Oxford HT10, JK

Regularity Theory in the Calculus of Variations : Part II

Wednesdays 11.15 am – 1.15 pm (Start: 3/2/10)

Contents

References

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] A. Bensoussan and J. Frehse. *Regularity results for nonlinear elliptic systems and applications.* Applied Mathematical Sciences, 151. Springer-Verlag, Berlin, 2002.

[3] L. Caffarelli and I. Peral. On $W^{1,p}$ estimates for elliptic equations in divergence form. *Comm. Pure Appl. Math.* 51 (1998), 1–21.

[4] L.C. Evans and R.F. Gariepy. *Measure theory and fine properties of functions.* CRC Press, 1992.

[5] I. Fonseca and G. Leoni. *Modern methods in the Calculus of Variations: L^p spaces.* Springer–Verlag, 2007.

[6] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Annals of Mathematics Studies, 105. Princeton University Press, Princeton, NJ, 1983.

[7] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order.* Springer–Verlag, 1977.

[8] E. Giusti. *Direct methods in the calculus of variations.* World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[9] T. Iwaniec and G. Martin. *Geometric function theory and non–linear analysis.* Oxford University Press, 2001.

[10] A. Koshelev. *Regularity problem for quasilinear elliptic and parabolic systems.* Lecture Notes in Mathematics, 1614. Springer–Verlag, Berlin, 1995.

[11] J. Kristensen and G. Mingione. The singular set of ω-minima. *Arch. Ration. Mech. Anal.* 177 (2005), 93–114.

[12] J. Kristensen and G. Mingione. The singular set of minima of integral functionals. *Arch. Ration. Mech. Anal.* 180 (2006), 331–398.

[13] J. Malý and W.P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations.* Amer. Math. Soc., 1997.

[14] G. Mingione. Regularity of minima: an invitation to the dark side of the Calculus of Variations. *Applications of Mathematics* 51 (2006), 355–425.

[15] C.B. Morrey. *Multiple integrals in the Calculus of Variations.* Springer–Verlag, 1966.

[16] E.M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press, Princeton, NJ, 1993.

[17] W.P. Ziemer. *Weakly differentiable functions.* Springer–Verlag, 1989.

Let $1<p<\infty$, $\Omega \subset \mathbb{R}^n$ bounded and open, $F = F(x,y,z)$

$F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ normal integrand

(Borel measurable and $(y,z) \mapsto F(x,y,z)$ lsc for each fixed $x \in \Omega$) and

(1) $\ell |z|^p \leq F(x,y,z) \leq L(|z|^p + 1) \quad \forall x,y,z$

where $0 < \ell \leq L < \infty$ are constants.

Then $\mathcal{F}[v,\Omega] := \int_\Omega F(x,v(x),Dv(x))\,dx$ is well-defined for all $v \in W^{1,p}(\Omega,\mathbb{R}^N)$.

Assume $z \mapsto F(x,y,z)$ is convex for each fixed $(x,y) \in \Omega \times \mathbb{R}^N$ and let $g \in W^{1,p}(\Omega,\mathbb{R}^N)$. Then $\mathcal{F}[v,\Omega]$ admits a minimizer $u \in W^{1,p}_g(\Omega,\mathbb{R}^N)$, ie

$$\mathcal{F}[u,\Omega] \leq \mathcal{F}[v,\Omega] \quad \forall v \in W^{1,p}_g .$$

(Pf by direct method)

— Don't expect uniqueness nor regularity on scale $C^{1,\alpha}_{loc}$. Note $\mathcal{F}$ isn't convex due to y-dependence in integrand F.

However things aren't entirely bad, there is some regularity — not all $W^{1,p}$-maps are

good enough to ~~be~~ $\mathcal{F}$-minimizers
and it basically comes down to the coercivity & growth condition (1). In Part I of the course we discussed the notion of quasiminimizer — recall its definition:

Let $Q \geq 1$.

$u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a Q-minimizer for $\int_\Omega (1+|Dv|^p)$ iff

$$\int_\omega (1+|Du|^p) \leq Q \int_\omega (1+|Dv|^p) \quad \forall v \in W^{1,p}_u(\omega, \mathbb{R}^N)$$

for all open $\omega \subseteq \Omega$.

~~If the con~~

Remarks If the condition only holds for all balls $\omega = B(x,R) \subset \Omega$ then we call u a spherical Q-minimizer. (As it turns out this is a strictly weaker condition.)

Observation. u is a Q-min for $\int_\Omega (1+|Dv|^p)$.

Fix open $\omega \subseteq \Omega$ and $v \in W^{1,p}_u(\omega, \mathbb{R}^N)$.

$$\ell \int_\omega |Du|^p \leq \mathcal{F}(u,\omega) \leq \mathcal{F}[v,\omega] \leq L \int_\omega (1+|Dv|^p)$$

ie $\ell \int_\omega (1+|Du|^p) \leq \ell|\omega| + L\int_\omega (1+|Dv|^p)$

$$\leq (\ell+L)\int_\omega (1+|Du|^p)$$

so we can use $Q = 1+\frac{L}{\ell}$.

We can now forget about the nasty integrand F — we sacrifice minimality for Q-minimality of a nice integral. In the following it would suffice if u was merely a spherical Q-minimizer.

Let $B_{2R} = B(x,2R) \subset \Omega$ and take a Lipschitz cut-off ρ s.t. for $R<r<s<2R$ we have $\mathbb{1}_{B_r} \leq \rho \leq \mathbb{1}_{B_s}$ and $|D\rho| \leq \frac{1}{s-r}$.

Note $v := u - \rho(u-a) \in W_0^{1,p}(B_s, \mathbb{R}^N)$ $(a \in \mathbb{R}^N)$

ie $\int_{B_s} (1+|Du|^p) \leq Q\int_{B_s}(1+|Dv|^p) =$

$Q\int_{B_s}(1+|(1-\rho)Du - (u-a)\otimes D\rho|^p)$

<u>Notation</u> If $a\in\mathbb{R}^N, b\in\mathbb{R}^n$ then $a\otimes b = (a_i b_j)^{1\leq i\leq N}_{1\leq j\leq n} \in \mathbb{R}^{N\times n}$

Use $|z_1+z_2|^p \leq 2^{p-1}(|z_1|^p + |z_2|^p)$ $\forall z_1, z_2$

$\int_{B_s}(1+|Du|^p) \leq Q\int_{B_s}(1+2^{p-1}(1-\rho)^{p-1}|Du|^p + 2^{p-1}|u-a|^p|D\rho|^p)$

$$\leq Q|B_s| + Q2^{p-1}\int_{B_s\setminus B_r}|Du|^p + 2^{p-1}Q\int_{B_s}\left(\frac{|u-a|}{s-r}\right)^p$$

ie

$$\int_{B_s}|Du|^p \leq (Q-1)|B_s| + Q2^{p-1}\int_{B_s\setminus B_r}|Du|^p + Q2^{p-1}\int_{B_s}\left(\frac{|u-a|}{s-r}\right)^p$$

seek to remove this term; can't just take $r\uparrow s$

<u>Widman's hole filling trick</u>: add $Q2^{p-1}\int_{B_r}|Du|^p$

$$(1+Q2^{p-1})\int_{B_r}|Du|^p \leq (Q-1)|B_s| + Q2^{p-1}\int_{B_s}|Du|^p + Q2^{p-1}\int_{B_s}\frac{|u-a|^p}{(s-r)^p}$$

ie

$$\int_{B_r}|Du|^p \leq \frac{Q-1}{1+Q2^{p-1}}|B_s| + \frac{Q2^{p-1}}{1+Q2^{p-1}}\int_{B_s}|Du|^p + \frac{Q2^{p-1}}{1+Q2^{p-1}}\int_{B_s}\frac{|u-a|^p}{(s-r)^p}$$

Of form:

$$\int_{B_r}|Du|^p \leq \theta\int_{B_s}|Du|^p + A + \frac{B}{(s-r)^p}$$

for all $R < r < s < 2R$, where

$$\theta := \frac{Q2^{p-1}}{1+Q2^{p-1}} \in (0,1),$$

$$A := \frac{Q-1}{1+Q2^{p-1}}|B_{2R}|, \quad B := \frac{Q2^{p-1}}{1+Q2^{p-1}}\int_{B_{2R}}|u-u_{x,2R}|^p$$

We now use iteration to get rid of the θ-term.

<u>Iteration Lemma</u> If $f : [R, 2R] \to [0, \infty)$ is nondecreasing, and

$$f(r) \le \theta f(s) + A + \frac{B}{(s-r)^p}$$

for all $R < r < s < 2R$, where $A, B, p \ge 0$ and $0 < \theta < 1$ are constants, then

$$f(R) \le c\left(A + \frac{B}{R^p}\right)$$

for a constant $c = c(\theta)$.

$\boxed{\text{Pf}}$ Exercise. $\square$ (Or see [8], Lemma 6.1, p.191.)

Hereby we find constant $c = c(\theta)$ s.t.

$$(*) \quad ⨍_{B_R} |Du|^p \le c + c ⨍_{B_{2R}} \frac{|u - u_{x,2R}|^p}{R^p}$$

(taking $a = u_{x,2R}$).

This is a Caccioppoli inequality (of the 1st kind): integral averages of derivative bounded by corresponding integral averages of function. It's an inhomogeneous reverse Poincaré inequality on increasing supports:

- 'inhomogeneous' because of additive constant c on RHS
- 'increasing supports' because we have B_R on LHS and B_{2R} on RHS.

Note: There is nothing special about B_R and B_{2R} — we could easily modify the proof with B_R and $B_{\sigma R}$ for any $\sigma > 1$ to get

$$\fint_{B_R} |Du|^p \leq c + \frac{c}{(\sigma-1)^p} \fint_{B_{\sigma R}} \frac{|u - u_{x,\sigma R}|^p}{R^p}.$$

Recall Poincaré–Sobolev's inequality

$$\fint_{B_{2R}} |u - u_{x,2R}|^p \leq c R^p \left(\fint_{B_{2R}} |Du|^{\frac{np}{n+p}} \right)^{\frac{n+p}{n}}$$

where $c = c(n,p)$ is a constant.

(Since $\frac{np}{n+p} < n$ and $W^{1,\frac{np}{n+p}} \hookrightarrow L^p \cdots$)

Consequently we get from ⊛

$$\fint_{B_R} |Du|^p \leq c + c \left(\fint_{B_{2R}} |Du|^{\frac{np}{n+p}} \right)^{\frac{n+p}{n}}$$

or equivalently:

$$(**) \quad \fint_{B_R} f^q \leq c + c \left(\fint_{B_{2R}} f \right)^q$$

for all balls $B_{2R} \subset \Omega$, where

$f := |Du|^{\frac{np}{n+p}}$, $q = \frac{n+p}{n} = 1 + \frac{p}{n} > 1$.

Inhomogeneous reverse Hölder inequality on increasing supports.

Gehring: (**) self-improves!

$\exists\, q_0 = q_0(n,c,q) > q$ s.t. $f \in L^{q_0}_{loc}(\Omega)$

and $$\fint_{B_R} f^{\tilde q} \leq \tilde c + \tilde c \left(\fint_{B_{2R}} f\right)^{\tilde q}$$

for all $\tilde q \leq q_0$ and all balls $B_{2R} \subset \Omega$.

<u>TH 1</u> Let $1<p<\infty$ and $Q \geq 1$.

There exists $q_0 = q_0(n,p,Q) > p$ s.t. any (spherical) Q-minimizer $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ for $\int_\Omega (1+|Dv|^p)$ belongs to $W^{1,q_0}_{loc}(\Omega,\mathbb{R}^N)$,

and $$\fint_{B_R} |Du|^q \leq c + c\left(\fint_{B_{2R}} |Du|^p\right)^{\frac{q}{p}}$$

for all $q \leq q_0$ and all balls $B_{2R} \subset \Omega$.

<u>Remarks.</u>

- When Ω is a bounded Lipschitz domain and $u \in W^{1,p}_g(\Omega,\mathbb{R}^N)$ with $g \in W^{1,q_0}(\Omega,\mathbb{R}^N)$ then $u \in W^{1,q_0}(\Omega,\mathbb{R}^N)$ with estimates involving g.
- It can be shown that $q_0 \nearrow \infty$ as $Q \searrow 1$.