} \eta^m \mathcal{L}(Df_y(x)) dx$  using integration by parts. The ensuing calculation is similar to the derivation of reverse Hölder inequalities for quasiregular maps [13], and was brought to this context in [23, 24]. Recall that  $\mathcal{L}(Df) = \sum_{I,J} \alpha_{IJ} \frac{\partial f^I}{\partial x_J}$ , where  $I = (i_1, i_2, \dots, i_m)$ ,  $J = (j_1, j_2, \dots, j_m)$  represents  $m$ -tuples. Consider the minor  $\frac{\partial f^I}{\partial x_J}$  and assume without loss of generality that  $i_1 = 1$ . Let us integrate by parts:

$$\begin{aligned}
\left| \int_{\Omega} \eta^m \frac{\partial f^I}{\partial x_J} dx \right| &= \left| \int_{\Omega} f_y^1 \eta^{m-1} \frac{\partial(\eta, f_y^{i_2}, \dots, f_y^{i_m})}{\partial(x_{j_1}, x_{j_2}, \dots, x_{j_m})} dx \right| \\
&\leq C(\Omega') \int_{\Omega} |\eta^{m-1}| |f_y| |Df_y|^{m-1} dx
\end{aligned} \tag{13}$$

By the properties of  $\psi_K$  we can find constants  $\alpha > 0, \beta \geq 0$  such that  $\psi_K(t) \geq \alpha t - \beta$  holds for all  $t \geq 0$ . The function  $\mathbb{R} \ni t \mapsto \psi_K(|t|^{\frac{m}{m-1}})$  is therefore super-linear, increasing and convex, and so its conjugate function  $\psi_m^*$  is convex and real-valued. Also note that as  $\psi_m^*(0) = 0$ ,  $\psi_m^*(t) \geq (\frac{m}{m-1})^{1-m} \frac{1}{m} |t|^m \geq 0$  and  $\psi_m^*$  is convex, it is also increasing on  $[0, \infty)$ . Let  $\epsilon > 0$  and

observe that by Fenchel–Young’s inequality for convex functions we have for fixed  $y$  as above and for almost every  $x$ ,

$$\begin{aligned} \eta(x)^{m-1}|f_y(x)||Df_y(x)|^{m-1} &\leq \psi_K\left(\frac{\epsilon}{3^m}\eta^m|Df_y|^m\right) + \psi_m^*\left(\frac{3^{m-1}}{\epsilon^{\frac{m-1}{m}}}|f_y|\right) \\ &\leq \epsilon\psi_K(3^{-m}\eta^m|Df_y|^m) + \psi_m^*\left(\frac{3^{m-1}}{\epsilon^{\frac{m-1}{m}}}L|y|\right), \end{aligned} \quad (14)$$

where in the last inequality we have used the monotonicity of  $\psi_m^*$  on  $[0, \infty)$ , that  $f$  is  $L$ -Lipschitz and (10). Thus combining (13) and (14) yield

$$\int_{\Omega} \eta^m \frac{\partial f^I}{\partial x_J} dx \leq \epsilon \int_{\Omega} \psi_K(3^{-m}\eta^m|Df_y|^m) dx + \int_{\Omega} \psi_m^*\left(\frac{3^{m-1}}{\epsilon^{\frac{m-1}{m}}}L|y|\right) dx.$$

Let us write  $v_K(t) = \sum_{IJ} |\alpha_{IJ}| \psi_m^*\left(\frac{3^{m-1}}{\epsilon^{\frac{m-1}{m}}}Lt\right)$ . Then summing in all the subdeterminants in the definition of  $\mathcal{L}$  and inserting the result in (12) we arrive at

$$\begin{aligned} \int_{\Omega} \psi_K(3^{-m}\eta^m|Df_y|^m) dx &\leq C(\Omega', K)\epsilon \int_{\Omega} \psi_K(3^{-m}\eta^m|Df_y|^m) dx \\ &\quad + |\Omega|v_K(|y|) \\ &\quad + C_K \int_{\Omega} \text{dist}_K^m(Df) dx. \end{aligned} \quad (15)$$

Choosing  $\epsilon$  small enough we can absorb the first term in the left–hand side and obtain the following inequality of the reverse Hölder type:

$$\int_{\Omega'} \psi_K(3^{-m}|Df_y(x)|^m) dx \leq C(K, \Omega') \left( |\Omega|v_K(|y|) + \int_{\Omega} \text{dist}_K^m(Df) dx \right). \quad (16)$$

Finally for aesthetics reasons we repeat all the argument with  $\psi'_K(\cdot) = \psi_K(3^m \cdot)$  and a last use of Jensen’s inequality yields (11).  $\square$

**Remark 2.** *We have stated the Lemma for functions in  $W^{1,m}$  for simplicity. A similar result holds for mappings  $f$  in  $W^{1,1}$  with  $\int_{\Omega} \text{dist}_K^m(Df) dx$  replaced by  $\int_{\Omega} \text{dist}_K(Df) dx + (\int_{\Omega} \text{dist}_K(Df) dx)^m$ . It is also possible to adapt the above proof and obtain a similar result when the null Lagrangian is inhomogeneous.*

*Proof of Theorem 2.* By [29] we can reduce to the case where  $(f_j)$  is a sequence of uniformly  $L$ -Lipschitz maps such that

$$\lim_{j \rightarrow \infty} \int_{\Omega} \text{dist}_K^m(Df_j) dx = 0.$$

Let  $0 < |y| \leq \text{diam}(\Omega)$ . Then we want to show that for every  $\epsilon$  there exists  $\delta > 0$  such that

$$|y| \leq \delta \Rightarrow \sup_j \psi_K(\omega_{f_j}(t)) \leq \epsilon.$$

Choose first  $J$  so that for  $j \geq J$ ,

$$\int_{\Omega} \text{dist}_K^m(Df_j) \leq \frac{\epsilon}{2}.$$

Then since  $|Df_j| \in L^m(\Omega)$  there exists  $\delta_J$  such that for each  $j < J$  and  $t < \delta_J$  it holds that  $\omega_{f_j}(t) \leq \frac{\epsilon}{2}$ . Declare  $\delta \leq \min\{\delta_J, v_K^{-1}(\frac{\epsilon}{2})\}$ , apply Lemma 2.2 and the proof follows.  $\square$

The arguments in Lemma 2.2 expresses also how regular are exact solutions  $Df \in K$ . The following is a result of this type where the compactness assumption on  $K$  is not needed.

**Proposition 1.** *Let  $n < p < \infty$  and  $f \in W_{\text{loc}}^{1,p}(\Omega, \mathbb{R}^n)$  be a solution to the inclusion*

$$Df \in K$$

*Let us assume that for a constant  $\epsilon > 0$  and for all  $A, B$  in  $K$  it holds that*

$$\det(A - B) \geq \epsilon |A - B|^p, \quad (17)$$

*then  $f \in W_{\text{loc}}^{1+\theta,p}(\Omega, \mathbb{R}^n)$  for  $\theta < \frac{1}{p-n+1}$ .*

*Sketch of Proof.* The proof follows the same lines as the arguments of Lemma 2.2 but it is simpler because we have the exact inclusion. In the following we choose  $|y|$  small enough and, for  $\Omega' \Subset \Omega'' \Subset \Omega$ , a cut-off function  $\eta \in C_0^\infty(\Omega'')$  with  $\eta = 1$  in  $\Omega'$ . The starting point is the pointwise inequality (17) which implies that,

$$\epsilon \int_{\Omega} |Df_y|^p \eta^n \, dx \leq \int_{\Omega} \eta^n \det(Df_y(x)) \, dx.$$

As before the gain comes from integrating by parts the determinant (here  $m = n$ ). We arrive at the estimate

$$\epsilon \int_{\Omega} |Df_y|^p \eta^n \, dx \leq C \int_{\Omega} |f_y| \eta^{n-1} |Df_y|^{n-1} \, dx.$$

Next, we directly use Hölder's inequality with exponents  $\frac{p}{n-1}, \frac{p}{p-n+1}$ . Manipulating the resulting terms conveniently we obtain that

$$\int_{\Omega'} |Df_y|^p \, dx \leq c \int_{\Omega''} |f_y|^{\frac{p}{p-n+1}} \, dx. \quad (18)$$

Finally, we use that since  $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ , it follows that  $\int_{\Omega''} |f_y|^p \, dx \leq \|f\|_{W^{1,p}}^p |y|^p$ . Thus, another use of Hölder's inequality in (18), this time with exponents  $p-n+1, \frac{p-n+1}{p-n}$ , yields that

$$\int_{\Omega'} |Df_y|^p \, dx \leq C(\Omega', \Omega'') \|f\|_{W^{1,p}}^p \left( \int_{\Omega''} |f_y|^p \right)^{\frac{1}{p-n+1}} \leq |y|^{p \frac{1}{p-n+1}} C(\Omega', \Omega'') \|f\|_{W^{1,p}}^p. \quad (19)$$

Hence  $f \in B_p^{1+\frac{1}{p-n+1}, \infty}(\Omega', \mathbb{R}^n)$  and by the relation between Besov and fractional Sobolev spaces the thesis follows.

**Remark 3.** *We have stated the result for homogeneous null Lagrangians. However it is clear that the proof would work equally well with other functions  $\mathcal{L}: \mathbb{R}^{k \times n} \times \mathbb{R}^{k \times n} \rightarrow \mathbb{R}$  which can be suitably integrated by parts. Besides inhomogeneous null Lagrangians one could also treat functions of the form  $(A, B) \mapsto \langle \text{Cof}(A) - \text{Cof}(B), A - B \rangle$  as also suggested in [24].*

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